EECS 217

Lecture 3: Distributed Circuits

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Fields of a Circuit

- Last lecture we made a connection between field theory and circuit theory using Poynting's Theorem. Let's take a different approach by noting that in any arbitrary circuit, the total electric field $\mathbf{E}$ can be written as

$$\mathbf{E} = \mathbf{E_0} + \mathbf{E}'$$

- The term E_0 is the applied field and the term E' is the response field. Since $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t}$, within a gradient, $\mathbf{E}$ and $\frac{\partial \mathbf{A}}{\partial t}$ are equal.

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

Response Field

- By Guass' Thm, adopting a Coulomb Gauge, $\nabla \cdot \mathbf{A} = 0$, the term ϕ is identified as the electric potential

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = 0 = \rho/\epsilon$$

- For a conductor, the total field is related to the current

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \mathbf{E_0} - \nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

- Integrate the above equation along the conductive path about the circuit and identify each term

$$\underbrace{\oint \mathbf{E_0} \cdot d\ell}_{\text{applied voltage}} = \underbrace{\oint \frac{\mathbf{J}}{\sigma} \cdot d\ell}_{\text{internal impedance}} - \underbrace{\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\ell}_{\text{external inductance}} - \underbrace{\oint \nabla \phi \cdot d\ell}_{\text{capacitance}}$$

Impedance Terms

- At high frequency the current $\mathbf{J}$ falls out of phase with the applied field. This is due to the internal impedance of the conductor.
- The external inductance term is easy to identify

$$\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\ell = \frac{\partial}{\partial t} \oint \mathbf{A} \cdot d\ell$$

$$= \frac{\partial}{\partial t} \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} = L \frac{\partial I}{\partial t}$$

- If there are other currents in the vicinity of the circuit, then mutual inductance terms account for the magnetic coupling

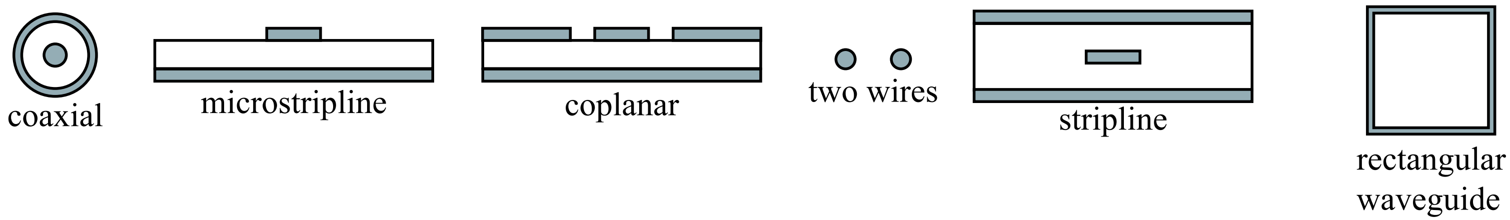
Distributed Elements

- A lumped inductor has zero loss and stores only magnetic energy. A lumped capacitor, likewise, only stores electric energy. In reality, any real element has distributed loss and stores some electric and magnetic energy. But a good lumped element (say an inductor) stores mostly magnetic energy and all losses can be lumped into an equivalent series R.
- A distributed element is a circuit element where we cannot easily lump all magnetic effects into an L and all electric effects into a C and all loss into R without introducing error.
- An IC resistor is a good example of a distributed element.

Distributed $RLGC$ Transmission Line

- We can model many elements using *distributed* circuits. The loss and energy storage are continuously intermingled through the space of the structure.
- If the structure is uniform along the length of the conductor, then we can model it as a transmission line.

