EECS 240
Analog Integrated Circuits

Lecture 6: Noise Analysis

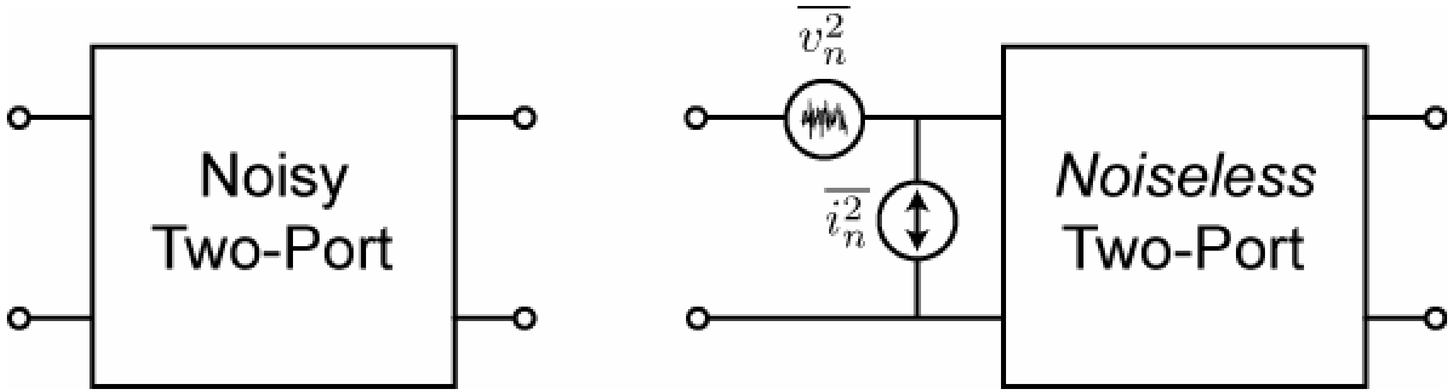
Ali M. Niknejad and Bernhard E. Boser

Department of Electrical Engineering and Computer Sciences

Input Equivalent Noise

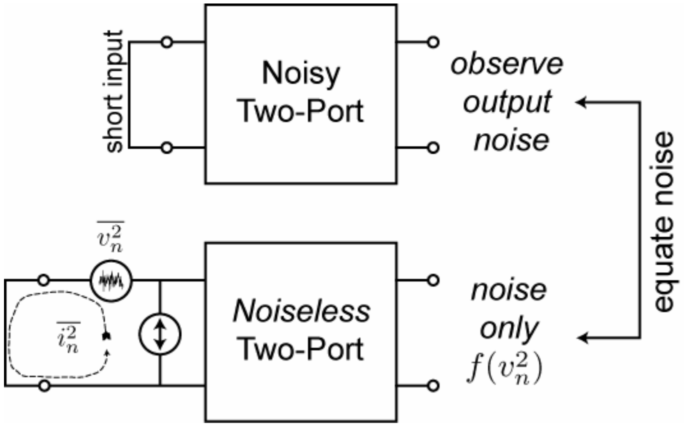
- Two-port representation
- Fictitious noise sources
- Effect of source resistance
- Correlation

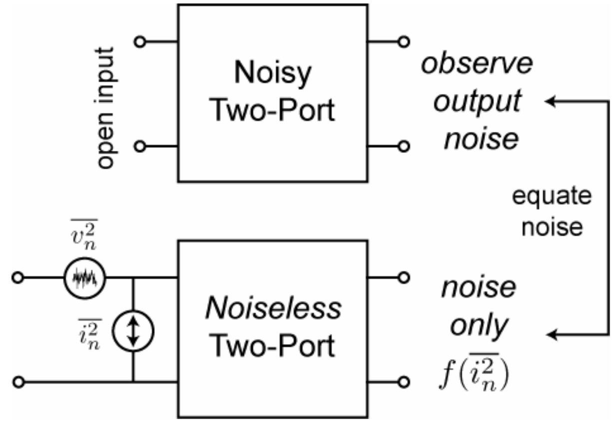
Equivalent Noise Generators



- Any noisy two port can be replaced with a *noiseless* two-port and equivalent input noise sources
- In general, these noise sources are correlated. For now let's neglect the correlation.

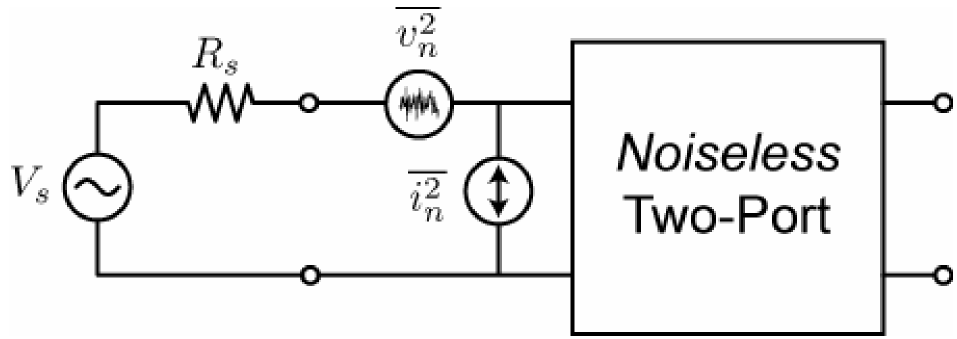
Finding the Equivalent Generators





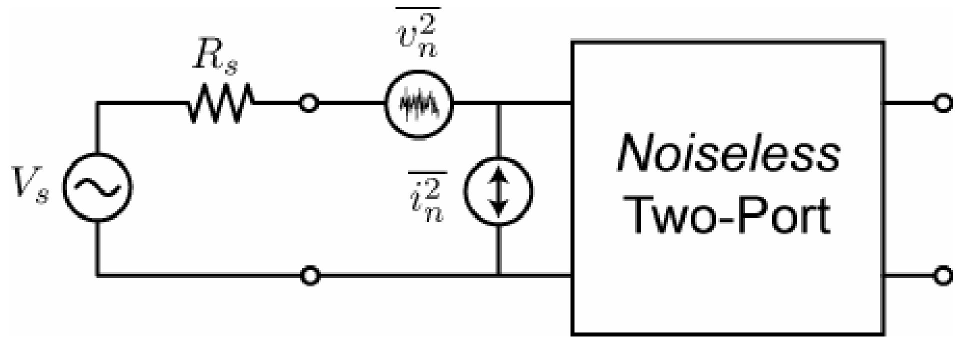
- The equivalent sources are found by opening and shorting the input and equating the noise.
- For a shorted input, the input current flows through the short and the output is due only to the input noise.
- For an open input, the dangling voltage does not contribute to the output noise.

Role of Source Resistance



- If $R_s = 0$, only the voltage noise v_n is important. Likewise, if $R_s = \infty$, only the current noise i_n is important.
- Amplifier Selection: If R_s is large, then select an amp with low i_n (MOS). If R_s is low, pick an amp with low v_n (BJT).
- For a given R_s, there is an optimal v_n/i_n ratio.
- Alternatively, for a given amp, there is an optimal R_s.

