NODAL ANALYSIS: RECAP

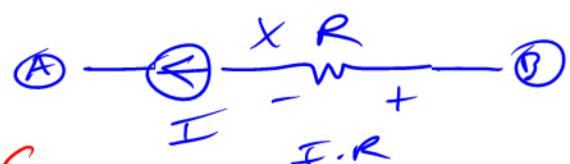
GOAL: TO FIND ALL NODAL VOLTAGES

APPLY: KCL TO A SUBSET OF THE NODES

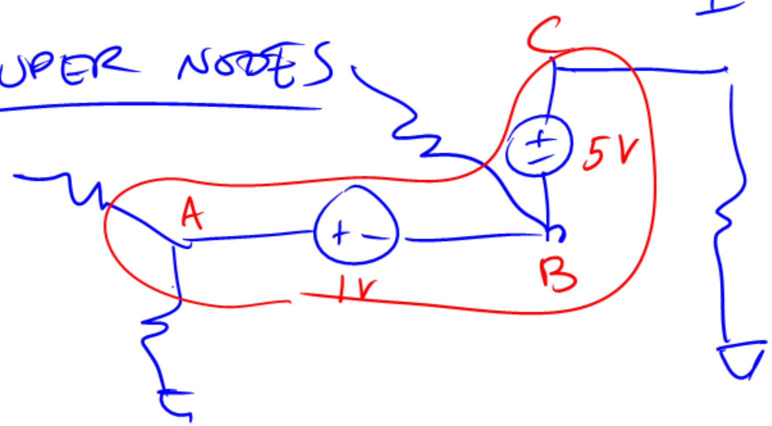
CURRENTS ARE CALC BASED ON POTENTIAL DIFF

1. IF NOT PROVIDED, CHOOSE A REF NODE (GROUND)
2. ELIMINATE TRIVIAL NODES

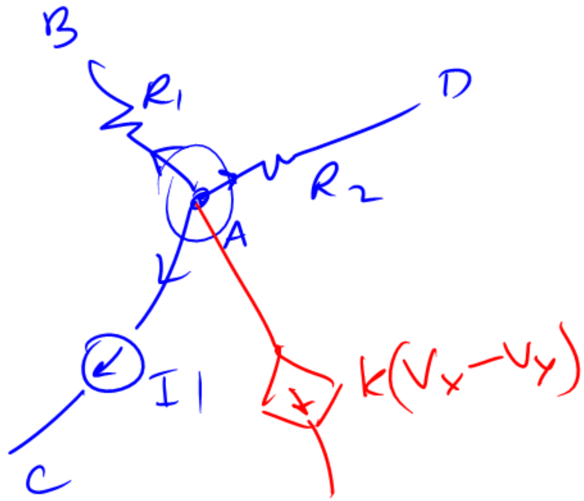
3. GROUP NODES WITH A FIXED POTENTIAL DIFF



⇒ CIRCLE <u>SUPER NODES</u>



4. APPLY KCL TO REMAINING NODES & SUPERNODES



$$\sum I_{(IN)} = \sum I_{(OUT)} \equiv 0$$

$$\frac{V_A - V_B}{R_1} + \frac{V_A - V_D}{R_2} + I_1 + \equiv 0$$

KNOWN

$$+ \underbrace{K\left(V_x - V_y\right)}_{\text{UNKNOWN}}$$

5. FOR DEPENDENT SOURCE, RESOLVE INTO AN EQ INVOLVING <u>ONLY</u> THE NODE VOLTAGES OF INTEREST

EE 42/100
Lecture 6: Network Theorems

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Superposition

- If a circuit is linear, then by the principal of superposition, we can analyze the circuit one source at a time. The total response is the sum of the outputs due to the individual sources.
- This is clear if we re-write the matrix equation as follows

$$b = \begin{pmatrix} I_{s1} \\ I_{s2} \\ I_{s3} + 2I_{s1} \\ V_{s5}/R_2 \end{pmatrix} = \begin{pmatrix} I_{s1} \\ 0 \\ 2I_{s1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_{s2} \\ 0 \\ 0 \end{pmatrix} + \cdots$$

$$Ax = b = b_1 + b_2 + \cdots$$

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\text{MATRIX}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_{\text{UNKNOWN NODE VOLTAGES}} = \begin{bmatrix} \text{SOURCES} \end{bmatrix}$$

- Note that we have partitioned the source terms so that each b_k only contains a single source. Clearly, the solution is given by

