EECS 217

Lecture 6: Microwave Networks

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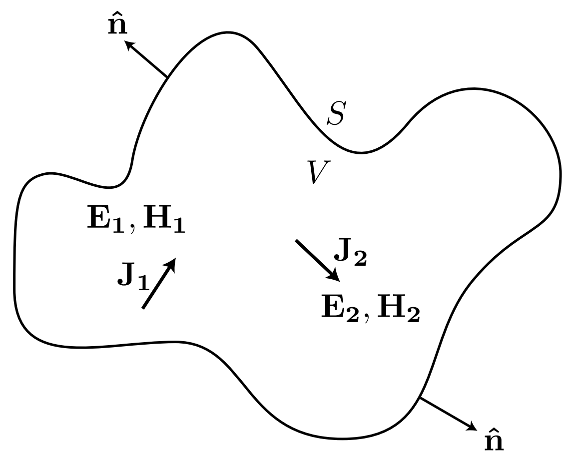
Lorentz Reciprocity Theorem



- Consider two sources $\mathbf{J_1}$ and $\mathbf{J_2}$ inside S. Inside S and on the boundary the fields satisfy Maxwell's Eq.

$$\nabla \times \mathbf{E_1} = -j\omega \mathbf{H_1}$$

$$\nabla \times \mathbf{H_1} = \mathbf{J_1} + j\omega\epsilon_1 \mathbf{E_1}$$

- Now use the following vector identify

$$\nabla \cdot (\mathbf{E_1} \times \mathbf{H_2} - \mathbf{E_2} \times \mathbf{H_1}) =$$

$$= (\nabla \times \mathbf{E_1}) \cdot \mathbf{H_1} - (\nabla \times \mathbf{H_2}) \cdot \mathbf{E_1} - (\nabla \times \mathbf{E_2}) \cdot \mathbf{H_1} + (\nabla \times \mathbf{H_1}) \cdot \mathbf{E_2}$$

$$= -j\omega \mathbf{H_1} \cdot \mathbf{H_2} - (\mathbf{J_2} + j\omega\epsilon \mathbf{E_2}) \cdot \mathbf{E_1} + j\omega \mathbf{H_2} \cdot \mathbf{H_1} + (\mathbf{J_1} + j\omega\epsilon \mathbf{E_1}) \cdot \mathbf{E_2}$$

Reciprocity (cont)

- After the massacre, only a few terms survive. Apply the divergence theorem (you saw it coming)

$$\int_V \nabla \cdot (\mathbf{E_1} \times \mathbf{H_2} - \mathbf{E_2} \times \mathbf{H_1}) dV = \oint_S (\mathbf{E_1} \times \mathbf{H_2} - \mathbf{E_2} \times \mathbf{H_1}) \cdot d\mathbf{S} =$$

$$= \int_V (\mathbf{J_1} \cdot \mathbf{E_2} - \mathbf{J_2} \cdot \mathbf{E_1}) dV$$

- For a sourceless region, the RHS is identically zero and we have one form of reciprocity

$$\oint_S \mathbf{E_1} \times \mathbf{H_2} \cdot d\mathbf{S} = \oint_S \mathbf{E_2} \times \mathbf{H_1} \cdot d\mathbf{S}$$

- On the other hand, if the integral encloses all of the sources, we can show that the surface integral term is zero.
- Let's take a few cases. Say S is a perfectly conducting surface so that $E_t = 0$. Then $\mathbf{n} \times \mathbf{E} = 0$ and

$$(\mathbf{E_1} \times \mathbf{H_2}) \cdot \hat{\mathbf{n}} = (\hat{\mathbf{n}} \times \mathbf{E_1}) \cdot \mathbf{H_2} = 0$$