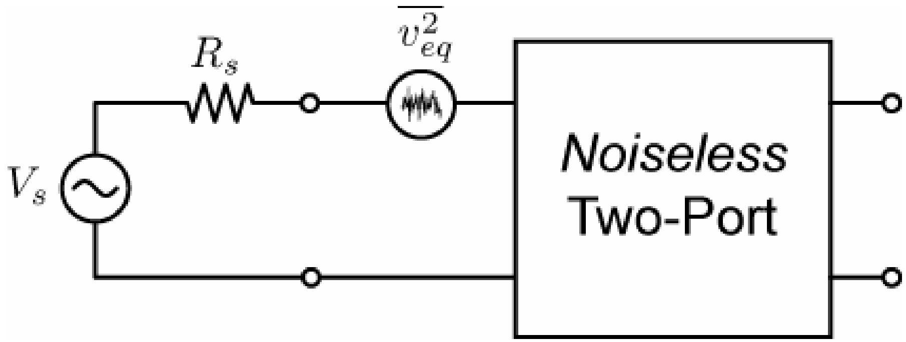
Total Output Noise



$$\overline{v_o^2} = (\overline{v_n^2}A_v^2 + \overline{v_{R_s}^2}A_v^2)\left(\frac{R_{in}}{R_{in}+R_s}\right)^2 + \left(\frac{R_{in}}{R_{in}+R_s}\right)^2 R_s^2\overline{i_n^2}A_v^2$$

$$= (\overline{v_n^2} + \overline{i_n^2}R_s^2 + \overline{v_{R_s}^2})\left(\frac{R_{in}}{R_{in}+R_s}\right)^2 A_v^2$$

New Equivalent Generator



$$\overline{v_{eq}^2} = \overline{v_n^2} + \overline{i_n^2} R_s^2$$

- We see that the total noise can be lumped into one equivalent voltage once R_s is known.
- Be careful! Up to now we ignored the correlation.

Optimum Source Impedance

$$R_{opt} = \sqrt{\frac{R_n}{G_n}} = \sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}}$$

- For a given two-port (amplifier), what's the optimum source impedance? Find the total output noise and find the minima for R_s to find R_{opt}.

Correlated Noise Sources

- Let's partition the input noise current into two components, a component correlated ("parallel") to the noise voltage and a component uncorrelated ("perpendicular") of the noise voltage

$$i_n = i_c + i_u$$

$$< i_u, v_n > = 0 \qquad i_c = Y_C v_n$$

$$v_{eq} = v_n(1 + Y_C Z_S) + Z_S i_u$$

Equivalent Noise Voltage (cor)

- Since the above expression is the sum of two uncorrelated noise voltages, we have

$$\overline{v_{eq}^2} = \overline{v_n^2}|1 + Y_C Z_S|^2 + |Z_s|^2 \overline{i_u^2}$$

- Now we can continue as before to find

$$B_{opt} = B_s = -B_c$$

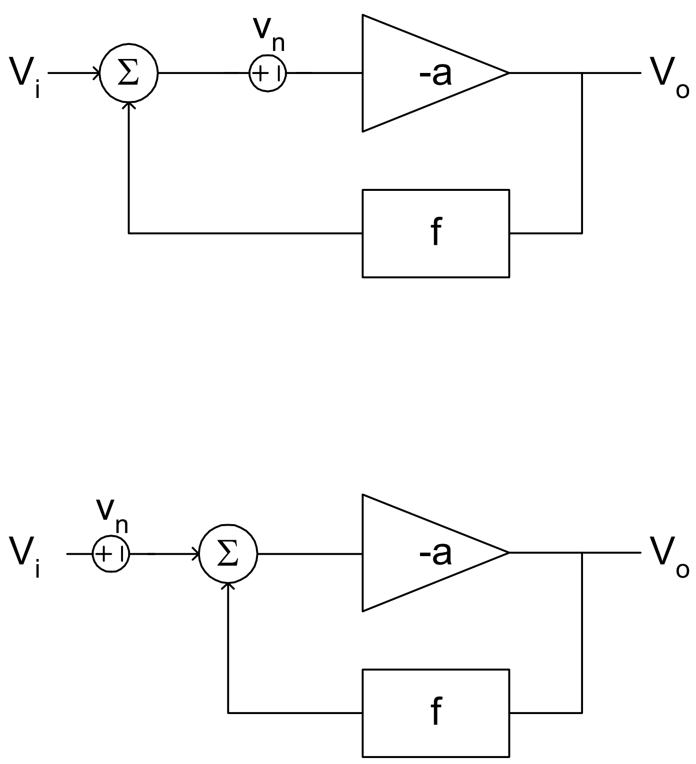
$$G_{opt} = G_s = \sqrt{\frac{G_u}{R_n} + G_c^2}$$

Noise and Feedback

- Ideal feedback:
  - No increase of input referred noise
  - No decrease of SNR at output
- Practical feedback: increased noise
  - Noise from feedback network
  - Noise gain from elements outside feedback loop
- System level: feedback can mitigate noise problems
  - E.g. under-damped accelerometer

    Ref: M. Lemkin and B. E. Boser, "A Three-Axis Micromachined Accelerometer with a CMOS Position-Sense Interface and Digital Offset-Trim Electronics," *IEEE J. Solid-State Circuits*, vol. SC-34, pp. 456-468, April 1999.

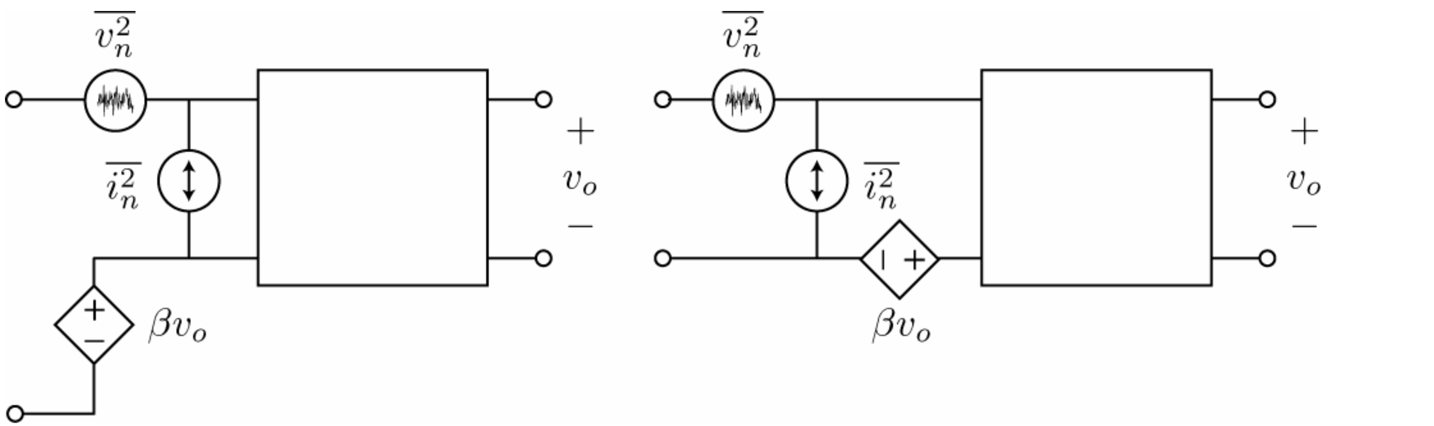
Ideal Feedback



Order of summation irrelevant:

→ Circuits are identical

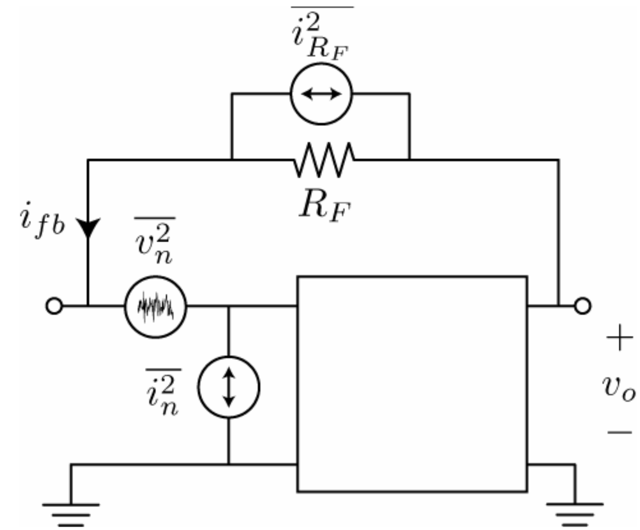
Ideal Feedback and Noise



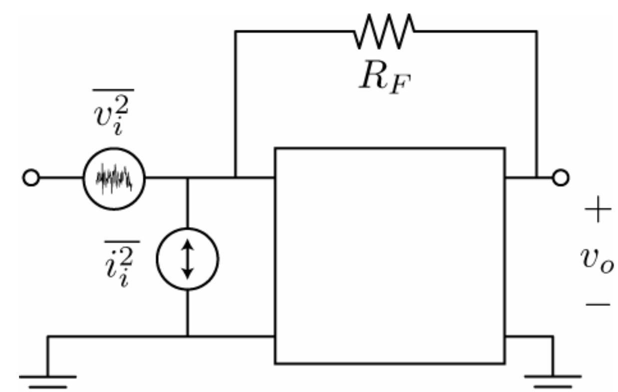
- It's clear that the ideal feedback network does not alter the noise of the system. Real feedback elements, though, have noise.

