EECS 217

Lecture 19: Mason's U Function/Examples

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Mason's Invariant U Function

- Mason discovered the function U given by

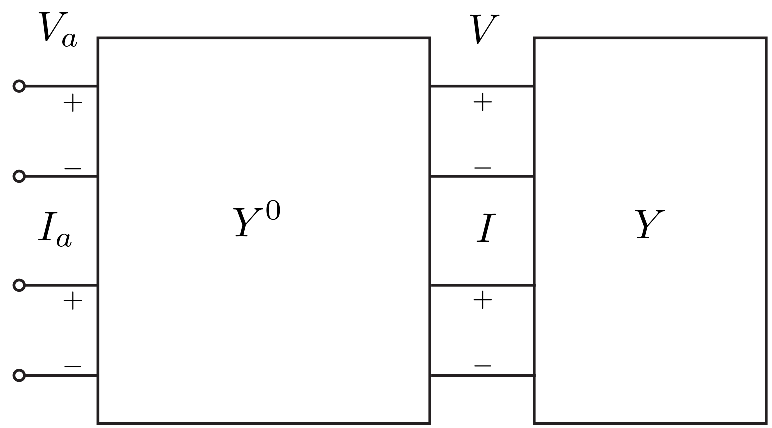
$$U = \frac{|k_{21} - k_{12}|^2}{4(\Re(k_{11})\Re(k_{22}) - \Re(k_{12})\Re(k_{21}))}$$

- For the hybrid matrix formulation (H or G), the U function is given by

$$U = \frac{|k_{21} + k_{12}|^2}{4(\Re(k_{11})\Re(k_{22}) + \Re(k_{12})\Re(k_{21}))}$$

- where k_{ij} are the two-port Y or Z parameters.
- This function is invariant under lossless reciprocal embeddings. Stated differently, any two-port can be *embedded* into a lossless and reciprocal circuit and the resulting two-port will have the same U function.
- This is a very important property, because this invariant property does not depend on any lossless matching circuitry that we employ before or after the two-port, or any lossless feedback. What does U signify?

Properties of U



- The invariant property is shown above. The U of the original two-port is the same as U_a of the overall two-port when a four port lossless reciprocal four-port is added.
- The U function has several important properties:
    1. If $U > 1$, the two-port is active. Otherwise, if $U \leq 1$, the two-port is passive.
    2. U is the maximum unilateral power gain of a device under a lossless reciprocal embedding.
    3. U is the maximum gain of a three-terminal device regardless of the common terminal.

Invariance of U

- With regards to the previous diagram, any lossless reciprocal embedding can be seen as an interconnection of the original two-port to a four-port, with the following block admittance matrix

$$\begin{pmatrix} I_a \\ -I \end{pmatrix} = \begin{pmatrix} Y_{11}^0 & Y_{12}^0 \\ Y_{21}^0 & Y_{22}^0 \end{pmatrix} \begin{pmatrix} V_a \\ V \end{pmatrix}$$

- Note that Y_{ij} is a 2×2 imaginary symmetric sub-matrix

$$Y_{jk}^0 = jB_{jk}$$

$$B_{jk} = B_{kj}^T$$

- Since $I = YV$, we can solve for V from the second equation

$$-I = Y_{21}^0 V_a + Y_{22}^0 V = -YV$$

$$V = -(Y + Y_{22}^0)^{-1} Y_{21}^0 V_a$$

U Invariance (cont)

- From the first equation we have the composite two-port matrix

$$I_a = (Y_{11}^0 - Y_{12}^0 (Y + Y_{22}^0)^{-1} Y_{21}^0) V_a = Y_a V_a$$

- By definition, the U function is given by

$$U = \frac{\det(Y_a - Y_a^T)}{\det(Y_a + Y_a^*)}$$

- Note that Y_a can be written as

$$Y_a = jB_{11} - jB_{12}(Y + jB_{22})^{-1} jB_{12}^T$$

$$Y_a = jB_{11} + B_{12}(Y + jB_{22})^{-1} B_{12}^T$$

- Focus on the denominator of U

$$Y_a + Y_a^* = B_{12}(W^{-1} + (W^*)^{-1}) B_{12}^T$$

- where $W = Y + Y_{22}^0 = Y + jB_{22}$.

