EECS 217

Lecture 17: Stability and Passivity of Two-Ports

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Stability of a Two-Port

- A two-port is unstable if the admittance of either port has a negative conductance for a passive termination on the second port. Under such a condtion, the two-port can oscillate.
- Consider the input admittance

$$Y_{in} = G_{in} + jB_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_L}$$

- Using the following definitions

$$Y_{11} = g_{11} + jb_{11} \qquad Y_{12}Y_{21} = P + jQ = L\angle\phi$$

$$Y_{22} = g_{22} + jb_{22} \qquad Y_L = G_L + jB_L$$

- Now substitute real/imag parts of the above quantities into Y_{in}

$$Y_{in} = g_{11} + jb_{11} - \frac{P + jQ}{g_{22} + jb_{22} + G_L + jB_L}$$

$$= g_{11} + jb_{11} - \frac{(P + jQ)(g_{22} + G_L - j(b_{22} + B_L))}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2}$$

Input Conductance

- Taking the real part, we have the input conductance

$$\Re(Y_{in}) = G_{in} = g_{11} - \frac{P(g_{22}+G_L) + Q(b_{22}+B_L)}{(g_{22}+G_L)^2 + (b_{22}+B_L)^2}$$

$$= \frac{(g_{22}+G_L)^2 + (b_{22}+B_L)^2 - \frac{P}{g_{11}}(g_{22}+G_L) - \frac{Q}{g_{11}}(b_{22}+B_L)}{D}$$

- Since $D > 0$ if $g_{11} > 0$, we can focus on the numerator. Note that $g_{11} > 0$ is a requirement since otherwise oscillations would occur for a short circuit at port 2.
- The numerator can be factored into several positive terms

$$N = (g_{22}+G_L)^2 + (b_{22}+B_L)^2 - \frac{P}{g_{11}}(g_{22}+G_L) - \frac{Q}{g_{11}}(b_{22}+B_L)$$

$$= \left(G_L + \left(g_{22} - \frac{P}{2g_{11}}\right)\right)^2 + \left(B_L + \left(b_{22} - \frac{Q}{2g_{11}}\right)\right)^2 - \frac{P^2+Q^2}{4g_{11}^2}$$

Input Conductance (cont)

- Now note that the numerator can go negative only if the first two terms are smaller than the last term. To minimize the first two terms, choose $G_L = 0$ and $B_L = -\left(b_{22} - \frac{Q}{2g_{11}}\right)$ (reactive load)

$$N_{min} = \left(g_{22} + \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2}$$

- And thus the above must remain positive, $N_{min} > 0$, so

$$\left(g_{22} + \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} > 0$$

$$g_{11}g_{22} > \frac{P + L}{2} = \frac{L}{2}(1 + \cos\phi)$$

Linvill/Llewellyn Stability Factors

- Using the above equation, we define the Linvill stability factor

$$L < 2g_{11}g_{22} - P$$

$$C = \frac{L}{2g_{11}g_{22} - P} < 1$$

- The two-port is stable if $0 < C < 1$.
- It's more common to use the inverse of C as the stability measure

$$\frac{2g_{11}g_{22} - P}{L} > 1$$

- The above definition of stability is perhaps the most common

$$K = \frac{2\Re(Y_{11})\Re(Y_{22}) - \Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} > 1$$

- The above expression is identical if we interchnage ports 1/2. Thus it's the general condition for stability.
- Note that $K > 1$ is the same condition for the maximum stable gain derived last lecture. The connection is now more obvious. If $K < 1$, then the maximum gain is infinity!

Stability From Another Perspective

- We can also derive stability in terms of the input reflection coefficient. For a general two-port with load Γ_L we have

$$v_2^- = \Gamma_L^{-1} v_2^+ = S_{21} v_1^+ + S_{22} v_2^+$$

$$v_2^+ = \frac{S_{21}}{\Gamma_L^{-1} - S_{22}} v_1^-$$

$$v_1^- = \left(S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}} \right) v_1^+$$

$$\Gamma = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}}$$

- If $|\Gamma| < 1$ for all Γ_L, then the two-port is stable

$$\Gamma = \frac{S_{11}(1 - S_{22}\Gamma_L) + S_{12} S_{21} \Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} + \Gamma_L (S_{21} S_{12} - S_{11} S_{22})}{1 - S_{22}\Gamma_L}$$