Conductive Surface

- It takes a bit more work, but we can also show that the above holds if the surface S has surface impedance Z_s

$$\mathbf{E_t} = Z_s \mathbf{J_s} = -Z_s \hat{\mathbf{n}} \times \mathbf{H}$$

$$\hat{\mathbf{n}} \times \mathbf{E} = -Z_s \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{H})$$

$$(\hat{\mathbf{n}} \times \mathbf{E_1}) \cdot \mathbf{H_2} - (\hat{\mathbf{n}} \times \mathbf{E_2}) \cdot \mathbf{H_1} = -Z_s (\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H_1}) \cdot \mathbf{H_2} + Z_s (\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H_2}) \cdot \mathbf{H_1}$$

$$= -Z_s (\hat{\mathbf{n}} \times \mathbf{H_2}) \cdot (\hat{\mathbf{n}} \cdot \mathbf{H_1}) + Z_s (\hat{\mathbf{n}} \times \mathbf{H_1}) \cdot (\hat{\mathbf{n}} \cdot \mathbf{H_2}) = 0$$

Radiation Boundary

- Consider a sphere at infinity. At infinity the fields become TEM

$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}} \hat{\mathbf{a_r}} \times \mathbf{E}$$

$$(\hat{\mathbf{n}} \times \mathbf{E_1}) \cdot \mathbf{H_2} - (\hat{\mathbf{n}} \times \mathbf{E_2}) \cdot \mathbf{H_1} =$$

$$= \sqrt{\frac{\epsilon}{\mu}} ((\hat{\mathbf{a_r}} \times \mathbf{E_1}) \cdot (\hat{\mathbf{a_r}} \times \mathbf{E_2}) - (\hat{\mathbf{a_r}} \times \mathbf{E_2}) \cdot (\hat{\mathbf{a_r}} \times \mathbf{E_1})) = 0$$

- One can actually show that for any surface enclosing all the sources, the integral vanishes so that

$$\int_V \mathbf{E_1} \cdot \mathbf{J_2} dV = \int_V \mathbf{E_2} \cdot \mathbf{J_1} dV$$

- For point sources

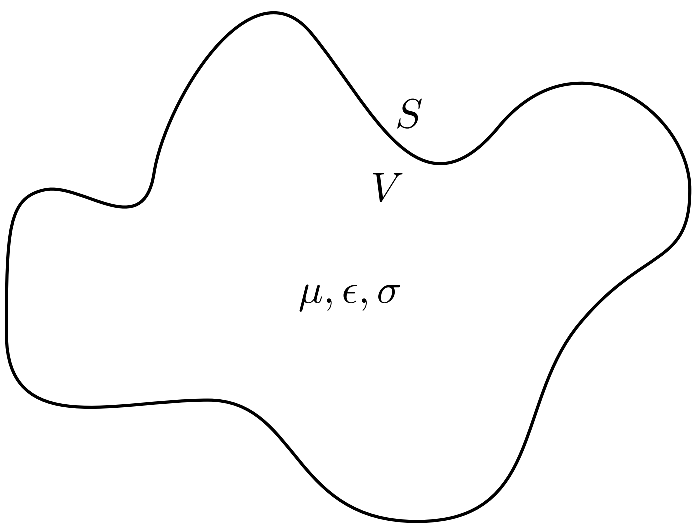
$$E_1 J_2 = E_2 J_1$$

Uniqueness Theorem

- An electromagnetic field is uniquely determined within a bounded region V at all times $t > 0$ by the initial values of the electric and magnetic vectors through V and the values of the tangential component of the electric vector or the magnetic vector over the boundaries for $t \geq 0$.
- *Proof:* Assume the solution is not unique and two distinct solutions are $\mathbf{E_1}/\mathbf{H_1}$ and $\mathbf{E_2}/\mathbf{H_2}$.

$$\mathbf{E_1}(t=0) = \mathbf{E_2}(t=0) \qquad \mathbf{H_1}(t=0) = \mathbf{H_2}(t=0)$$