Invariance (cont)

- Factoring W^{-1} from the left and $(W^*)^{-1}$ from the right, we have

$$= B_{12} W^{-1} (W^* + W)(W^*)^{-1} B_{12}^T$$

- But $W + W^* = Y + Y^*$ resulting in

$$Y_a + Y_a^* = B_{12} W^{-1} (Y + Y^*)(W^*)^{-1} B_{12}^T$$

- In a like manner, one can show that

$$Y_a - Y_a^T = B_{12} W^{-1} (Y^T - Y)(W^*)^{-1} B_{12}^T$$

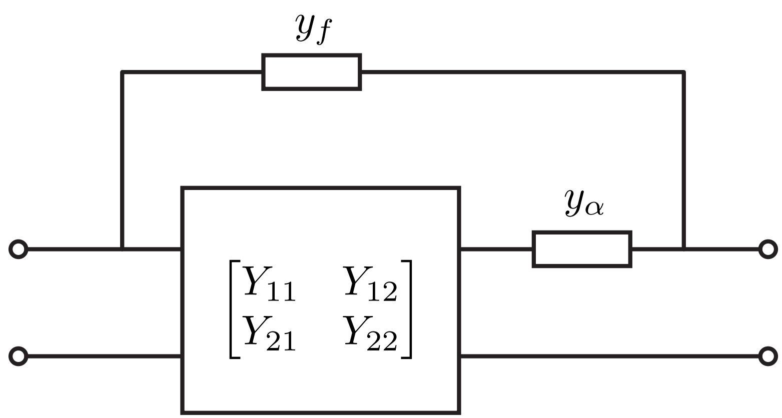
- Taking the determinants and ratios

$$\det(Y_a + Y_a^*) = \frac{(\det B_{12})^2 \det(Y + Y^*)}{(\det W)^2}$$

$$\det(Y_a - Y_a^T) = \frac{(\det B_{12})^2 \det(Y^T - Y)}{(\det W)^2}$$

$$U = \frac{\det(Y_a - Y_a^T)}{\det(Y_a + Y_a^*)} = \frac{\det(Y - Y^T)}{\det(Y + Y^*)}$$

Maximum Unilateral Gain



- Consider the above feedback structure where y_f and y_α are lossless reactances. We can derive the overall two-port equations by a cascade connection followed by a shunt connection of two-ports

$$Y_a = \frac{y_\alpha}{y_\alpha + y_{22}} \begin{bmatrix} y_{11} + \Delta_y / y_\alpha & y_{12} \\ y_{21} & y_{22} \end{bmatrix} + \begin{bmatrix} y_f & -y_f \\ -y_f & y_f \end{bmatrix}$$

Unilaterization

- To unilaterize the device, we select

$$y_f = \frac{y_{12} y_\alpha}{y_{22} + y_\alpha}$$

- We can solve for b_α and b_f

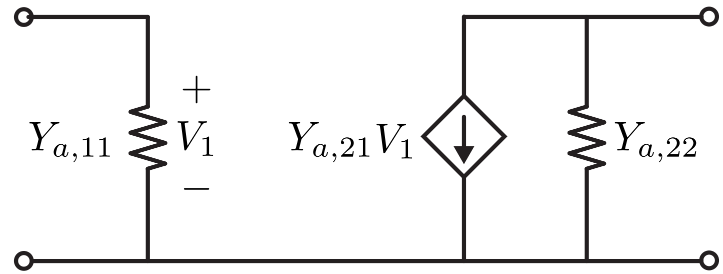
$$b_f = \Im(y_{12}) - \frac{\Re(y_{12})}{\Re(y_{22})} \Im(y_{22})$$

$$b_\alpha = b_f \frac{\Re(y_{22})}{\Re(y_{12})}$$

- It can be shown that the overall Y_a matrix is given by

$$Y_a = \frac{j\Im(y_{22}^* y_{12})}{y_{12}\Re(y_{22})} \begin{bmatrix} y_{11} + y_{12} - j\frac{\Delta_y \Re(y_{12})}{\Im(y_{22}^* y_{12})} & 0 \\ y_{21} - y_{12} & y_{22} + y_{12} \end{bmatrix}$$