Example: Shunt Feedback

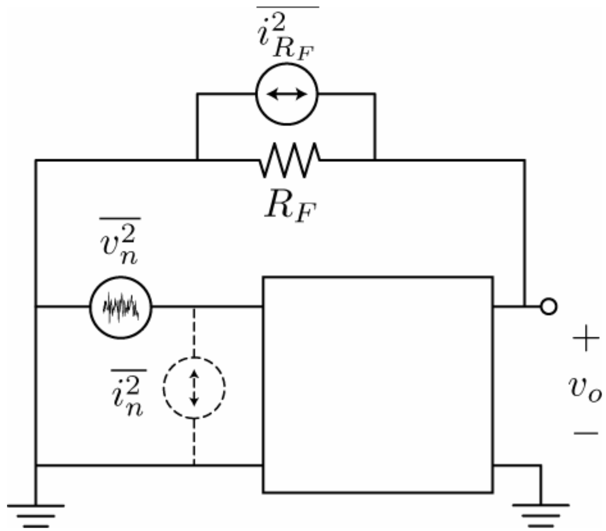
- Shunt feedback samples the output voltage and subtracts from the input current. It's thus most effective in a trans-resistance amplifier configuration.

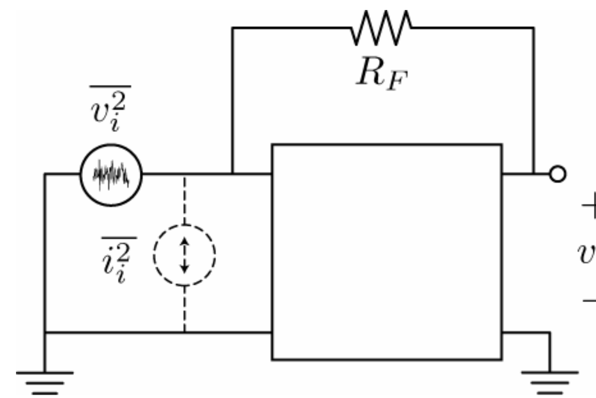


- The action of the feedback is to lower the input and output impedance. In a typical implementation, the resistor R_F adds thermal noise to the input.



Shunt FB Analysis





- To find the equiv input noise voltage, we short the input (and thus the noise current).
- The output noise is clearly given by the two-port voltage gain squared. Even though we don't think of this as a voltage amplifier, we can still use the trans-resistance gain since $i_{in} = v_{in}/Z_{in}$

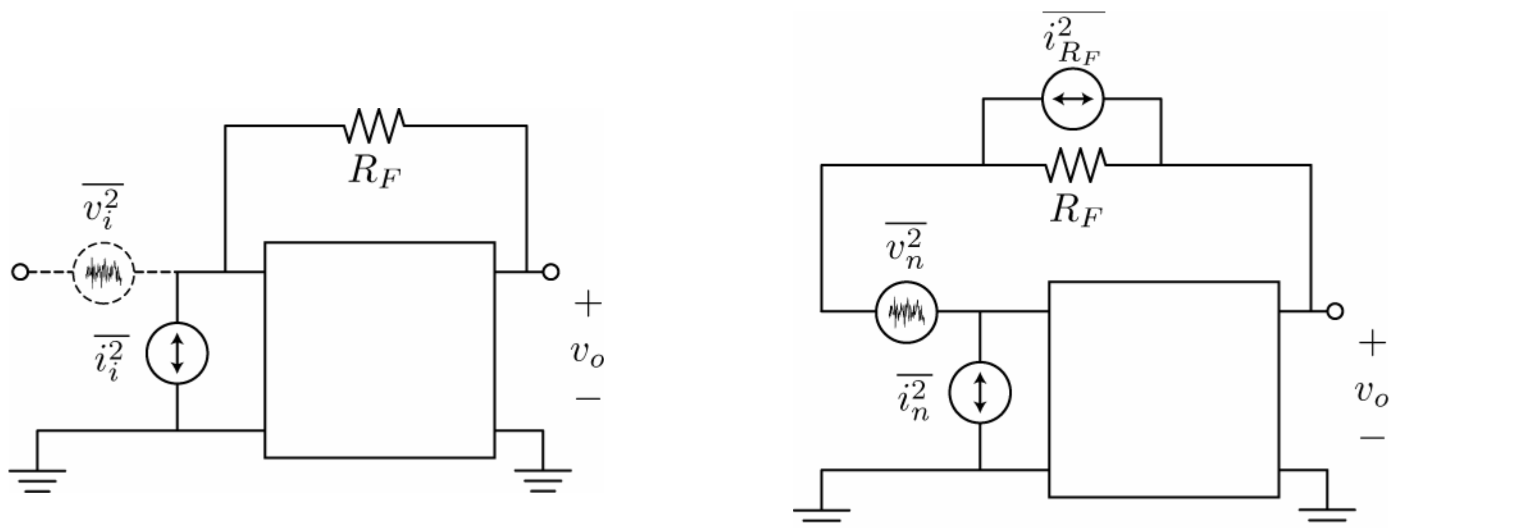
Shunt FB Equiv Voltage Noise

$$\left.\begin{aligned}\overline{v_o^2} &= \overline{v_i^2}|A_v|^2 \\ \overline{v_o^2} &= \overline{v_n^2}|A_v|^2\end{aligned}\right\} \quad \overline{v_i^2} = \overline{v_n^2}$$

- The *voltage gain* does not change with feedback since we are voltage driving the circuit.
- Since the input current is independent of the feedback noise current, it does not alter the output noise.

Shunt FB Equiv Noise Current

- If we leave the input terminal open-circuited, then the input voltage noise for the equiv circuit is disabled. For the real circuit, though, the input noise is active through R_F.



Shunt Noise Current

- Suppose the closed-loop gain of the circuit is given by $Z_{cl} \approx R_F$. Then we have

$$\overline{v_o^2} = |Z_{cl}|^2 \overline{i_i^2}$$

- For the full circuit, let α represent the input current division between the two-port and the feedback network. If the two-port output imepdance is small, we have $\alpha = Z_{in}/(Z_{in} + R_F) \approx 1$

$$\overline{v_o^2} = |Z_{cl}|^2 \overline{i_{ia}^2} |\alpha|^2 + \overline{i_F^2} |Z_{cl}|^2 |\alpha|^2 + \frac{\overline{v_{ia}^2}}{R_F^2} |Z_{cl}|^2$$

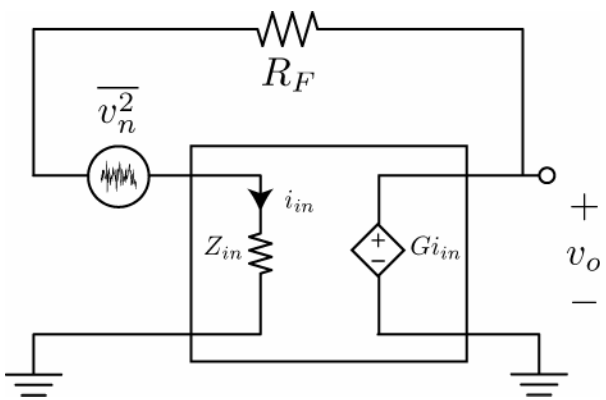
Shunt Current (cont)

- The first two terms are pretty obvious. The last term requires a small calculation as shown below