- Assume linear field equations (exclude ferromagnetic materials) so that the difference fields are also a solution



$$\mathbf{E} \triangleq \mathbf{E_1} - \mathbf{E_2}$$

$$\mathbf{H} \triangleq \mathbf{H_1} - \mathbf{H_2}$$

Uniqueness (cont)

- Assume sources are outside of V so that withing V Poynting's Thm is satisfied

$$\frac{\partial}{\partial t}\int_V \tfrac{1}{2}(\epsilon|E|^2 + \mu|H|^2)dV + \int_V \rho|J|^2 dV = -\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}$$

- Since both solutions satisfy the boundary conditions, $\mathbf{n}\times\mathbf{E} = 0$ or $\mathbf{n}\times\mathbf{H} = 0$, and so the RHS is zero.
- We have the following

$$\frac{\partial}{\partial t}\int_V \tfrac{1}{2}\underbrace{(\epsilon|E|^2 + \mu|H|^2)}_{\geq 0}dV = -\underbrace{\int_V \rho|J|^2 dV}_{\leq 0}$$

- Since integrand is zero at time $t = 0$ and non-zero for $t > 0$, the only consistent solution is $\mathbf{E} = 0$ and $\mathbf{H} = 0$ for $t \geq 0$.

The Network Formulation

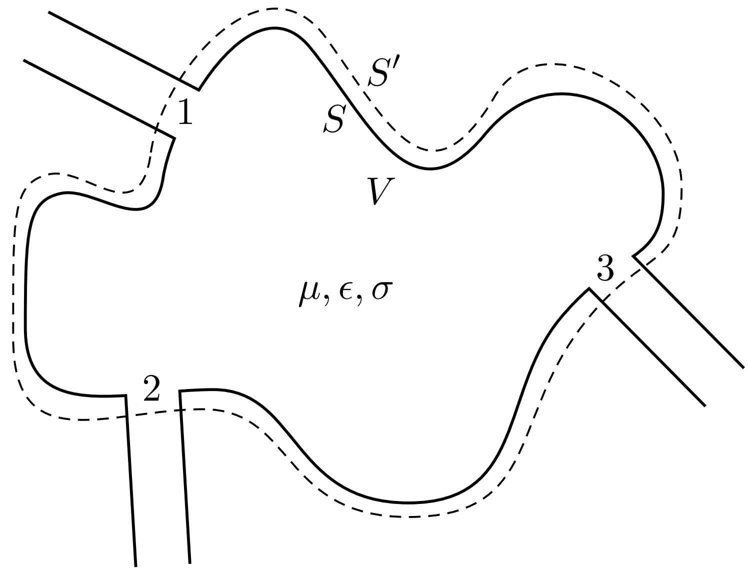
- Imagine that S is a perfectly conductor. Then $E_t \equiv 0$ on the surface of S except at the reference planes. Thus if the voltage or current is given at each reference plane, the fields are uniquely determined inside V. For instance if voltages V_1, V_2,$\cdots$ are specified, then the currents into each port are a linear combination of the voltages (if the materials are linear):

$$I_1 = Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 + \cdots$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2 + Y_{23}V_3 + \cdots$$

$$I_3 = Y_{31}V_1 + Y_{32}V_2 + Y_{33}V_3 + \cdots$$

$$\vdots$$



- Or in general, we can define an $N \times N$ complex matrix Y such that

$$i = Yv$$

Network Formulation (cont)

- Similarly if the currents at the ref. planes, then tangential magnetic fields are also known and unique sol'n of Maxwell's eq. inside V follows. The tangential E-fields, or the voltages, can then be computed as a linear combination of the voltages

$$V_1 = Z_{11} I_1 + Z_{12} I_2 + Z_{13} I_3 + \cdots$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2 + Z_{23} I_3 + \cdots$$

$$V_3 = Z_{31} I_1 + Z_{32} I_2 + Z_{33} I_3 + \cdots$$

$$\vdots$$

- What if the boundary is not a perfect conductor? Then introduce a new surface S' several skin depths within the conductor so that the tangential fields are essentially zero. Then the same argument as above applies except now the conductive portion will led to loss and contribute to the real part of Y_{ij} or Z_{ij}.

