EECS 217

Lecture 16: Properties of Two-Ports

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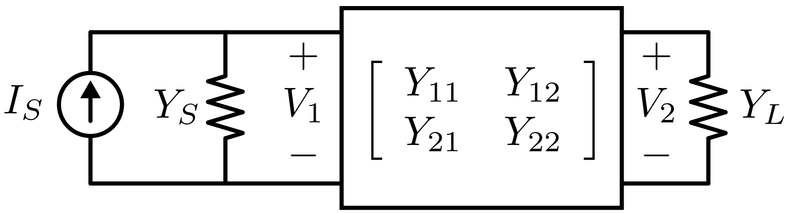
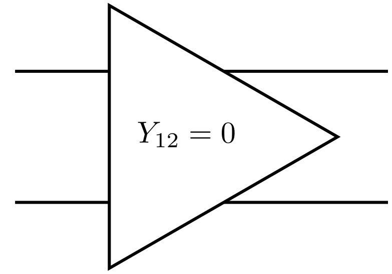
Why Two-Ports?





- A large array of important devices are two-ports. Examples include amplifiers, filters, and matching networks.
- A general two-port is shown above and represented by the 8 real numbers of the two-port matrix. A *unilateral* two-port is shown with as a triangle to emphasize the fact that only signals from the input appear at the output. In terms of the two-port matrix, the complex coefficient $m_{12} = 0$ (e.g. $y_{12} = 0$, $z_{12} = 0$, etc).

Input/Output Admittance

- The input and output impedance of a two-port will play an important role in our discussions. The stability and power gain of the two-port is determined by these quantities.
- In terms of y-parameters

$$Y_{in} = \frac{I_1}{V_1} = \frac{Y_{11}V_1 + Y_{12}V_2}{V_1} = Y_{11} + Y_{12}\frac{V_2}{V_1}$$

- The voltage gain of the two-port is given by solving the following equations

$$-I_2 = V_2 Y_L = -(Y_{21}V_1 + V_2 Y_{22})$$

$$\frac{V_2}{V_1} = \frac{-Y_{21}}{Y_L + Y_{22}}$$

- Note that for a simple transistor $Y_{21} = g_m$ and so the above reduces to the familiar $g_m R_o || R_L$.

Input/Output Admittance (cont)

- We can now solve for the input and output admittance

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}$$

$$Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}}$$

- Note that if $Y_{12} = 0$, then the input and output impedance are de-coupled

$$Y_{in} = Y_{11}$$

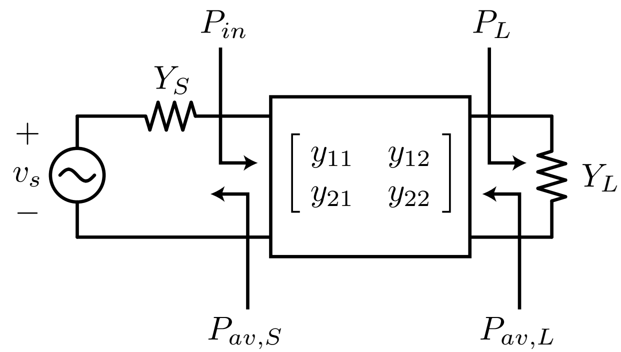
$$Y_{out} = Y_{22}$$

- But in general they are coupled and changing the load will change the input admittance.
- It's interesting to note the same formula derived above also works for the input/output impedance

$$Z_{in} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_L + Z_{22}}$$

- The same is true for the hybrid and inverse hybrid matrices.

Power Gain



- We can define power gain in many different ways. The *power gain* G_p is defined as follows

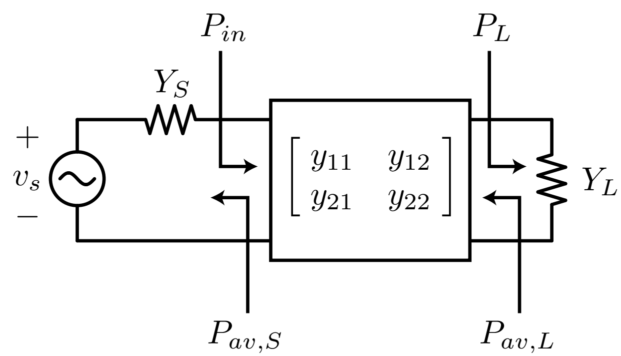
$$G_p = \frac{P_L}{P_{in}} = f(Y_L, Y_{ij}) \neq f(Y_S)$$

- We note that this power gain is a function of the load admittance Y_L and the two-port parameters Y_{ij}.
- The *available power gain* is defined as follows

$$G_a = \frac{P_{av,L}}{P_{av,S}} = f(Y_S, Y_{ij}) \neq f(Y_L)$$

- The available power from the two-port is denoted $P_{av,L}$ whereas the power available from the source is $P_{av,S}$.