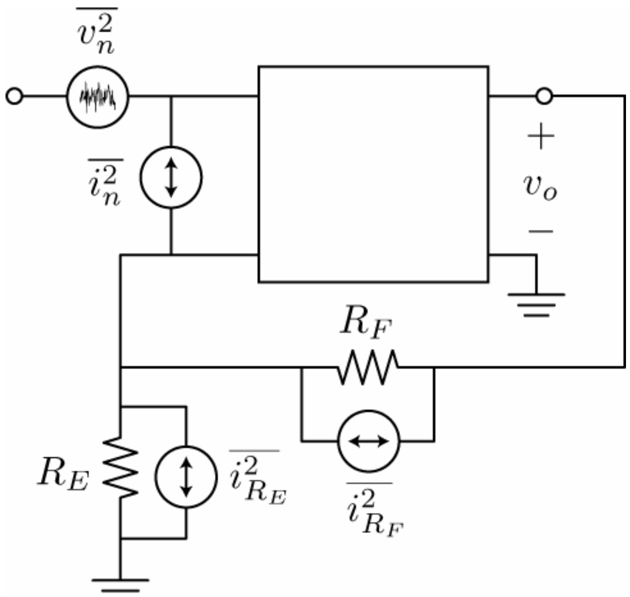
$$v_o = G i_{in} = G \frac{v_o - v_{ia}}{R_F + Z_{in}}$$

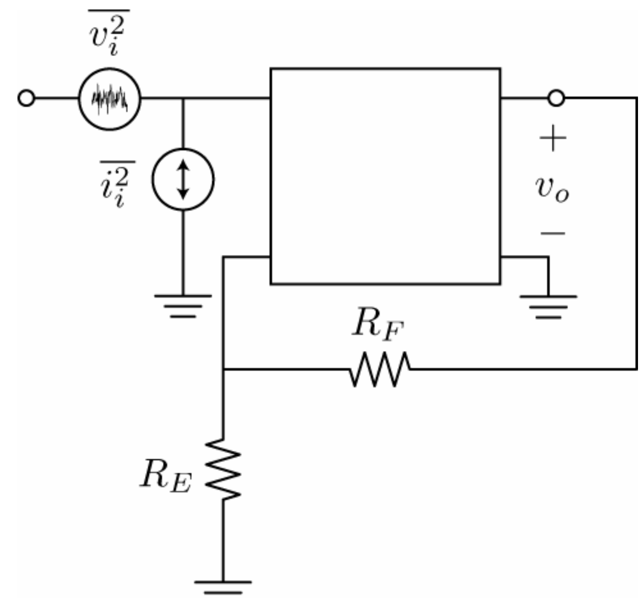
$$v_o \left(1 - \frac{G}{R_F + Z_{in}}\right) = -G \frac{v_{ia}}{R_F + Z_{in}}$$

$$v_o = -\frac{G}{1+T} \frac{v_{ia}}{R_F + Z_{in}} \approx -Z_{cl} \frac{v_{ia}}{R_F}$$



Series-Shunt Feedback

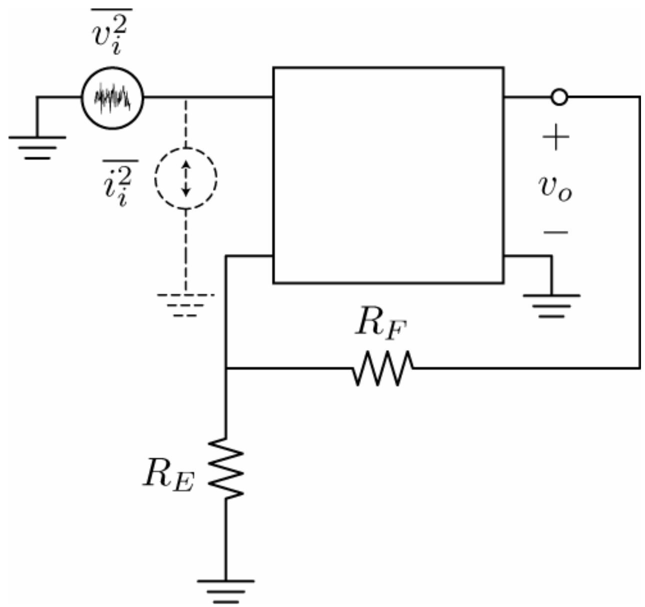


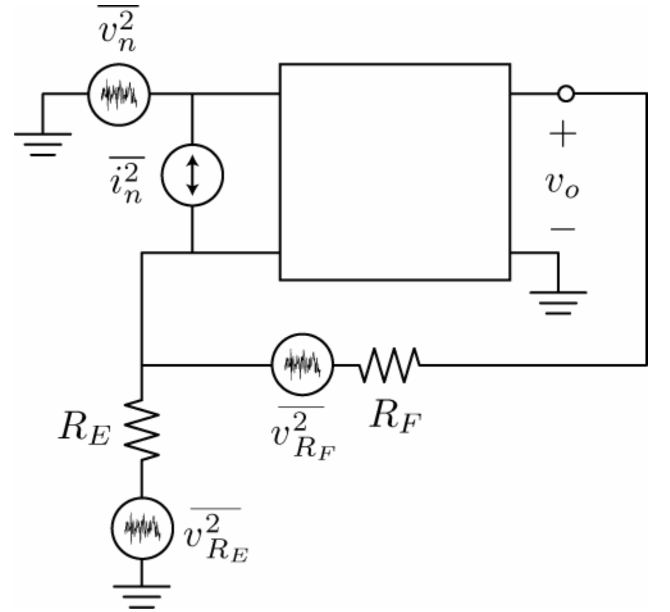


- In a feedback voltage amplifier, open-circuit the input to find the equivalent input noise. It's clear from the above figure that

$$\overline{i_i^2} = \overline{i_n^2}$$

Series-Shunt Shorted Input





- Assuming high/low input/output impedance, the current i_n develops a voltage across $R_E||R_F$.
- The noise voltages due to R_E and R_F generate an input that is readily calculated by voltage division.

Series-Shunt Noise Voltage

- We compute the transfer function from each noise source to the output using superposition. Let A_{cl} represent the closed-loop voltage gain

$$\overline{v_o^2} = |A_{cl}|^2 \left\{ \overline{v_{ia}^2} + \overline{i_{ia}^2} R_E || R_F + \left(\frac{R_F}{R_E + R_F} \right)^2 \overline{v_{R_E}^2} + \left(\frac{R_E}{R_E + R_F} \right)^2 \overline{v_{R_F}^2} + \right\}$$

- The above can be simplified to

$$\overline{v_o^2} = |A_{cl}|^2 \left\{ \overline{v_{ia}^2} + \overline{i_{ia}^2} (R_E || R_F) + 4kt\Delta f (R_E || R_F) \right\}$$

Series-Shunt Voltage (cont)

- Since the same closed-loop gain is used for the equivalent noise generator, we have

$$\overline{v_i^2} = \overline{v_{ia}^2} + \overline{i_{ia}^2}(R_E||R_F) + 4kt\Delta f(R_E||R_F)$$

- If the loop gain and closed-loop gain is large, we have

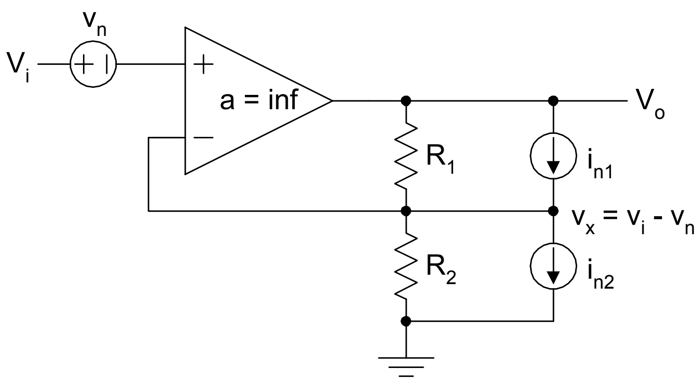
$$A_{cl} \approx \frac{1}{f} = 1 + \frac{R_F}{R_E} \gg 1$$

- Which means that the noise of R_E dominates.