Symmetry of Impedance Matrix

- Suppose all terminals (ref planes, ports) are shorted except at the i'th plane. The solution to Maxwell's eq. is $\mathbf{E_i}$,$\mathbf{H_i}$. Similarly $\mathbf{E_j}$,$\mathbf{H_j}$ corresponds to the case when all terminals except the j'th plane are shorted. By the Lorentz Reciprocity Them:

$$\oint_S (\mathbf{E_i} \times \mathbf{H_j} - \mathbf{E_j} \times \mathbf{H_i}) \cdot d\mathbf{S} = 0$$

- For a sourceless region bounded by S. Let S consist of conducting walls bounding the junction and the N terminal planes. The integral over the walls vanishes if the walls are perfectly conducting or if the walls exhibit a surface impedance Z_m. So the above reduces to

$$\sum_{n=1}^{N} \int_{t_n} (\mathbf{E_i} \times \mathbf{H_j} - \mathbf{E_j} \times \mathbf{H_i}) \cdot d\mathbf{S} = 0$$

- But $\mathbf{n} \times \mathbf{E_i}$ and $\mathbf{n} \times \mathbf{E_j}$ are zero at all terminal planes except i and j

$$\int_{t_i} \mathbf{E_i} \times \mathbf{H_j} \cdot d\mathbf{S} = \int_{t_j} \mathbf{E_j} \times \mathbf{H_i} \cdot d\mathbf{S}$$

or

$$V_i (I_i)_j = V_j (I_j)_i$$

Symmetry (cont)

- $(I_i)_j$ is the current at the terminal plane i arising from an applied voltage at plane j

$$I_i = (I_i)_j = Y_{ij} V_j$$

$$I_j = (I_j)_i = Y_{ji} V_i$$

Therefore

$$V_i V_j Y_{ij} = V_j V_i Y_{ji}$$

or

$$Y_{ij} = Y_{ji}$$

- Since $Z = Y^{-1}$, the inverse of a symmetric matrix is also symmetric

$$AA^{-1} = I$$

$$I^t = I = A^t (A^{-1})^t$$

$$(A^{-1})^t = (A^t)^{-1}$$

$$A^t = A \rightarrow A^{-1} = (A^{-1})^t$$

Loss Free Networks

- For any network we have

$$-\oint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = \sum_{m=1}^{N} V_m I_m^* = 2W_L + 4j\omega(W_m - W_e)$$

- Since $V_m = \sum_{n=1}^{N} Z_{mn} I_n$ and $W_L \equiv 0$

$$\sum_{m=1}^{N} \sum_{n=1}^{N} Z_{mn} I_n I_m^* = 4j\omega(W_m - W_e)$$

- Let all ports be open except port i:

$$Z_{ii} I_i I_i^* = 4j\omega(W_m - W_e)$$

- Thus the diagonal terms are imaginary. Now let all ports be open circuited except port i and j

$$Z_{ij} I_i I_j^* + Z_{ji} I_j I_i^* + Z_{ii} |I_i|^2 + Z_{jj} |I_j|^2 = 4j\omega(W_m - W_e)$$

Loss Free Networks (cont)