Transmission Line Menagerie

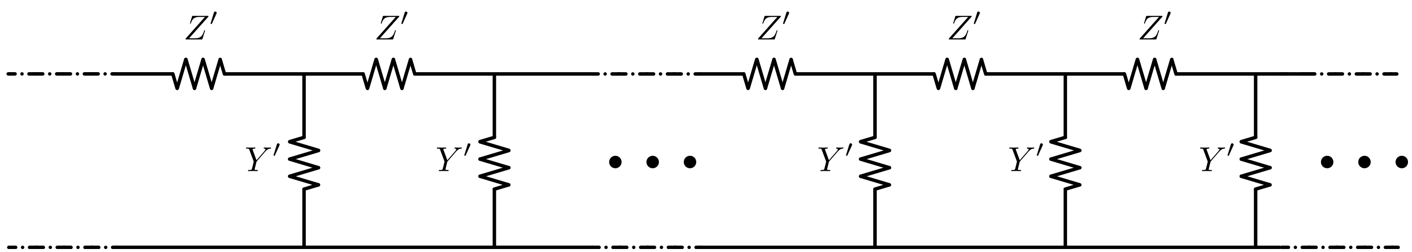


- T-Lines come in many shapes and sizes
- Coaxial usually 75Ω or 50Ω (cable TV, Internet)
- Microstrip lines are common on printed circuit boards (PCB) and integrated circuit (ICs)
- Coplanar also common on PCB and ICs
- Twisted pairs is almost a T-line, ubiquitous for phones/Ethernet

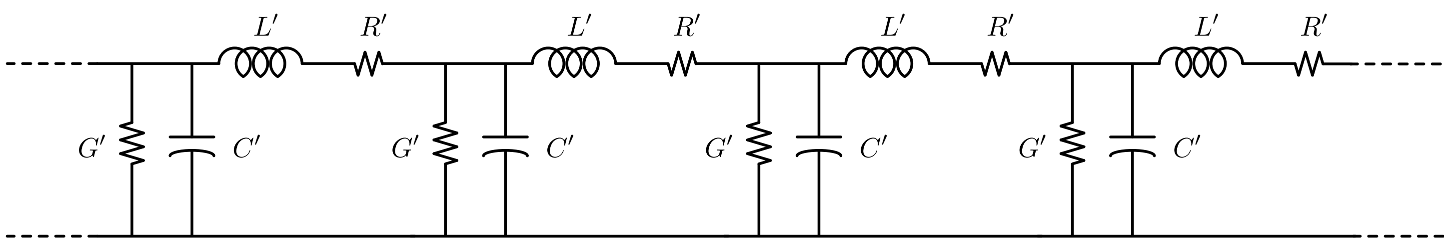
Waveguides and Transmission Lines

- The transmission lines we've been considering have been propagating the "TEM" mode or Transverse Electro-Magnetic. Later we'll see that they can also propagation other modes
- Waveguides cannot propagate TEM but propagation TM (Transverse Magnetic) and TE (Transverse Electric)
- In general, *any* set of more than one lossless conductors with uniform cross-section can transmit TEM waves. Low loss conductors are commonly approximated as lossless.

Generalized Distributed Circuit Model



- Z': impedance per unit length (e.g. $Z' = j\omega L' + R'$)
- Y': admittance per unit length (e.g. $Y' = j\omega C' + G'$)
- A lossy T-line might have the following form (but we'll analyze the general case)



Calculating Distributed Elements

- Equating the energy stored/dissipated in a field to the energy stored/dissipated by the distributed elements, we have

$$L = \frac{\mu}{I_0 I_0^*} \int_S \mathbf{H} \cdot \mathbf{H}^* dS$$

$$C = \frac{\epsilon'}{V_0 V_0^*} \int_S \mathbf{E} \cdot \mathbf{E}^* dS$$

$$R = \frac{R_s}{I_0 I_0^*} \int_{\ell_1 + \ell_2} \mathbf{H} \cdot \mathbf{H}^* d\ell$$

$$G = \frac{\omega \epsilon''}{V_0 V_0^*} \int_S \mathbf{E} \cdot \mathbf{E}^* dS$$

Time Harmonic Telegrapher's Equations

- Applying KCL and KVL to a infinitesimal section

$$v(z+\delta z) - v(z) = -Z'\delta z i(z)$$

$$i(z+\delta z) - i(z) = -Y'\delta z v(z)$$

- Taking the limit as before ($\delta z \to 0$)

$$\frac{dv}{dz} = -Zi(z)$$

$$\frac{di}{dz} = -Yv(z)$$

Sin. Steady-State (SSS) Voltage/Current

- Taking derivatives (notice z is the only variable) we arrive at

$$\frac{d^2v}{dz^2} = -Z\frac{di}{dz} = YZv(z) = \gamma^2 v(z)$$

$$\frac{d^2i}{dz^2} = -Y\frac{dv}{dz} = YZi(z) = \gamma^2 i(z)$$

- Where the propagation constant γ is a complex function

$$\gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

- The general solution to $D^2G - \gamma^2 G = 0$ is $e^{\pm\gamma z}$

Lossless Line for SSS

- The voltage and current are related (just as before, but now easier to derive)

$$v(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$i(z) = \frac{V^+}{Z_0} e^{-\gamma z} - \frac{V^-}{Z_0} e^{\gamma z}$$