Unilaterized Two-Port

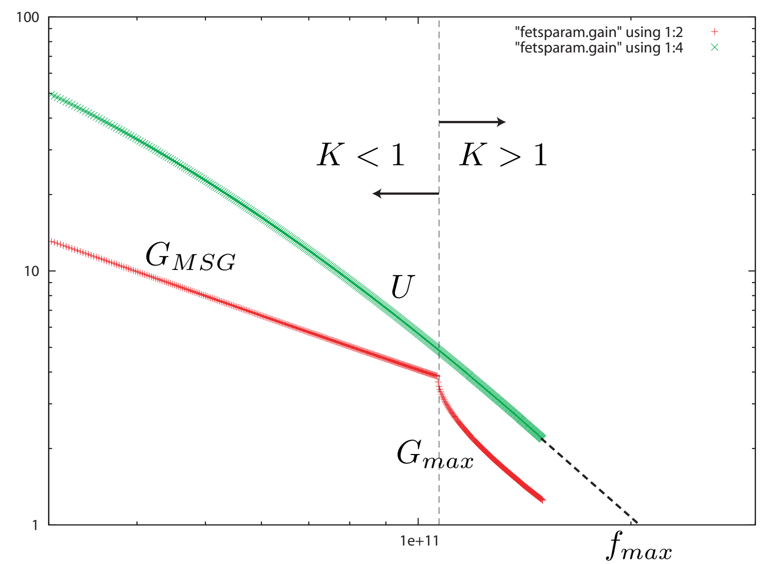


- The two-port equivalent circuit under unilaterization is shown above. Notice now that the maximum power gain of this circuit is given by

$$G_{U,max} = \frac{|Y_{a_{21}}|^2}{4\Re(Y_{a_{11}})\Re(Y_{a_{22}})} = U_a$$

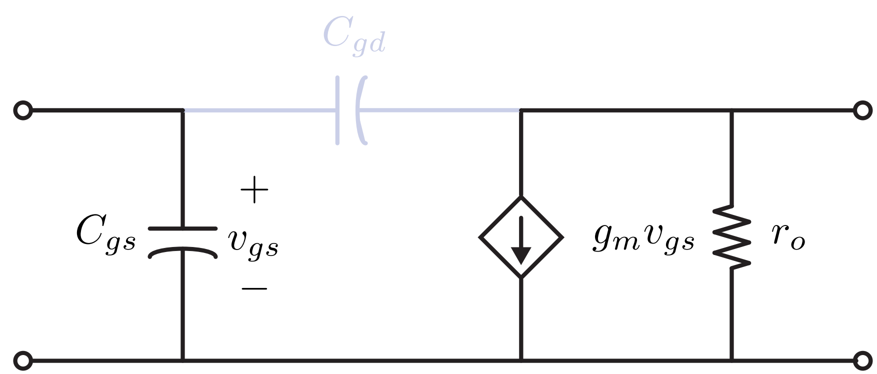
- Thus we can attribute physical significance to U_a as the maximum unilateral gain. Furthermore, due to the invariance of U, $U_a = U$ for the original two-port network.
- It's important to note that *any* unilaterization scheme will yield the same maximum power! Thus U is a good metric for the device.

Gain Plots



- U is often used as a metric for a two-port device. It represents the maximum gain that the device can deliver if we use lossless reciprocal embeddings to unilaterize the device. U is also a good metric for characterizing a three terminal device with a common-terminal, such as a transistor.
- U is invariant to the common terminal, so a common-gate amplifier has the same U as a common-source amplifier.
- In the figure above, the device G_{MSG} is plotted for low frequencies where $K < 1$. At the breakpoint, $K > 1$ and the device is unconditionally stable and thus G_{max} is plotted. Note that the U curve is always larger than G_{max} but both curves cross $0\,\mathrm{dB}$ together. At this point, the f_{max} of the device, the two-port becomes passive.