- We thus have

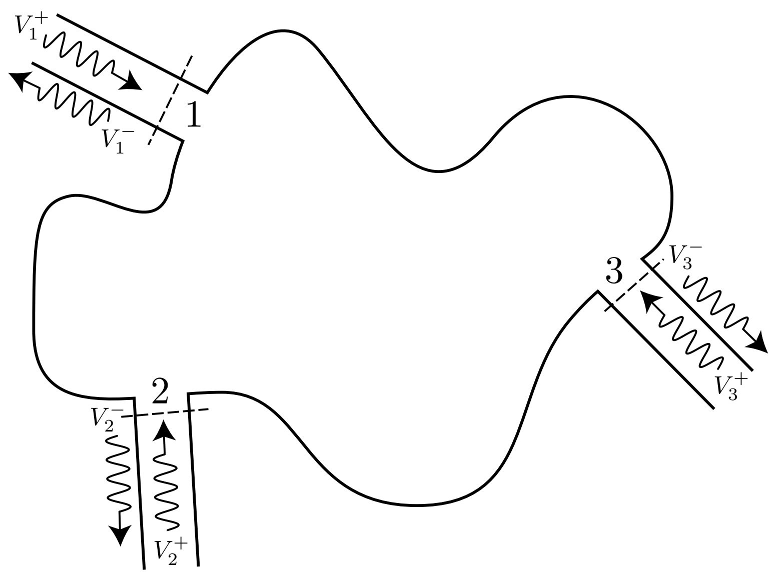
$$\Re\left(Z_{ij} I_i I_j^* + Z_{ji} I_j I_i^*\right) = 0$$

- Since the network is reciprocal, $Z_{ij} = Z_{ji}$, so

$$\Re\left(Z_{ij} \underbrace{\left(I_i I_j^* + I_j I_i^*\right)}_{\text{real}} \right) = 0$$

- That means that Z_{ij} has to be imaginary.
- In conclusion, for a *lossless reciprocal* network, Z is imaginary. Since $Y = Z^{-1}$, Y is also imaginary.

Scattering Matrix



- Voltages and currents are difficult to measure directly at microwave freq. Z matrix requires "opens", and it's hard to create an ideal open (parasitic capacitance and radiation). Likewise, a Y matrix requires "shorts", again ideal shorts are impossible at high frequency due to the finite inductance.
- Many active devices could oscillate under the open or short termination.
- S parameters are easier to measure at high frequency. The measurement is direct and only involves measurement of relative quantities (such as the SWR or the location of the first minima relative to the load).
- It's important to realize that although we associate S parameters with high frequency and wave propagation, the concept is valid for any frequency.

Incident and Scattering Waves

- Let's define the vector v^+ as the incident "forward" waves on each transmission line connected to the N port. Define the reference plane as the point where the transmission line terminates onto the N port.
- The vector v^- is then the reflected or "scattered" waveform at the location of the port.

$$v^+ = \begin{pmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \\ \vdots \end{pmatrix} \qquad v^- = \begin{pmatrix} V_1^- \\ V_2^- \\ V_3^- \\ \vdots \end{pmatrix}$$

- Because the N port is linear, we expect that scattered field to be a linear function of the incident field

$$v^- = Sv^+$$

- S is the scattering matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & \cdots \\ S_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

Relation to Voltages

- The fact that the S matrix exists can be easily proved if we recall that the voltage and current on each transmission line termination can be written as

$$V_i = V_i^+ + V_i^- \qquad\qquad I_i = Y_0(I_i^+ - I_i^-)$$

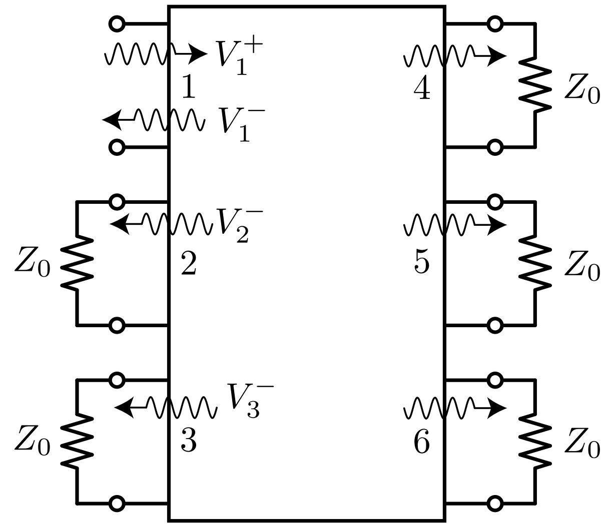
- Inverting these equations

$$V_i + Z_0 I_i = V_i^+ + V_i^- + V_i^+ - V_i^- = 2V_i^+$$

$$V_i - Z_0 I_i = V_i^+ + V_i^- - V_i^+ + V_i^- = 2V_i^-$$

- Thus v^+,v^- are simply linear combinations of the port voltages and currents. By the uniqueness theorem, then, $v^- = Sv^+$.