- Where $Z_0 = \sqrt{\frac{Z'}{Y'}}$ is the characteristic impedance of the line (function of frequency with loss)
- For a lossless line we discussed before, $Z' = j\omega L'$ and $Y' = j\omega C'$
- Propagation constant is imaginary

$$\gamma = \sqrt{j\omega L' j\omega C'} = j\sqrt{L'C'}\omega$$

Back to Time-Domain

- Recall that the *real* voltages and currents are the $\Re$ and $\Im$ parts of

$$v(z,t) = e^{\pm\gamma z} e^{j\omega t} = e^{j\omega t \pm \beta z}$$

- Thus the voltage/current waveforms are sinusoidal in space and time
- Sinusoidal source voltage is transmitted unaltered onto T-line (with delay)
- If there is loss, then γ has a real part α, and the wave decays or grows on the T-line

$$e^{\pm\gamma z} = e^{\pm\alpha z} e^{\pm j\beta z}$$

- The first term represents amplitude response of the T-line

Passive T-Line/Wave Speed

- For a passive line, we expect the amplitude to decay due to loss on the line
- The speed of the wave is derived as before. In order to follow a constant point on the wavefront, you have to move with velocity

$$\frac{d}{dt}\left(\omega t \pm \beta z = \text{constant}\right)$$

- Or, $v = \frac{dz}{dt} = \pm\frac{\omega}{\beta} = \pm\sqrt{\frac{1}{L'C'}}$

Field Theory for T-Lines

- In more general terms, a transmission line is a structure that supports transverse electromagnetic (TEM) wave propagation. By “transverse” we mean that the fields are always perpendicular to the direction of propagation.
- Since currents flow into and out of the conductors of a transmission line, $J_z \neq 0$ but $E_z \equiv 0$. This implies that the conductors must be infinitely conductive, or lossless.
- Furthermore, since the axial magnetic field is zero, it follows that the E-field behaves statically in the transverse plane

$$\int_{\text{t-plane}} \mathbf{E} \cdot d\ell = -\frac{d}{dt} \int_{\text{t-plane}} \mathbf{B} \cdot d\mathbf{S} \equiv 0$$

Field Theory (cont)

- We can thus define a unique voltage in the transverse plane

$$v(z,t) = -\int_1^2 \mathbf{E} \cdot d\ell$$

- Similarly, since the axial electric field is zero, the magnetic field is solely dependent on the current

$$\int_{\text{t-plane}} \mathbf{H} \cdot d\ell = -\frac{d}{dt}\int_{\text{t-plane}} \mathbf{E} \cdot d\mathbf{S} + \mu I$$

Since the displacement current is identically zero, we define a unique current

$$\int_{\text{t-plane}} \mathbf{H} \cdot d\ell = \mu I$$

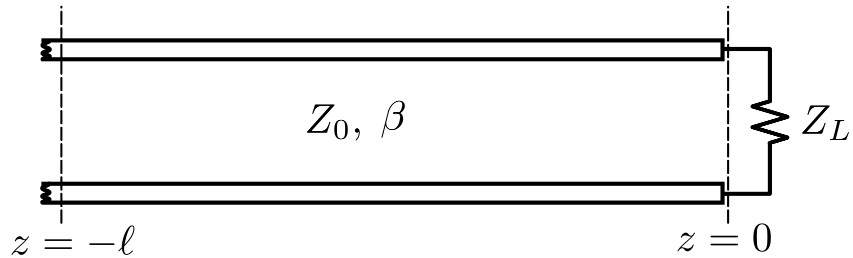
TEM Modes

- In summary, for TEM propagation, the fields have zero axial component, $E_z \equiv H_z \equiv 0$, and the transverse components of the fields behave like static fields.
- Thus, for a uniform structure the laws of electromagnetics reduce to statics regardless of the frequency of excitation!
- While only lossless conductors can truly support TEM waves, in practice, low-loss conductors are often employed and behave essentially like TEM guides.
- Finally, it can be proved that TEM waves can only be supported by *two* or more conductors. Any single conductor cannot support TEM waves because a single ideal conductor cannot support a static field solution.