Stability Examples



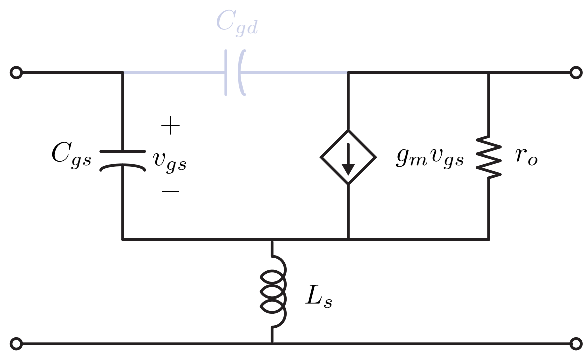
- A simple equivalent circuit for a FET without any feedback is of course absolutely stable if the resistors of the model are positive. The Z matrix for the circuit is given by

$$Z = \begin{bmatrix} \frac{1}{j\omega C_{gs}} & 0 \\ \frac{-g_m r_o}{j\omega C_{gs}} & r_o \end{bmatrix}$$

- Since $Z_{12} = 0$, the stability factor $K = \infty$

$$K = \frac{2\Re(Z_{11})\Re(Z_{22}) - \Re(Z_{12}Z_{21})}{|Z_{12}Z_{21}|}$$

Inductive Degeneration



- Although $Z_{12} \approx 0$ for a FET at low frequency, the input impedance is purely capacitive. To introduce a real component, inductive degeneration is commonly employed. The Z matrix for the inductor is simply

$$Z = j\omega L_s \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Adding the Z matrix (due to series connection)

$$Z = \begin{bmatrix} j\omega L_s + \frac{1}{j\omega C_{gs}} & j\omega L_s \\ j\omega L_s - \frac{g_m r_o}{j\omega C_{gs}} & r_o + j\omega L_s \end{bmatrix}$$

Inductive Degen (cont)

- This feedback introduces a Z_{12} and thus the stability must be carefully examined

$$K = \frac{2 \cdot 0 \cdot r_o - \left(-\omega^2 L_s^2 - \frac{g_m L_s r_o}{C_{gs}}\right)}{\omega^2 L_s^2 + \frac{g_m r_o L_s}{C_{gs}}} = 1$$

- We see that this circuit is unconditionally stable. More importantly, the stability factor is frequency independent. In reality parasitics can destabilize the transistor.
- The maximum gain is thus given by

$$G_{max} = \left|\frac{Z_{21}}{Z_{12}}\right| \left(K - \sqrt{K^2 - 1}\right) = \left|\frac{Z_{21}}{Z_{12}}\right|$$

$$= \frac{\omega L_s + \frac{g_m r_o}{\omega C_{gs}}}{\omega L_s} = 1 + \frac{g_m r_o}{\omega^2 L_s C_{gs}}$$

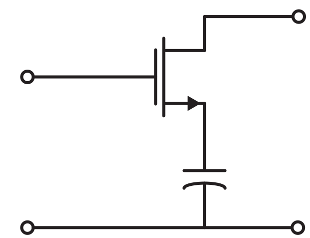
$$= 1 + \left(\frac{\omega_T}{\omega_0}\right)^2 \left(\frac{r_o}{\omega_T L_s}\right)$$

- It's easy to show that the synthesize real input resistance is $\omega_T L_s$, and so the last term is the ratio of r_o/R_S under matched conditions.

Capacitive Degeneration

- The Z matrix for capacitive degeneration is given by

$$Z = \begin{bmatrix} \frac{1}{j\omega C_s} + \frac{1}{j\omega C_{gs}} & \frac{1}{j\omega C_s} \\ \frac{1}{j\omega C_s} - \frac{g_m r_o}{j\omega C_{gs}} & r_o + \frac{1}{j\omega C_s} \end{bmatrix}$$



- The stability factor is given by

$$K = \frac{2 \cdot 0 \cdot r_o - \left(\frac{g_m r_o}{\omega^2 C_s C_{gs}} - \frac{1}{\omega^2 C_s^2} \right)}{\left| \frac{g_m r_o}{\omega^2 C_s C_{gs}} - \frac{1}{\omega^2 C_s^2} \right|}$$

- Note this is simply

$$K = \frac{-a+b}{|a-b|} = \begin{cases} \frac{b-a}{a-b} < 0 & a > b \\ \frac{b-a}{b-a} = 1 & b < a \end{cases}$$

- The condition for stability is therefore

$$\frac{g_m r_o}{C_{gs}} > \frac{1}{C_s}$$

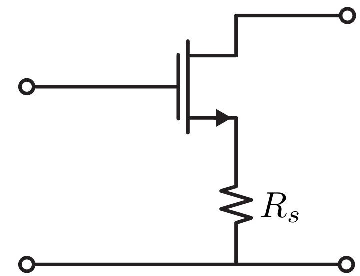
Range of K

- So far we have dealt with $K > 0$. Suppose that $|\Delta| > 1$. We know that for $0 < K < 1$ the two-port is conditionally stable. In other words, the stability circle intersects with the unit circle with the overlap (usually) corresponding to the unstable region. Instability can also occur if $K > 1$ and $|\Delta| > 1$, but this is less common (occurs with FB).
- On the other hand, if $-1 < K < 0$, one can show graphically that the entire unit circle on the Smith Chart is unstable. In other words, the stability circle does not intersect with the unit circle or the instability circle contains the entire circle.