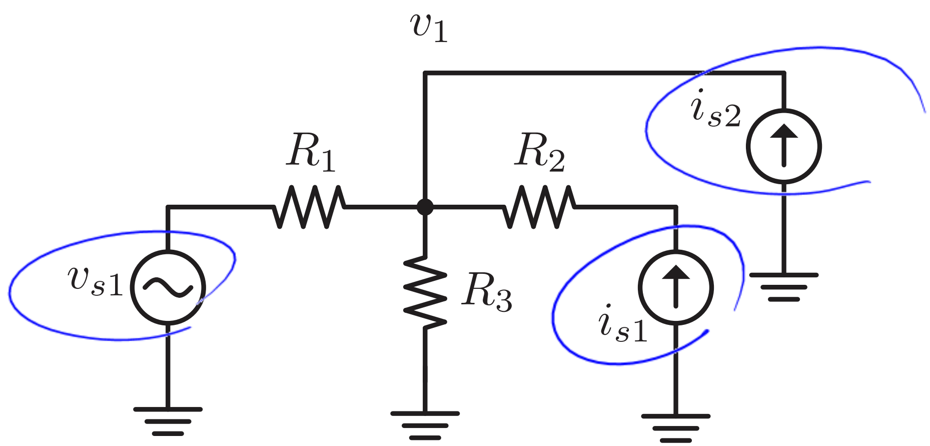
$$Ax = b_1 + b_2 + \cdots \qquad \underbrace{A^{-1}A}_{I}x = A^{-1}b_1 + A^{-1}b_2 + \cdots$$

$$x = \underbrace{A^{-1}b_1}_{} + \underbrace{A^{-1}b_2}_{} + \cdots = x_1 + x_2 + \cdots$$

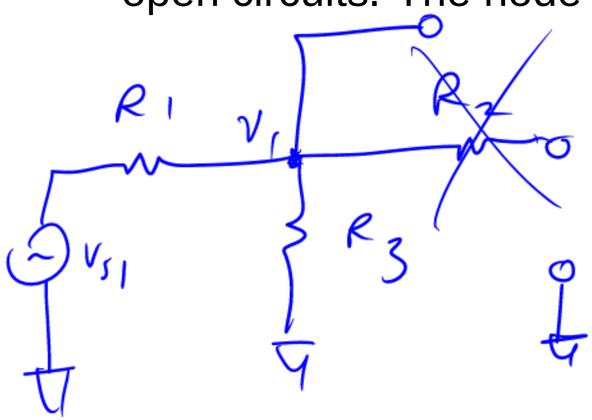
$$x = \quad + $$

- where x_k is the solution with source k turned on and all other sources set to zero. That means that other voltage sources are short circuited (zero voltage) and other current sources are open circuited (zero current).

Example: Analyzing a Circuit with Superposition

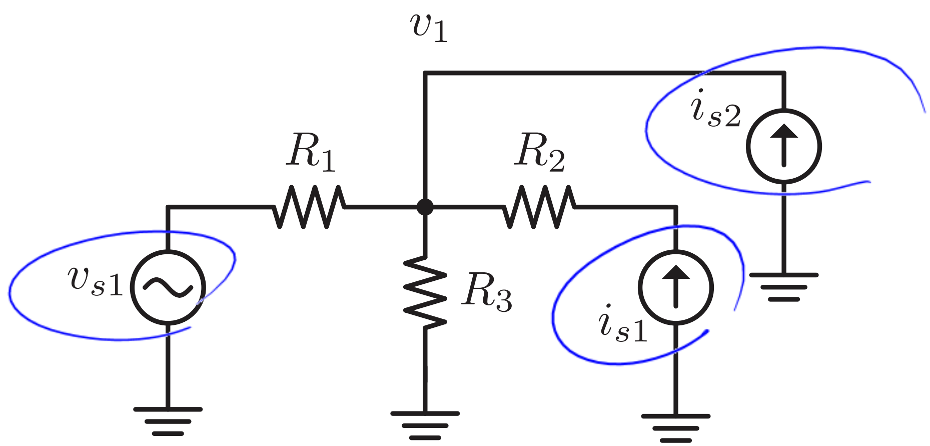


- In this example there are three independent sources. When we analyze the circuit source by source, the circuit is often simple enough that we can solve the equations directly by inspection.
- First turn of i_{s1} and i_{s2}. Zero current means that we replace these sources with open circuits. The node voltage v_1 is therefore by inspection