$$= \frac{S_{11} - \Delta \Gamma_L}{1 - S_{22}\Gamma_L}$$

Stability Circle

- To find the boundary between stability/instability, let's set $|\Gamma| = 1$

$$\left| \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} \right| = 1$$

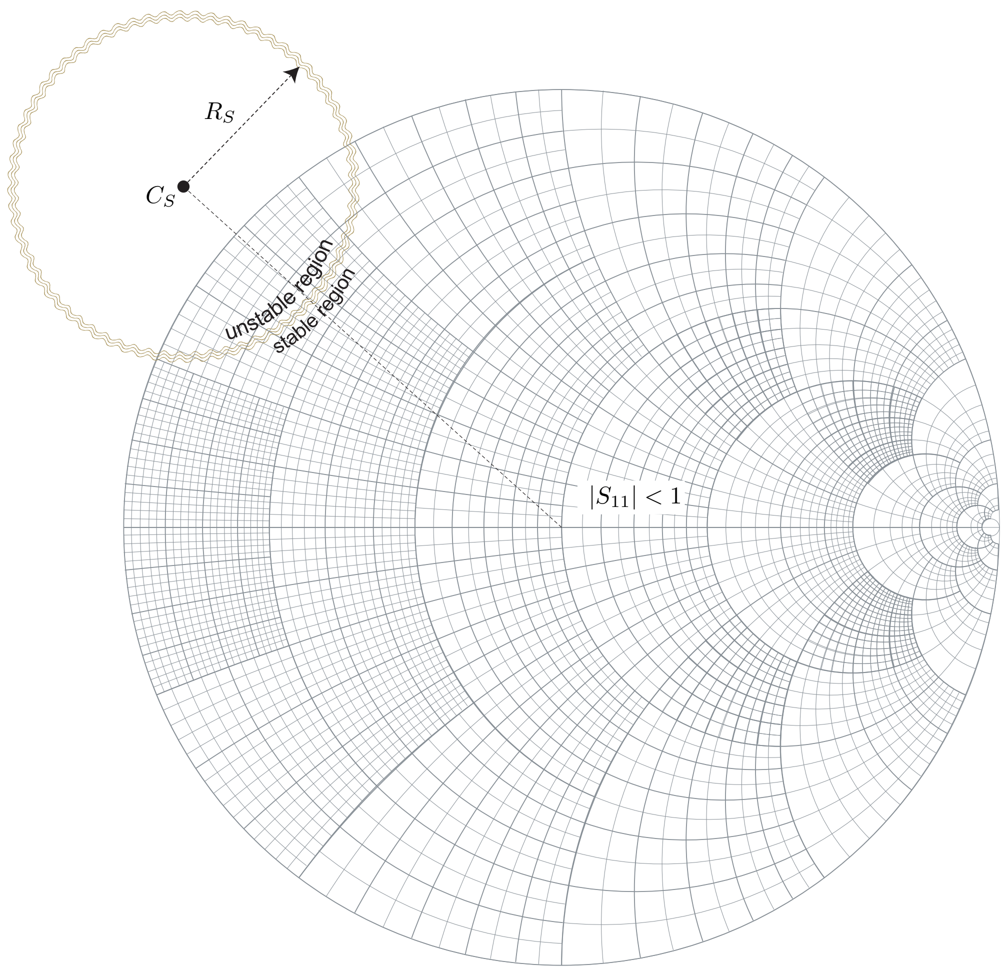
$$|S_{11} - \Delta\Gamma_L| = |1 - S_{22}\Gamma_L|$$

- After some algebraic manipulations, we arrive at the following equation

$$\left| \Gamma - \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2} \right| = \frac{|S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2}$$

- This is of course an equation of a circle, $|\Gamma - C| = R$, in the complex plane with center at C and radius R
- Thus a circle on the Smith Chart divides the region of instability from stability.

Example: Stability Circle



- In this example, the origin of the circle lies outside the stability circle but a portion of the circle falls inside the unit circle. Is the region of stability inside the circle or outside?
- This is easily determined if we note that if $\Gamma_L = 0$, then $\Gamma = S_{11}$. So if $S_{11} < 1$, the origin should be in the stable region. Otherwise, if $S_{11} > 1$, the origin should be in the unstable region.

Stability: Unilateral Case

- Consider the stability circle for a unilateral two-port

$$C_S = \frac{S_{11}^* - (S_{11}^* S_{22}^*) S_{22}}{|S_{11}|^2 - |S_{11} S_{22}|^2} = \frac{S_{11}^*}{|S_{11}|^2}$$

$$R_S = 0$$

$$|C_S| = \frac{1}{|S_{11}|}$$

- The cetner of the circle lies outside of the unit circle if $|S_{11}|$ < 1. The same is true of the load stability circle. Since the radius is zero, stability is only determined by the location of the center.
- If $S_{12} = 0$, then the two-port is unconditionally stable if $S_{11} < 1$ and $S_{22} < 1$.
- This result is trivial since

$$\Gamma_S \Big|_{S_{12}=0} = S_{11}$$

- The stability of the source depends only on the device and not on the load.