Resistive Degeneration

- Resistive degeneration is commonly employed to stabilize the bias point of a transistor. The Z matrix is given by

$$Z = \begin{bmatrix} R_s + \frac{1}{j\omega C_{gs}} & R_s \\ R_s - \frac{g_m r_o}{j\omega C_{gs}} & r_o + R_s \end{bmatrix}$$



- The K factor is computed as before

$$K = \frac{2R_s(r_o + R_s) - R_s^2}{R_s\sqrt{R_s^2 + \frac{g_m^2 r_o^2}{\omega^2 C_{gs}^2}}}$$

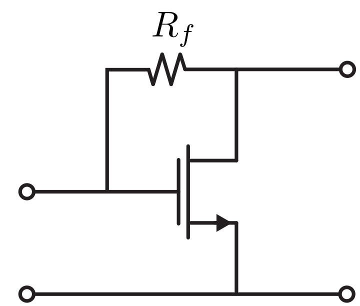
- At low frequencies, we have

$$K = \frac{2r_o + R_s}{\frac{g_m r_o}{\omega C_{gs}}} \approx \frac{2\omega C_{gs}}{g_m} = \frac{2\omega}{\omega_T} < 1$$

Shunt Feedback

- Shunt feedback is a common broadband matching approach. Now working with the Y matrix of the transistor (simplified as before)

$$Y_{fet} = \begin{bmatrix} j\omega C_{gs} & 0 \\ g_m & G_o + j\omega C_{ds} \end{bmatrix}$$



- The feedback element has a Y matrix

$$Y_f = G_f \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

- And thus the overall amplifier

$$Y = \begin{bmatrix} G_f + j\omega C_{gs} & -G_f \\ g_m - G_f & G_f + G_o + j\omega C_{ds} \end{bmatrix}$$

Shunt Feedback (cont)

- The stability factor for the shunt feedback amplifier is given by

$$K = \frac{2G_f(G_o + G_f) - G_f(G_f - g_m)}{G_f|g_m - G_f|}$$

- Suppose that $g_m R_f > 1$

$$= \frac{g_m + G_f}{g_m - G_f} = \frac{g_m R_f + 1}{g_m R_f - 1} > 1$$

- The choice of R_f and g_m is governed by the current consumption, power gain, and impedance matching. For a bi-conjugate match

$$G_{max} = \left|\frac{Y_{21}}{Y_{12}}\right| \left(K - \sqrt{K^2 - 1}\right)$$

$$= \frac{g_m - G_f}{G_f} \left(\left(\frac{g_m R_f + 1}{g_m R_f - 1}\right) - \sqrt{\left(\frac{g_m R_f + 1}{g_m R_f - 1}\right)^2 - 1} \right) = \left(1 - \sqrt{g_m R_F}\right)^2$$

Shunt Feedback Input Admittance

- The input admittance is calculated as follows

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_L}$$

$$= j\omega C_{gs} + G_f - \frac{-G_f(g_m - G_f)}{G_o + G_f + G_L + j\omega C_{ds}}$$

$$= j\omega C_{gs} + G_f + \frac{G_f(g_m - G_f)(G_o + G_f + G_L - j\omega C_{ds}}{(G_o + G_f + G_L)^2 + \omega^2 C_{ds}^2}$$

- At lower frequencies, $\omega < \frac{1}{C_{ds}R_f||R_L}$ we have (neglecting G_o)

$$\Re(Y_{in}) = G_f + \frac{G_f(g_m - G_f)}{G_f + G_L}$$

$$= \frac{1 + g_m R_L}{R_F + R_L}$$

$$\Im(Y_{in}) = \omega \left(C_{gs} - \frac{C_{ds}}{1 + \frac{R_f}{R_L}} \right)$$