Feedback Summary

- For quick approximations, simply consider the loading effect of the feedback network on the input and associate a noise to this element.
- For shunt-shunt feedback, the loading at the input is R_F. Since the input is a current, represent this as an input noise current.
- For series-shunt feedback, the loading is $R_F || R_E$ (short the output). Since the input is a voltage, we associate a noise voltage with this element.

Example: Non-Inverting Amp



$$v_o = R_1\left(\frac{v_x}{R_2} - i_{n1} + i_{n2}\right) + v_x$$

$$= v_x\left(1 + \frac{R_1}{R_2}\right) - \left(i_{n1} - i_{n2}\right)R_1$$

$$\overline{v_{ieq}^2} = \overline{v_n^2} + \left(\overline{i_{n1}^2} + \overline{i_{n2}^2}\right)\left(\frac{R_1 R_2}{R_1 + R_2}\right)^2$$

$$\cong \overline{v_n^2} + \underbrace{4k_B T R_o \Delta f}_{\overline{v_{nfb}^2}}$$

<u>Example</u>:

$$R_1 = R_o\left(A_{v0} - 1\right)$$

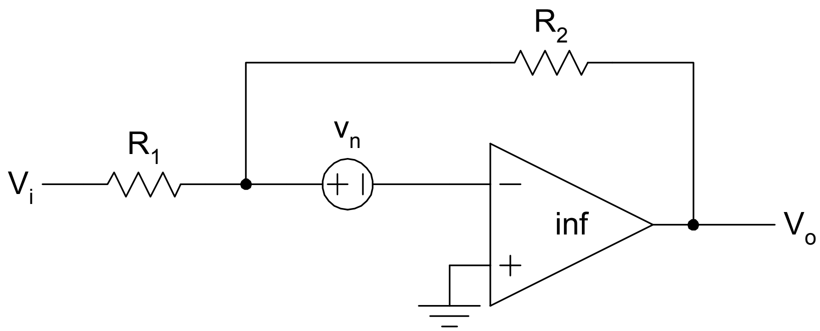
$$R_2 = R_o = 100\text{k}\Omega$$

$$A_{v0} = 10$$

$$\frac{\overline{v_{nfb}^2}}{\Delta f} = \left(40\frac{\text{nV}}{\sqrt{\text{Hz}}}\right)^2$$

- Decreasing R_o reduces noise but increases feedback current

Example: Inverting Amplifier



$$v_o = -v_i \underbrace{\frac{R_2}{R_1}}_{-A_{v0}} + v_n \frac{R_1 + R_2}{R_1}$$

$$\overline{v_{ieq}^2} = \overline{v_n^2}\left(\frac{R_1 + R_2}{R_1}\frac{R_1}{R_2}\right)^2$$

$$= \overline{v_n^2}\left(\frac{R_1 + R_2}{R_2}\right)^2$$

$$= \overline{v_n^2}\left(1 + \frac{1}{|A_{v0}|}\right)^2$$

Example:

$$A_{v0} = -1: \qquad \sqrt{\frac{\overline{v_{ieq}^2}}{\overline{v_n^2}}} = 2$$

$$A_{v0} = -10: \qquad \sqrt{\frac{\overline{v_{ieq}^2}}{\overline{v_n^2}}} = 1.1$$

$$A_{v0} = -0.1: \qquad \sqrt{\frac{\overline{v_{ieq}^2}}{\overline{v_n^2}}} = 11$$

Noise from R_1, R_2 ignored.

Note: R_1 is outside feedback loop → signal and noise have different gains to output.

$$\boxed{\overline{v_{ieq}^2} \neq \overline{v_n^2}}$$

Example: MOS S&H

- Sampling noise:

$$\overline{v_{on}^2(f)} = 4k_B TR\left|\frac{1}{1+sRC}\right|^2$$

$$\overline{v_{oT}^2} = \int_0^\infty \overline{v_{on}^2(f)}\,df$$

$$= \frac{k_B T}{C}$$

- Noise bandwidth:

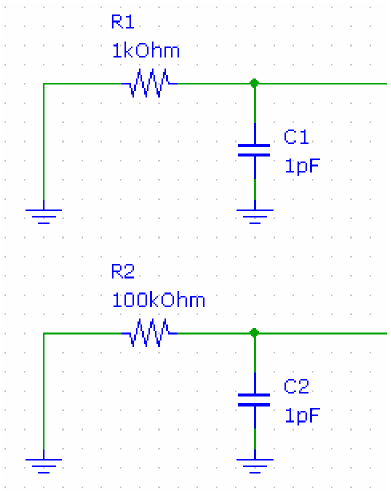
$$4k_B TRB = \frac{k_B T}{C}$$

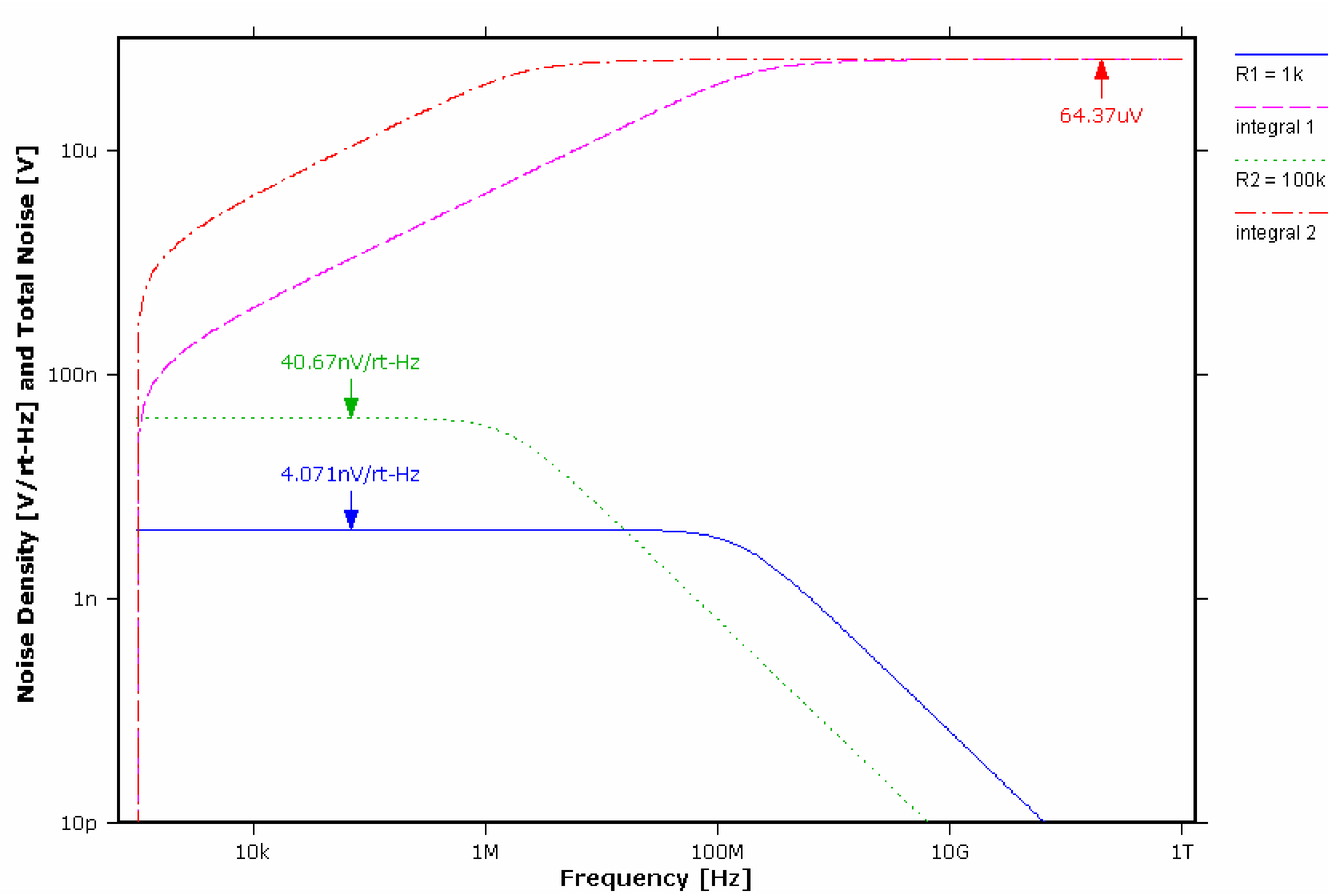
$$B = \frac{1}{4RC} = \frac{\pi}{2} f_o \qquad \text{with} \qquad f_o = \frac{1}{2\pi RC}$$