T-Line Char Impedance

- An ideal transmission line has a constant char. impedance Z_0 versus frequency. Since the fields are the solution of the static fields in the transverse plane, there is no freq. dep. in the fields and thus no variation in the ind. and cap.
- Since all conductors are perfectly conducting, no fields can penetrate the conductors and thus only external ind. contributes to magnetic energy storage in the line (ind. is constant with freq.). In practice, of course, all transmission lines have loss. We have modeled the loss with an eq. series and shunt resistor in the ladder network. In practice, the TEM line properties do depend on frequency. While it is possible to account for the loss in our field equations, it's much better to treat the loss as a perturbation to an ideal line rather than manipulate the full blown Maxwell's equations from the outset.

Lossless T-Line Termination



- Okay, lossless line means $\gamma = j\beta$ ($\alpha = 0$), and $\Im(Z_0) = 0$ (real characteristic impedance independent of frequency)
- The voltage/current phasors take the standard form

$$v(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$i(z) = \frac{V^+}{Z_0} e^{-\gamma z} - \frac{V^-}{Z_0} e^{\gamma z}$$

Lossless T-Line Termination (cont)

- At load $Z_L = \frac{v(0)}{i(0)} = \frac{V^+ + V^-}{V^+ - V^-} Z_0$
- The reflection coefficient has the same form

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- Can therefore write

$$v(z) = V^+ \left(e^{-j\beta z} + \rho_L e^{j\beta z} \right)$$

$$i(z) = \frac{V^+}{Z_0} \left(e^{-j\beta z} - \rho_L e^{j\beta z} \right)$$

Power on T-Line (I)

- Let's calculate the average power dissipation on the line at point z

$$P_{av}(z) = \frac{1}{2}\Re\left[v(z)i(z)^*\right]$$

- Or using the general solution

$$P_{av}(z) = \frac{1}{2}\frac{|V^+|^2}{Z_0}\Re\left(\left(e^{-j\beta z} + \rho_L e^{j\beta z}\right)\left(e^{j\beta z} - \rho_L^* e^{-j\beta z}\right)\right)$$

- The product in the $\Re$ terms can be expanded into four terms

$$1 + \underbrace{\rho_L e^{2j\beta z} - \rho_L^* e^{2j\beta z}}_{a-a^*} - |\rho_L|^2$$

- Notice that $a - a^* = 2j\Im(a)$

Power on T-Line (II)

- The average power dissipated at z is therefore

$$P_{av} = \frac{|V^+|^2}{2Z_0}\left(1 - |\rho_L|^2\right)$$

- Power flow is constant (independent of z) along line (lossless)
- No power flows if $|\rho_L| = 1$ (open or short)
- Even though power is constant, voltage and current are not!

Voltage along T-Line

- When the termination is matched to the line impedance $Z_L = Z_0$, $\rho_L = 0$ and thus the voltage along the line $|v(z)| = |V^+|$ is constant. Otherwise

$$|v(z)| = |V^+||1 + \rho_L e^{2j\beta z}| = |V^+||1 + \rho_L e^{-2j\beta\ell}|$$

- The voltage magnitude along the line can be written as

$$|v(-\ell)| = |V^+||1 + |\rho_L|e^{j(\theta - 2\beta\ell)}|$$

- The voltage is maximum when the $2\beta\ell$ is a equal to $\theta + 2k\pi$, for any integer k; in other words, the reflection coefficient phase modulo 2π

$$V_{max} = |V^+|(1 + |\rho_L|)$$

Voltage Standing Wave Ratio (SWR)

- Similarly, minimum when $\theta + k\pi$, where k is an integer $k \neq 0$

$$V_{min} = |V^+|(1 - |\rho_L|)$$

- The ratio of the maximum voltage to minimum voltage is an important metric and commonly known as the voltage standing wave ratio, VSWR (Sometimes pronounced viswar), or simply the standing wave ratio SWR

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{1 + |\rho_L|}{1 - |\rho_L|}$$

- It follows that for a shorted or open transmission line the VSWR is infinite, since $|\rho_L| = 1$.