Measure S_{ij}

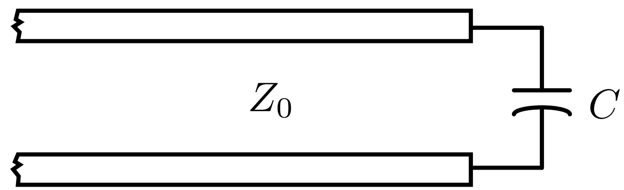


- The term S_{ij} can be computed directly by the following formula

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+ = 0 \, \forall \, k \neq j}$$

- In other words, to measure S_{ij}, drive port j with a wave amplitude of V_j^+ and terminate all other ports with the characteristic impedance of the lines (so that $V_k^+ = 0$ for $k \neq j$). Then observe the wave amplitude coming out of the port i

S Matrix for a 1-Port Capacitor



- Let's calculate the S parameter for a capacitor

$$S_{11} = \frac{V_1^-}{V_1^+}$$

- This is of course just the reflection coefficient for a capacitor

$$S_{11} = \rho_L = \frac{Z_C - Z_0}{Z_C + Z_0} = \frac{\frac{1}{j\omega C} - Z_0}{\frac{1}{j\omega C} + Z_0}$$

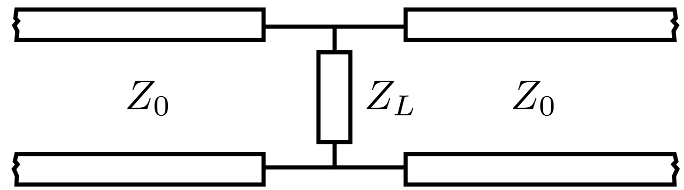
$$= \frac{1 - j\omega C Z_0}{1 + j\omega C Z_0}$$

S Matrix for a 2-Port Shunt Element

- Consider a shunt impedance connected at the junction of two transmission lines. The voltage at the junction is of course continuous. The currents, though, differ

$$V_1 = V_2$$

$$I_1 + I_2 = Y_L V_2$$



- To compute S_{11}, enforce $V_2^+ = 0$ by terminating the line. Thus we can be re-write the above equations

$$V_1^+ + V_1^- = V_2^-$$

$$Y_0(V_1^+ - V_1^-) = Y_0 V_2^- + Y_L V_2^- = (Y_L + Y_0)V_2^-$$

- We can now solve the above eq. for the reflected and transmitted wave

$$V_1^- = V_2^- - V_1^+ = \frac{Y_0}{Y_L + Y_0}(V_1^+ - V_1^-) - V_1^+$$

$$V_1^-(Y_L + Y_0 + Y_0) = (Y_0 - (Y_0 + Y_L))V_1^+$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \frac{Y_0 - (Y_0 + Y_L)}{Y_0 + (Y_L + Y_0)} = \frac{Z_0||Z_L - Z_0}{Z_0||Z_L + Z_0}$$

Shunt Element (cont)