Sampling Noise

- "kT/C" noise
- Application: ADC, SC circuits, …
- Aliasing
- Variance of noise sample
- Spectral density of sampled noise

SPICE Verification





Useful Integrals

$$\int_0^\infty \left| \frac{1}{1+\frac{s}{\omega_o}} \right|^2 df = \frac{\omega_o}{4}$$

$$\int_0^\infty \left| \frac{1}{1+\dfrac{s}{\omega_o Q}+\dfrac{s^2}{\omega_o^2}} \right|^2 df = \frac{\omega_o Q}{4}$$

$$\int_0^\infty \left| \frac{\dfrac{s}{\omega_o}}{1+\dfrac{s}{\omega_o Q}+\dfrac{s^2}{\omega_o^2}} \right|^2 df = \frac{\omega_o Q}{4}$$

Example 4: CS Amplifier

$$\overline{v_{on}^2(f)} = 4k_BT\left(\frac{1}{R_L}+\frac{2}{3}g_m\right)\left|\frac{R_L}{1+sR_LC_L}\right|^2$$

$$\overline{v_{oT}^2} = 4k_BT\left(\frac{1}{R_L}+\frac{2}{3}g_m\right)R_L^2\int_0^\infty\left|\frac{1}{1+sR_LC_L}\right|^2 df$$

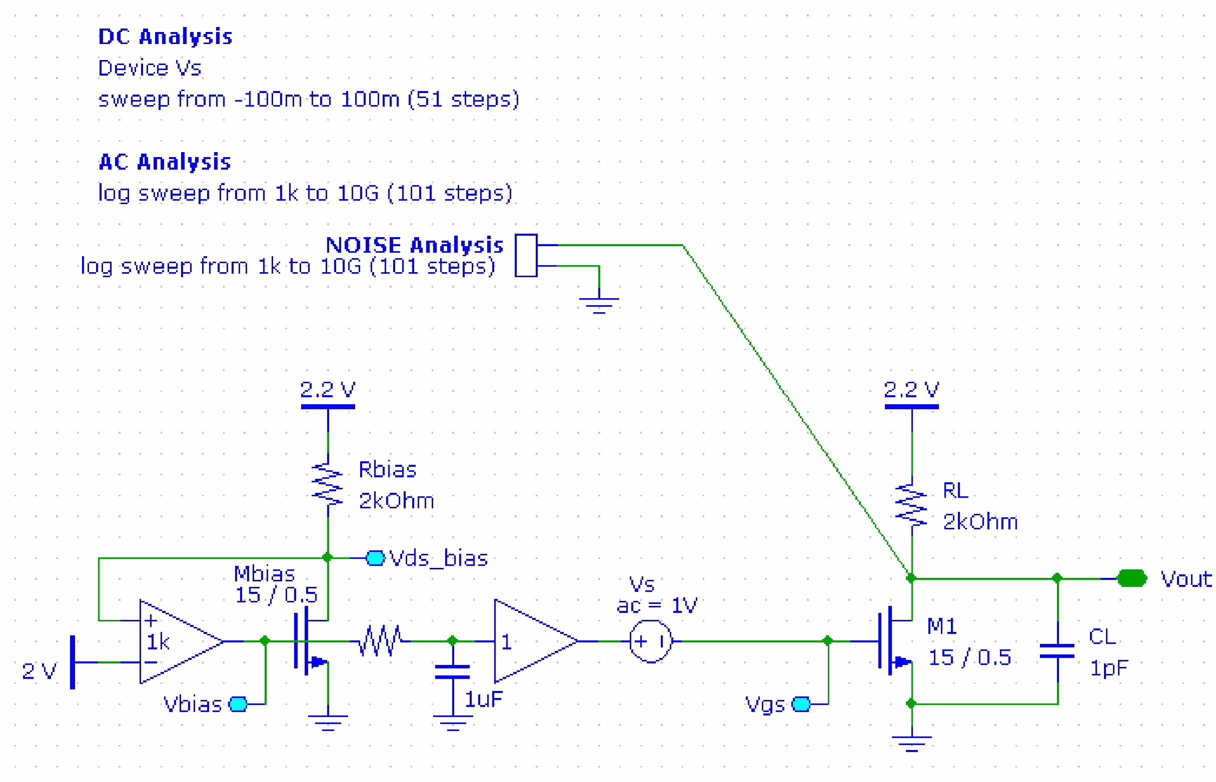
$$= 4k_BT\left(\frac{1}{R_L}+\frac{2}{3}g_m\right)R_L^2\frac{1}{4R_LC_L}$$