SWR Location

- Physically the maxima occur when the reflected wave adds in phase with the incoming wave, and minima occur when destructive interference takes place. The distance between maxima and minima is π in phase, or $2\beta\delta x = \pi$, or

$$\delta x = \frac{\pi}{2\beta} = \frac{\lambda}{4}$$

- VSWR is important because it can be deduced with a *relative* measurement. Absolute measurements are difficult at microwave frequencies. By measuring VSWR, we can readily calculate $|\rho_L|$.

VSWR → Impedance Measurement

- By measuring the location of the voltage minima from an unknown load, we can solve for the load reflection coefficient phase θ

$$\psi_{min} = \theta - 2\beta\ell_{min} = \pm\pi$$

- Note that

$$|v(-\ell_{min})| = |V^+||1 + |\rho_L|e^{j\psi_{min}}|$$

- Thus an unknown impedance can be characterized at microwave frequencies by measuring VSWR and ℓ_{min} and computing the load reflection coefficient. This was an important measurement technique that has been largely supplanted by a modern network analyzer with built-in digital calibration and correction.

VSWR Example

- Consider a transmission line terminated in a load impedance $Z_L = 2Z_0$. The reflection coefficient at the load is purely real

$$\rho_L = \frac{z_L - 1}{z_L + 1} = \frac{2 - 1}{2 + 1} = \frac{1}{3}$$

- Since $1 + |\rho_L| = 4/3$ and $1 - |\rho_L| = 2/3$, the VSWR is equal to 2.
- Since the load is real, the voltage minima will occur at a distance of $\lambda/4$ from the load

Impedance of T-Line (I)

- We have seen that the voltage and current along a transmission line are altered by the presence of a load termination. At an arbitrary point z, wish to calculate the input impedance, or the ratio of the voltage to current relative to the load impedance Z_L

$$Z_{in}(-\ell) = \frac{v(-\ell)}{i(-\ell)}$$

- It shall be convenient to define an analogous reflection coefficient at an arbitrary position along the line

$$\rho(-\ell) = \frac{V^- e^{-j\beta\ell}}{V^+ e^{j\beta\ell}} = \rho_L e^{-2j\beta\ell}$$

Impedance of T-Line (II)

- $\rho(z)$ has a constant magnitude but a periodic phase. From this we may infer that the input impedance of a transmission line is also periodic (relation btwn ρ and Z is one-to-one)

$$Z_{in}(-\ell) = Z_0 \frac{1 + \rho_L e^{-2j\beta\ell}}{1 - \rho_L e^{-2j\beta\ell}}$$

- The above equation is of paramount important as it expresses the input impedance of a transmission line as a function of position ℓ away from the termination.

Impedance of T-Line (III)

- This equation can be transformed into another more useful form by substituting the value of ρ_L

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_{in}(-\ell) = Z_0 \frac{Z_L(1 + e^{-2j\beta\ell}) + Z_0(1 - e^{-2j\beta\ell})}{Z_0(1 + e^{-2j\beta\ell}) + Z_L(1 - e^{-2j\beta\ell})}$$

Using the common complex expansions for sine and cosine, we have

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{(e^{jx} - e^{-jx})/2j}{(e^{jx} + e^{-jx})/2}$$

Impedance of T-Line (IV)

- The expression for the input impedance is now written in the following form

$$Z_{in}(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}$$

- This is the most important equation of the lecture, known sometimes as the “transmission line equation”

Shorted Line I/V

- The shorted transmission line has infinite VSWR and $\rho_L = -1$. Thus the minimum voltage $V_{min} = |V^+|(1 - |\rho_L|) = 0$, as expected. At any given point along the transmission line

$$v(z) = V^+(e^{-j\beta z} - e^{j\beta z}) = -2jV^+ \sin(\beta z)$$

whereas the current is given by

$$i(z) = \frac{V^+}{Z_0}(e^{-j\beta z} + e^{j\beta z})$$

or

$$i(z) = \frac{2V^+}{Z_0} \cos(\beta z)$$

Shorted Line Impedance (I)