Power Gain (cont)



- Finally, the *transducer gain* is defined by

$$G_T = \frac{P_L}{P_{av,S}} = f(Y_L, Y_S, Y_{ij})$$

- This is a measure of the efficacy of the two-port as it compares the power at the load to a simple conjugate match.

Derivation of Power Gain

- The power gain is readily calculated from the input admittance and voltage gain

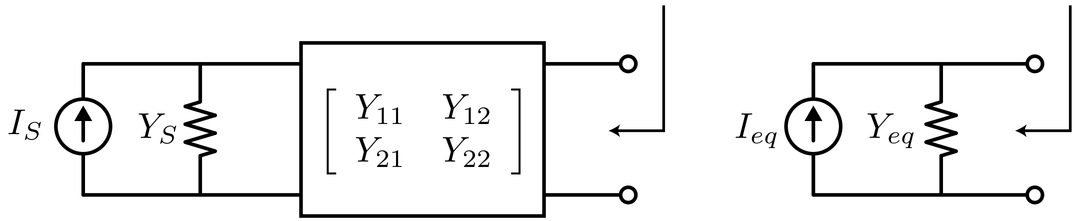
$$P_{in} = \frac{|V_1|^2}{2}\Re(Y_{in})$$

$$P_L = \frac{|V_2|^2}{2}\Re(Y_L)$$

$$G_p = \left|\frac{V_2}{V_1}\right|^2 \frac{\Re(Y_L)}{\Re(Y_{in})}$$

$$G_p = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \frac{\Re(Y_L)}{\Re(Y_{in})}$$

Derivation of Available Gain



- To derive the available power gain, consider a Norton equivalent for the two-port where

$$I_{eq} = I_2 = Y_{21}V_1 = \frac{Y_{21}}{Y_{11} + Y_S} I_S$$

- The Norton equivalent admittance is simply the output admittance of the two-port

$$Y_{eq} = Y_{22} - \frac{Y_{21}Y_{12}}{Y_{11} + Y_S}$$

- The available power at the source and load are given by

$$P_{av,S} = \frac{|I_S|^2}{8\Re(Y_S)} \qquad P_{av,L} = \frac{|I_{eq}|^2}{8\Re(Y_{eq})}$$

$$G_a = \frac{|Y_{21}|^2}{|Y_{11} + Y_S|^2} \frac{\Re(Y_S)}{\Re(Y_{eq})}$$

Transducer Gain Derivation

- The transducer gain is given by

$$G_T = \frac{P_L}{P_{av,S}} = \frac{\frac{1}{2}\Re(Y_L)|V_2|^2}{\frac{|I_S|^2}{8\Re(Y_S)}} = 4\Re(Y_L)\Re(Y_S)\left|\frac{V_2}{I_S}\right|^2$$

- We need to find the output voltage in terms of the source current. Using the voltage gain we have and input admittance we have

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right|$$

$$I_S = V(Y_S + Y_{in})$$

$$\left|\frac{V_2}{I_S}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right| \frac{1}{|Y_S + Y_{in}|}$$

$$|Y_S + Y_{in}| = \left|Y_S + Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right|$$

Transducer Gain (cont)

- We can now express the output voltage as a function of source current as

$$\left|\frac{V_2}{I_S}\right|^2 = \frac{|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

- And thus the transducer gain

$$G_T = \frac{4\Re(Y_L)\Re(Y_S)|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

- It's interesting to note that *all* of the gain expression we have derived are in the exact same form for the impedance, hybrid, and inverse hybrid matrices.

Comparison of Power Gains

- In general, $P_L \leq P_{av,L}$, with equality for a matched load. Thus we can say that

$$G_T \leq G_a$$

- The maximum transducer gain as a function of the load impedance thus occurs when the load is conjugately matched to the two-port output impedance

$$G_{T,max,L} = \frac{P_L(Y_L = Y_{out}^*)}{P_{av,S}} = G_a$$

- Likewise, since $P_{in} \leq P_{av,S}$, again with equality when the the two-port is conjugately matched to the source, we have

$$G_T \leq G_p$$

- The transducer gain is maximized with respect to the source when

$$G_{T,max,S} = G_T(Y_{in} = Y_S^*) = G_p$$

Bi-Conjugate Match

- When the input and output are simultaneously conjugately matched, or a *bi-conjugate match* has been established, we find that the transducer gain is maximized with respect to the source and load impedance

$$G_{T,max} = G_{p,max} = G_{a,max}$$

- This is thus the recipe for calculating the optimal source and load impedance in to maximize gain

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} = Y_S^*$$

$$Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}} = Y_L^*$$

- Solution of the above four equations (real/imag) results in the optimal $Y_{S,opt}$ and $Y_{L,opt}$.