$$= \frac{k_BT}{C_L}\left(1+\frac{2}{3}g_mR_L\right)$$

$$= \frac{k_BT}{C_L}\left(1+\frac{2}{3}\left|A_{vo}\right|\right)$$

$$= \frac{k_BT}{C_L}n_F$$

SPICE Circuit



$$\frac{g_m}{I_D} = 10V^{-1}$$

$$I_D = 100\mu A$$

$$g_m = 1mS$$

$$R_L = 2k\Omega$$

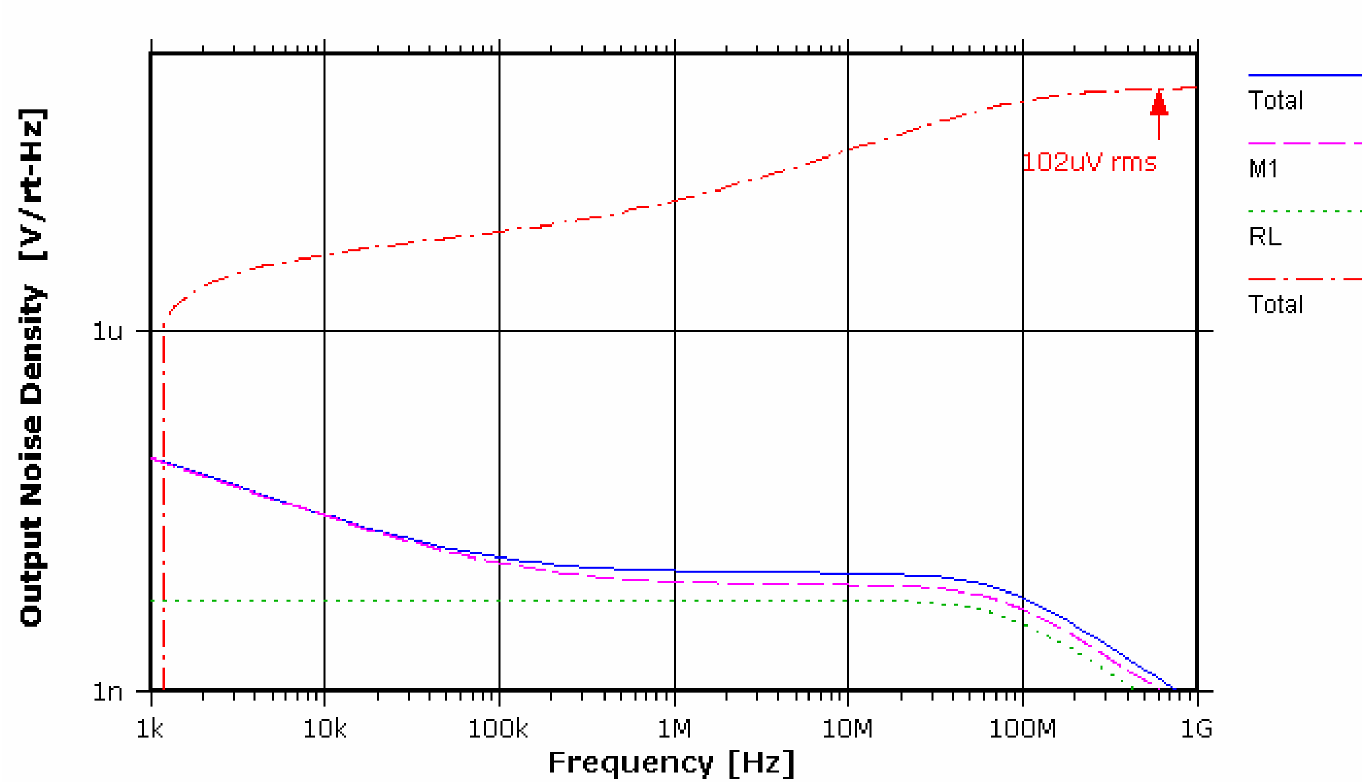
$$A_{vo} = -2$$

$$\sqrt{\overline{v_{oT}^2}} = \sqrt{\frac{k_B T}{C_L}}\sqrt{1+\tfrac{2}{3}\left|A_{vo}\right|}$$

$$= 64\mu V \times \sqrt{1+\tfrac{4}{3}}$$

$$= \underline{98\mu V \quad \text{rms}}$$

SPICE Result



Signal-To-Noise Ratio

- SNR

$$SNR = \frac{P_{sig}}{P_{noise}}$$

- Signal Power
  sinusoidal source

$$P_{sig} = \tfrac{1}{2} V_{zero-peak}^2 \qquad P_{sig} = 0.5V^2$$

$$P_{noise} = \frac{k_B T}{C} n_f \qquad P_{noise} = (64\mu V)^2$$

- Noise Power
  assuming thermal noise dominates

$$SNR = \frac{0.5}{(64\mu)^2} = 122 \times 10^6 = 80.9dB$$

- SNR = f(C)

$$C \quad \uparrow \times 4 \qquad SNR \quad \uparrow +6dB$$

dB versus Bits

- Quantization "noise"
  - Quantizer step size: Δ
  - Box-car pdf
  - Variance: $$S_Q = \frac{\Delta^2}{12}$$
- SNR of N-Bit sinusoidal signal
  - Signal power $$P_{sig} = \frac{1}{2}\left(2^N \frac{\Delta}{2}\right)^2$$
  - SNR
  - 6.02 dB per Bit $$SNR = \frac{P_{sig}}{S_Q} = 1.5 \times 2^{2N}$$ $$= \left[1.76 + 6.02N\right] \text{ dB}$$

N	dB
8	50
16	98
24	146

SNR versus Power

- 1 Bit → 6dB → 4x SNR
- 4x SNR → 4x C
- Circuit bandwidth ~g_m/C → 4x g_m
- Keeping V* constant → 4x I_D, 4x W

- Thermal noise limited circuit:
  Each additional Bit QUADRUPLES power dissipation.
  E.g. 15 Bit noise-limited ADC dissipates 100mW
  16 Bit redesign dissipates 400mW !

- Overdesign is very costly. We need design procedures that get us very close to the specifications.

Analog Circuit Dynamic Range

- The biggest signal we can ever expect at the output of a circuit is limited by the supply voltage, V_{DD}. Hence (for sinusoids)

$$V_{\max}(rms) = \frac{1}{\sqrt{2}} \frac{V_{DD}}{2}$$

- The noise is

$$V_n(rms) = \sqrt{n_f \frac{k_B T}{C}}$$

- So the dynamic range in dB is:

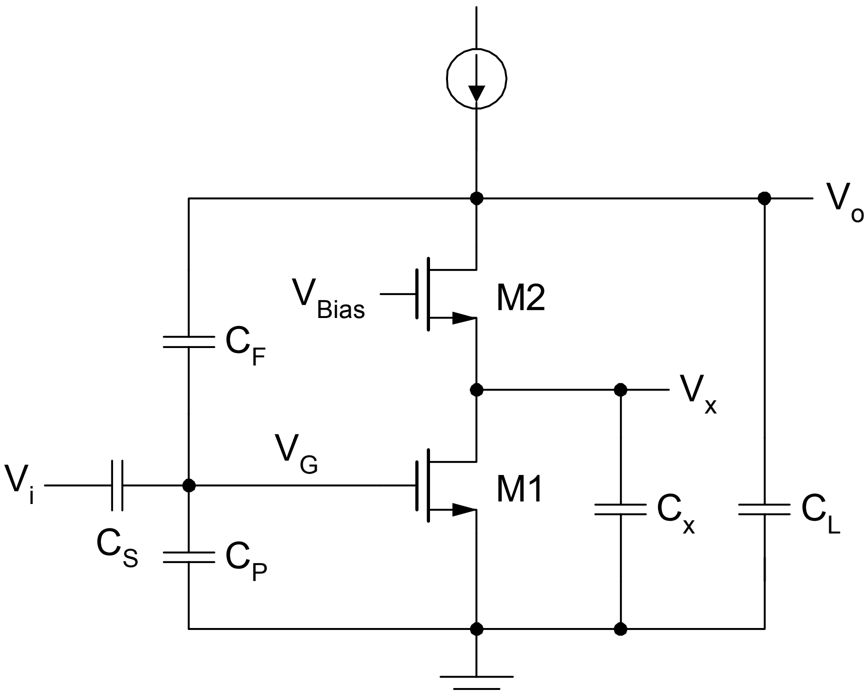
$$DR = \frac{V_{\max}(rms)}{V_n(rms)} = \frac{V_{DD}\sqrt{C}}{\sqrt{8 n_f k_B T}} \qquad [\mathrm{V/V}]$$

$$= 20\log_{10}\left(V_{DD}\sqrt{\frac{C}{n_f}} \right) + 75 \qquad [\mathrm{dB}] \text{ with C in } [\mathrm{pF}]$$

Analog Circuit Dynamic Range

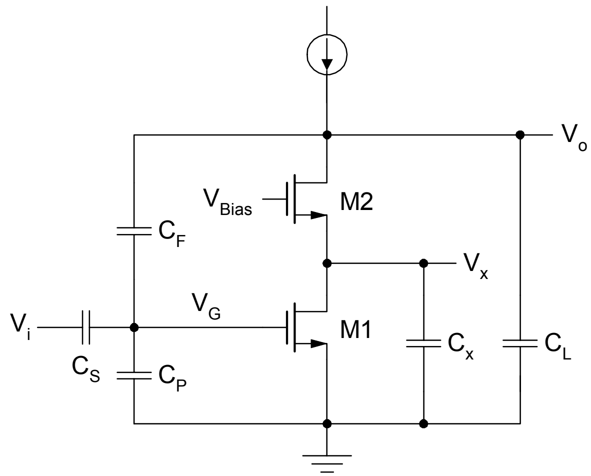
- For integrated circuits built in modern CMOS processes, $V_{DD} < 3V$ and $C < 1nF$ ($n_f = 1$)
  - DR < 110dB (18 Bits)
- For PC board circuits built with "old-fashioned" 30V opamps and discrete capacitors of < 100nF
  - DR < 140dB (23 Bits)
  - A 30dB (5 Bit) advantage!
- Note: oversampling ADCs break this barrier (cost: speed penalty)

"Big" Noise Example



- Cascoded common-source stage: what are the noise contributions from M1, M2?
- Simplified model for conclusive results:
  - Lump parasitic capacitors
  - Feedback sets gain, neglect r_o

Example (cont.)