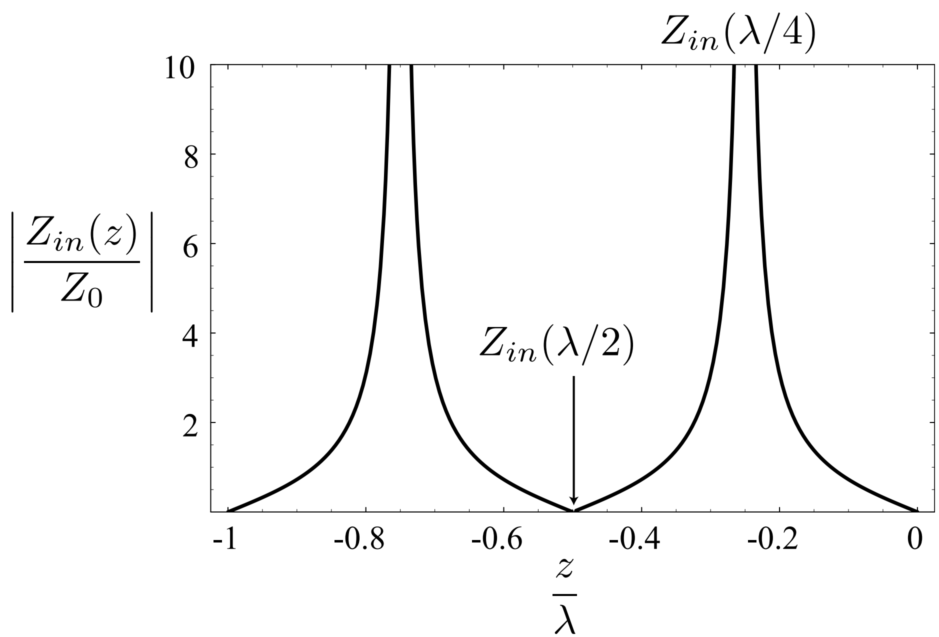
- The impedance at any point along the line takes on a simple form

$$Z_{in}(-\ell) = \frac{v(-\ell)}{i(-\ell)} = jZ_0 \tan(\beta\ell)$$

- This is a special case of the more general transmission line equation with $Z_L = 0$.
- Note that the impedance is purely imaginary since a shorted lossless transmission line cannot dissipate any power.
- We have learned, though, that the line stores reactive energy in a distributed fashion.

Shorted Line Impedance (II)

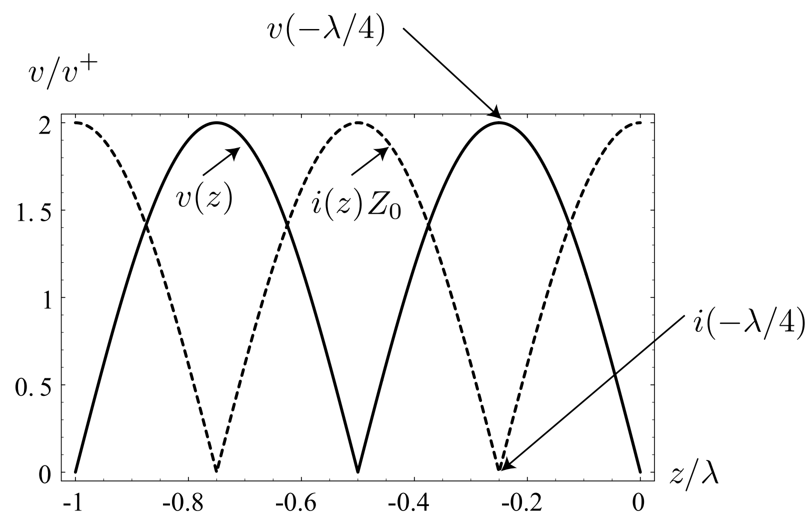
- A plot of the input impedance as a function of z is shown below



- The tangent function takes on infinite values when $\beta\ell$ approaches $\pi/2$ modulo 2π

Shorted Line Impedance (III)

- Shorted transmission line can have infinite input impedance!
- This is particularly surprising since the load is in effect transformed from a short of $Z_L = 0$ to an infinite impedance.
- A plot of the voltage/current as a function of z is shown below



Shorted Line Reactance

- $\ell \ll \lambda/4 \rightarrow$ inductor
- $\ell < \lambda/4 \rightarrow$ inductive reactance
- $\ell = \lambda/4 \rightarrow$ open (acts like resonant parallel LC circuit)
- $\ell > \lambda/4$ but $\ell < \lambda/2 \rightarrow$ capacitive reactance
- And the process repeats ...