- The above eq. can be written by inspection since $Z_0||Z_L$ is the effective load seen at the junction of port 1.
- Thus for port 2 we can write

$$S_{22} = \frac{Z_0||Z_L - Z_0}{Z_0||Z_L + Z_0}$$

- Likewise, we can solve for the transmitted wave, or the wave scattered into port 2

$$S_{21} = \frac{V_2^-}{V_1^+}$$

- Since $V_2^- = V_1^+ + V_1^-$, we have

$$S_{21} = 1 + S_{11} = \frac{2Z_0||Z_L}{Z_0||Z_L + Z_0}$$

- By symmetry, we can deduce S_{12} as

$$S_{12} = \frac{2Z_0||Z_L}{Z_0||Z_L + Z_0}$$

Conversion Formula

- Since V^+ and V^- are related to V and I, it's easy to find a formula to convert for Z or Y to S

$$V_i = V_i^+ + V_i^- \;\rightarrow\; v = v^+ + v^-$$

$$Z_{i0} I_i = V_i^+ - V_i^- \;\rightarrow\; Z_0 i = v^+ - v^-$$

- Now starting with $v = Zi$, we have

$$v^+ + v^- = Z Z_0^{-1}(v^+ - v^-)$$

- Note that Z_0 is the scalar port impedance

$$v^-(I + Z Z_0^{-1}) = (Z Z_0^{-1} - I)v^+$$

$$v^- = (I + Z Z_0^{-1})^{-1}(Z Z_0^{-1} - I)v^+ = S v^+$$

- We now have a formula relating the Z matrix to the S matrix

$$S = (Z Z_0^{-1} + I)^{-1}(Z Z_0^{-1} - I) = (Z + Z_0 I)^{-1}(Z - Z_0 I)$$

Conversion (cont)

- Recall that the reflection coefficient for a load is given by the same equation!

$$\overline{\rho} = \frac{Z/Z_0 - 1}{Z/Z_0 + 1}$$

- To solve for Z in terms of S, simply invert the relation

$$Z_0^{-1} Z S + I S = Z_0^{-1} Z - I$$

$$Z_0^{-1} Z(I - S) = S + I$$

$$Z = Z_0 (I + S)(I - S)^{-1}$$

- As expected, these equations degenerate into the correct form for a 1×1 system

$$Z_{11} = Z_0 \frac{1 + S_{11}}{1 - S_{11}}$$

Reciprocal Networks

- We have found that the Z and Y matrix are symmetric. Now let's see what we can infer about the S matrix.

$$v^+ = \frac{1}{2}(v + Z_0 i)$$

$$v^- = \frac{1}{2}(v - Z_0 i)$$

- Substitute $v = Zi$ in the above equations

$$v^+ = \frac{1}{2}(Zi + Z_0 i) = \frac{1}{2}(Z + Z_0)i$$

$$v^- = \frac{1}{2}(Zi - Z_0 i) = \frac{1}{2}(Z - Z_0)i$$

- Since $i = i$, the above eq. must result in consistent values of i. Or

$$2(Z + Z_0)^{-1} v^+ = 2(Z - Z_0)^{-1} v^-$$

Thus

$$S = (Z - Z_0)(Z + Z_0)^{-1}$$

Reciprocal Networks (cont)

- Consider the transpose of the S matrix

$$S^t = \left((Z + Z_0)^{-1}\right)^t (Z - Z_0)^t$$

- Recall that Z_0 is a diagonal matrix

$$S^t = (Z^t + Z_0)^{-1}(Z^t - Z_0)$$

- If $Z^t = Z$ (reciprocal network), then we have

$$S^t = (Z + Z_0)^{-1}(Z - Z_0)$$

- Previously we found that

$$S = (Z + Z_0)^{-1}(Z - Z_0)$$

- So that we see that the S matrix is also symmetric (under reciprocity)

$$S^t = S$$

Another Proof

- Note that in effect we have shown that

$$(Z+I)^{-1}(Z-I) = (Z-I)(Z+I)^{-1}$$

- This is easy to demonstrate if we note that

$$Z^2 - I = Z^2 - I^2 = (Z+I)(Z-I) = (Z-I)(Z+I)$$

- In general matrix multiplication does not commute, but here it does

$$(Z-I) = (Z+I)(Z-I)(Z+I)^{-1}$$

$$(Z+I)^{-1}(Z-I) = (Z-I)(Z+I)^{-1}$$

- Thus we see that $S^t = S$.