Mu Stability Test

- If we want to determine if a two-port is unconditionally stable, then we should use the μ test

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} > 1$$

- The μ test not only is a test for unconditional stability, but the magnitude of μ is a measure of the stability. In other words, if one two port has a larger μ, it is more stable.
- The advantage of the μ test is that only a single parameter needs to be evaluated. There are no auxiliary conditions like the K test derivation earlier.
- The derivaiton of the μ test can proceed as follows. First let $\Gamma_S = |\rho_s|e^{j\phi}$ and evaluate Γ_{out}

$$\Gamma_{out} = \frac{S_{22} - \Delta|\rho_s|e^{j\phi}}{1 - S_{11}|\rho_s|e^{j\phi}}$$

- Next we can manipulate this equation into the following eq. for a circle $|\Gamma_{out} - C| = R$

$$\left|\Gamma_{out} + \frac{|\rho_s|S_{11}^*\Delta - S_{22}}{1 - |\rho_s||S_{11}|^2}\right| = \frac{\sqrt{|\rho_s|}|S_{12}S_{21}|}{(1 - |\rho_s||S_{11}|^2)}$$

Mu Test (cont)

- For a two-port to be unconditionally stable, we'd like Γ_{out} to fall within the unit circle

$$||C| + R| < 1$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| < 1 - |\rho_s||S_{11}|^2$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| + |\rho_s||S_{11}|^2 < 1$$

- The worse case stability occurs when $|\rho_s| = 1$ since it maximizes the left-hand side of the equation. Therefore we have

$$\mu = \frac{1 - |S_{11}|^2}{|S_{11}^*\Delta - S_{22}| + |S_{12}S_{21}|} > 1$$

K-Δ *Test*

- The K stability test has already been derived using Y parameters. We can also do a derivation based on S parameters. This form of the equation has been attributed to Rollett and Kurokawa.
- The idea is very simple and similar to the μ test. We simply require that all points in the instability region fall outside of the unit circle.
- The stability circle will intersect with the unit circle if

$$|C_L| - R_L > 1$$

or

$$\frac{|S_{22}^* - \Delta^* S_{11}| - |S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2} > 1$$

- This can be recast into the following form (assuming $|\Delta| < 1$)

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}||S_{21}|} > 1$$

N-Port Passivity

- We would like to find if an N-port is active or passive. By definition, an N-port is passive if it can only absorb net power. The total net complex power flowing into or out of a N port is given by

$$P = (V_1^* I_1 + V_2^* I_2 + \cdots) = (I_1^* V_1 + I_2^* V_2 + \cdots)$$

- If we sum the above two terms we have

$$P = \frac{1}{2}(v^*)^T i + \frac{1}{2}(i^*)^T v$$

- For vectors of current and voltage i and v. Using the admittanc ematrix $i = Yv$, this can be recast as

$$P = \frac{1}{2}(v^*)^T Y v + \frac{1}{2}(Y^* v^*)^T v = \frac{1}{2}(v^*)^T Y v + \frac{1}{2}(v^*)^T (Y^*)^T v$$

$$P = (v^*)^T \frac{1}{2}(Y + (Y^*)^T) v = (v^*)^T Y_H v$$

- Thus for a network to be passive, the Hermitian part of the matrix Y_H should be positive semi-definite.

Two-Port Passivity

- For a two-port, the condition for passivity can be simplified as follows. Let the general hybrid admittance matrix for the two-port be given by

$$H(s) = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} + j \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$$

$$H_H(s) = \frac{1}{2}(H(s) + H^*(s))$$

$$= \begin{pmatrix} m_{11} & \frac{1}{2}((m_{12} + m_{21}) + j(n_{12} - n_{21})) \\ ((m_{12} + m_{21}) + j(n_{21} - n_{12})) & m_{22} \end{pmatrix}$$

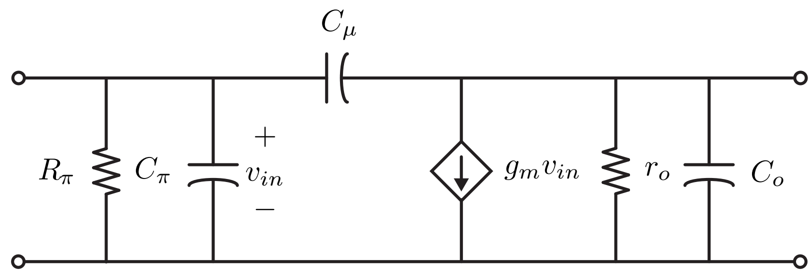
- This matrix is positive semi-definite if

$$m_{11} > 0 \qquad m_{22} > 0 \qquad det H_n(s) \geq 0$$ or

$$4m_{11}m_{22} - |k_{12}|^2 - |k_{21}|^2 - 2\Re(k_{12}k_{21}) \geq 0$$

$$4m_{11}m_{22} \geq |k_{12} + k_{21}^*|^2$$

Hybrid-Pi Example



- The hybrid-pi model for a transistor is shown above. Under what conditions is this two-port active? The hybrid matrix is given by

$$H(s) = \frac{1}{G_\pi + s(C_\pi + C_\mu)} \begin{pmatrix} 1 & sC_\mu \\ g_m - sC_\mu & q(s) \end{pmatrix}$$

$$q(s) = (G_\pi + sC_\pi)(G_0 + sC_\mu) + sC_\mu(G_\pi + g_m)$$

- Applying the condition for passivity we arrive at

$$4G_\pi G_0 \geq g_m^2$$

- The above equation is either satisfied for the two-port or not, regardless of frequency. Thus our analysis shows that the hybrid-pi model is not physical. We know from experience that real two-ports are active up to some frequency f_{max}.