Calculation of Optimal Source/Load

- Another approach is to simply equate the partial derivatives of G_T with respect to the source/load admittance to find the maximum point

$$\frac{\partial G_T}{\partial G_S} = 0 \qquad \frac{\partial G_T}{\partial G_L} = 0$$

$$\frac{\partial G_T}{\partial B_S} = 0 \qquad \frac{\partial G_T}{\partial B_L} = 0$$

- Again we have four equations. But we should be smarter about this and recall that the maximum gains are all equal. Since G_a and G_p are only a function of the source or load, we can get away with only solving two equations. For instance

$$\frac{\partial G_a}{\partial G_S} = 0 \qquad \frac{\partial G_a}{\partial B_S} = 0$$

- This yields $Y_{S,opt}$ and by setting $Y_L = Y_{out}^*$ we can find the $Y_{L,opt}$.
- Likewise we can also solve

$$\frac{\partial G_p}{\partial G_L} = 0 \qquad \frac{\partial G_p}{\partial B_L} = 0$$

- And now use $Y_{S,opt} = Y_{in}^*$.

Optimal Power Gain Derivation

- Let's outline the procedure for the optimal power gain. We'll use the power gain G_p and take partials with respect to the load. Let

$$Y_{jk} = m_{jk} + jn_{jk}$$

$$Y_L = G_L + jX_L$$

$$Y_{12}Y_{21} = P + jQ = Le^{j\phi}$$

$$G_p = \frac{|Y_{21}|^2}{D} G_L$$

$$\Re\left(Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right) = m_{11} - \frac{\Re(Y_{12}Y_{21}(Y_L + Y_{22})^*)}{|Y_L + Y_{22}|^2}$$

$$D = m_{11}|Y_L + Y_{22}|^2 - P(G_L + m_{22}) - Q(B_L + n_{22})$$

$$\frac{\partial G_p}{\partial B_L} = 0 = -\frac{|Y_{21}|^2 G_L}{D^2} \frac{\partial D}{\partial B_L}$$

Optimal Load (cont)

- Solving the above equation we arrive at the following solution

$$B_{L,opt} = \frac{Q}{2m_{11}} - n_{22}$$

- In a similar fashion, solving for the optimal load conductance

$$G_{L,opt} = \frac{1}{2m_{11}}\sqrt{(2m_{11}m_{22} - P)^2 - L^2}$$

- If we substitute these values into the equation for G_p (lot's of algebra ...), we arrive at

$$G_{p,max} = \frac{|Y_{21}|^2}{2m_{11}m_{22} - P + \sqrt{(2m_{11}m_{22} - P)^2 - L^2}}$$

Final Solution

- Notice that for the solution to exists, G_L must be a real number. In other words

$$(2m_{11}m_{22} - P)^2 > L^2$$

$$(2m_{11}m_{22} - P) > L$$

$$K = \frac{2m_{11}m_{22} - P}{L} > 1$$

- This factor K plays an important role as we shall show that it also corresponds to an unconditionally stable two-port. We can recast all of the work up to here in terms of K

$$Y_{S,opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{22})}$$

$$Y_{L,opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{11})}$$

$$G_{p,max} = G_{T,max} = G_{a,max} = \frac{Y_{21}}{Y_{12}} \frac{1}{K + \sqrt{K^2 - 1}}$$

Maximum Gain

- The maximum gain is usually written in the following insightful form

$$G_{max} = \frac{Y_{21}}{Y_{12}}(K - \sqrt{K^2 - 1})$$

- For a reciprocal network, such as a passive element, $Y_{12} = Y_{21}$ and thus the maximum gain is given by the second factor

$$G_{r,max} = K - \sqrt{K^2 - 1}$$

- Since $K > 1$, $|G_{r,max}| < 1$. The reciprocal gain factor is known as the efficiency of the reciprocal network.
- The first factor, on the other hand, is a measure of the non-reciprocity.

Unilateral Maximum Gain

- For a unilateral network, the design for maximum gain is trivial. For a bi-conjugate match

$$Y_S = Y_{11}^*$$

$$Y_L = Y_{22}^*$$

$$G_{T,max} = \frac{|Y_{21}|^2}{4m_{11}m_{22}}$$