Calculate noise transfer functions to amplifier output:

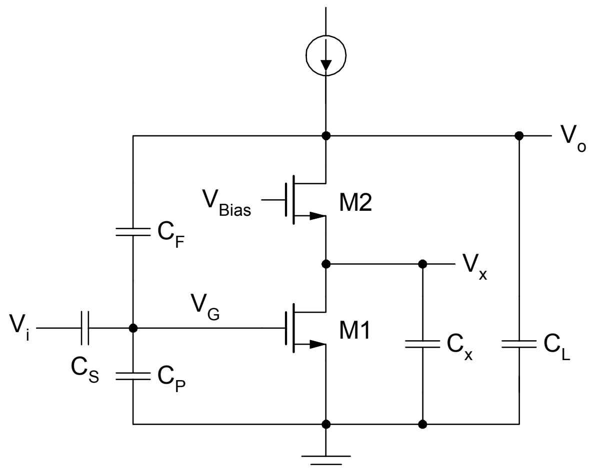
$$v_o: \quad s(C_L + C_F)v_o - sC_F v_g - g_{m2} v_x + i_{n2} = 0$$

$$v_x: \quad sC_x v_x + g_{m1} v_g + g_{m2} v_x + i_{n1} - i_{n2} = 0$$

$$v_g: \quad v_g = F v_o$$

$$\text{with} \qquad F = \frac{C_F}{C_F + C_S + C_P}$$

Example (cont.)



After some algebra …

$$v_o = -\frac{1}{Fg_{m1}}\left(i_{n1} + i_{n2}\frac{s}{\omega_p}\right)\frac{1}{1 + \frac{s}{\omega_o Q} + \frac{s^2}{\omega_o^2}}$$

with

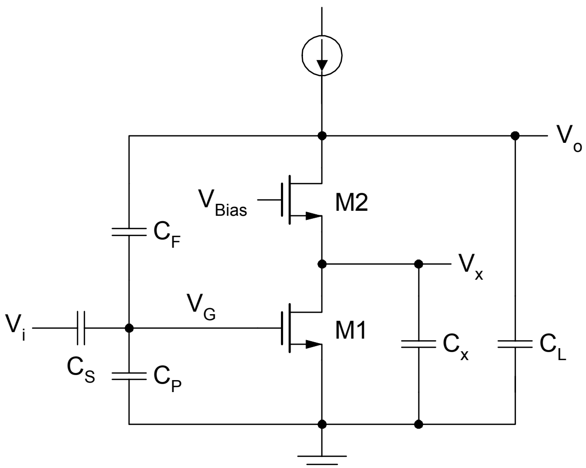
$$\omega_o^2 = \omega_u \omega_p$$

$$\omega_o Q = \omega_u$$

$$\omega_u = F\frac{g_{m1}}{C_{Leff}} = F\frac{g_{m1}}{C_L + C_F(1-F)}$$

$$\omega_p = \frac{g_{m2}}{C_x}$$

Example (cont.)



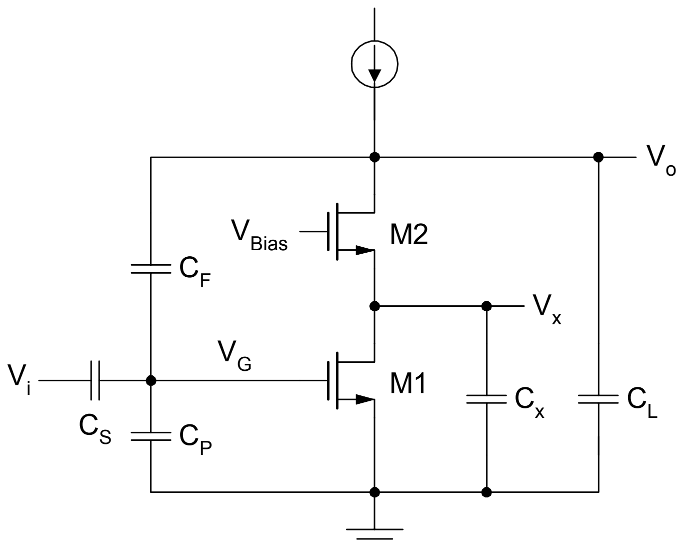
Spectral noise density at amplifier output:

$$\frac{\overline{v_{on}^2}}{\Delta f} = \frac{4k_b T}{F^2 g_{m1}} \left(\underbrace{1}_{M1} + \underbrace{\frac{g_{m2}}{g_{m1}} \left| \frac{s}{\omega_p} \right|^2}_{M2} \right) \left| \frac{1}{1 + \frac{s}{\omega_o Q} + \frac{s^2}{\omega_o^2}} \right|^2$$

- Noise from M2:
  - Circular current at low frequency → does not reach amplifier output
  - At high frequency C_x → short

- **Cascode contributes little noise at low frequency**
- **At high frequency the noise contribution can be significant**

Example (cont.)



Total noise at amplifier output:

$$\int_0^\infty \frac{\overline{v_{on}^2(f)}}{\Delta f} df = \frac{4k_bT\gamma}{F^2 g_{m1}} \int_0^\infty df \left(\underbrace{1}_{M1} + \underbrace{\frac{g_{m2}}{g_{m1}} \left| \frac{s}{\omega_p} \right|^2_{s=2\pi jf}}_{M2} \right) \left| \frac{1}{1 + \frac{s}{\omega_o Q} + \frac{s^2}{\omega_o^2}} \right|^2_{s=2\pi jf}$$

$$= \frac{4k_bT\gamma}{F^2 g_{m1}} \frac{\omega_o Q}{4} \left[1 + \frac{g_{m2}}{g_{m1}} \left(\frac{\omega_o}{\omega_p} \right)^2 \right]$$

$$= \frac{k_bT\gamma}{FC_{Leff}} \left(1 + \frac{g_{m2}}{g_{m1}} \frac{\omega_u}{\omega_p} \right)$$

$$= \frac{k_bT\gamma}{FC_{Leff}} \left(\underbrace{1}_{M1} + \underbrace{\frac{FC_x}{C_{Leff}}}_{M2} \right)$$

- **Total noise depends only on C!**
- **M1 always contributes noise**
- **Significant noise from M2 for "large" C_x → make C_x small (compared to C_{Leff})**

Design Example

- Track & Hold amplifier for 16-Bit ADC (B=16)
- f_s = 100MHz → $\omega_u \sim 2\pi f_s N$
  $N = \ln(2^B)$ … based on settling, see later
- Amplifier based on cascoded common-source, $A_v = -1$
- Choose
  - L = 0.35μm
  - M1 and M2 same size (not necessarily ideal)
  - $C_F = C_S = C_{GS}$ (reasonable tradeoff)
    → F = 1/3
  - Maximum signal amplitude V_s (peak-to-peak)

Design Equations

$$DR = \left(2^B\right)^2$$

$$= \frac{\frac{1}{2}\left(\frac{V_s}{2}\right)^2}{\frac{kT}{FC_{Leff}}\left(1 + \frac{C_x \approx C_{GS}}{C_{Leff}}\right)}$$

$$\omega_u = 2\pi f_s N \qquad \mathrm{N} \cong \ln\left(2^{\mathrm{B}}\right)$$

$$= F\frac{g_{m1}}{C_{Leff}}$$

$$C_{GS} = \tfrac{2}{3}WLC_{ox}$$

$$V^* = \frac{2I_D}{g_{m1}}$$

$$\xrightarrow{\text{solve}}$$

$$C_L \cong \frac{8 \times 2^{2B}}{F}\frac{k_B T_r}{V_s^2}$$

$$g_{m1} \cong 2\pi f_s N \frac{C_L}{F}$$

Design Examples

Parameter	14-Bit T&H	16-Bit T&H	
B	14	16	
f_s	100MHz	100MHz	
V_s	1V	1V	
V*	200mV	200mV	
W/I_D	0.1μm/μA	0.1μm/μA	
C_L	26pF	413pF	
g_m	81mS	1300mS	
I_D	10mA	167mA	
W	675μm	10800μm	
C_{GS}	0.84pF	13.4pF	