$$\boxed{v_1^{v_{s1}} = \frac{R_2}{R_1 + R_2} v_{s1}}$$

Example: Analyzing a Circuit with Superposition



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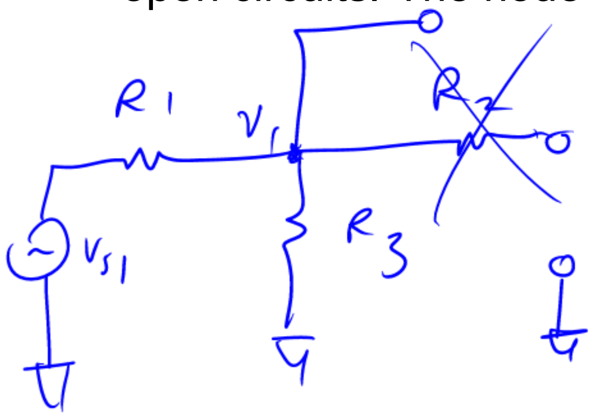


$$\boxed{v_1^{v_{s1}} = \frac{R_2}{R_1 + R_2} v_{s1}}$$

Superposition Example (2)

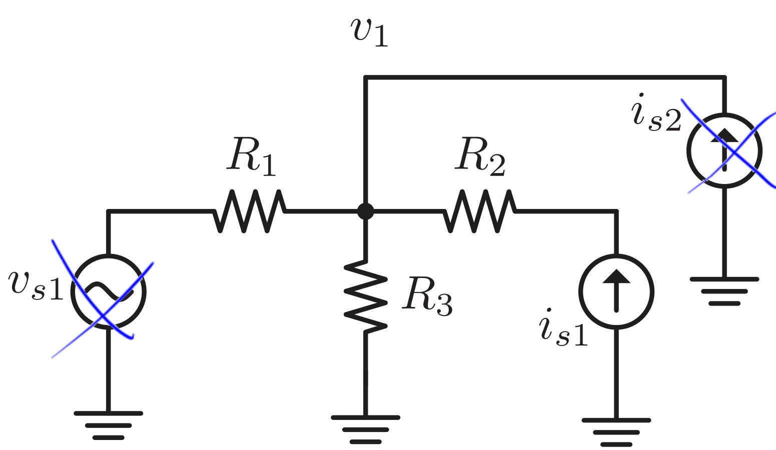
i_{R_1} = CURRENT DIV OF i_{s1}

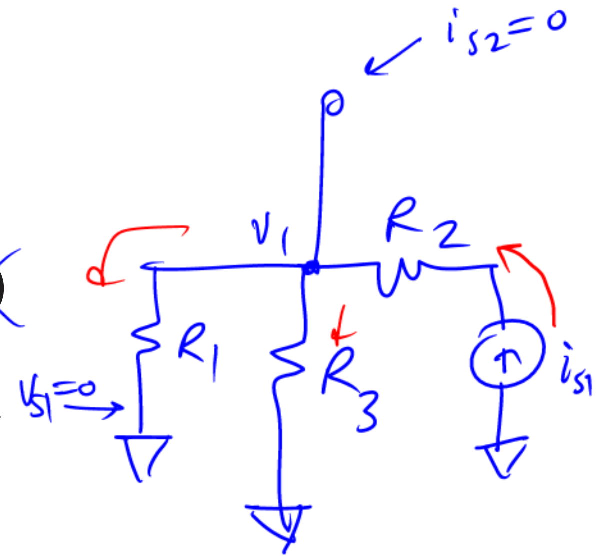
$$= \frac{G_1}{G_1+G_3} \times i_{s1}$$

$$v_1^{(i_{s1})} = i_{R_1} \cdot R_1$$

$$= \frac{1}{G_1+G_3}\, i_{s1}$$

$$= \frac{R_1 R_3}{R_1+R_3}\, i_{s1}$$



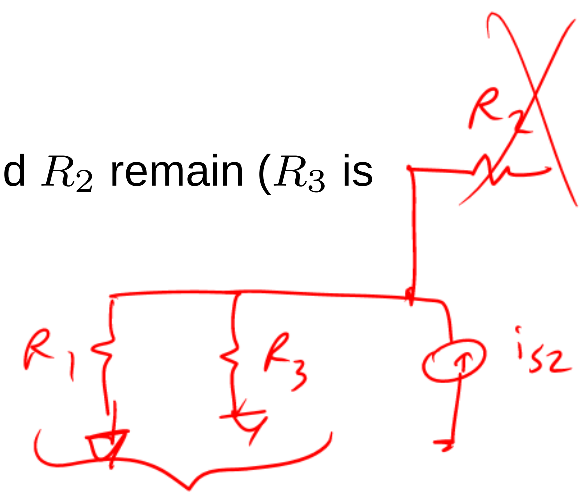


- Next turn off v_{s1} (short circuit) and i_{s2} (open circuit). The current i_{s1} will therefore divide between ~~r_2~~ R_3 and R_1 and establish a voltage at node v_1 (equivalently, it see's a parallel combination of R_1 and ~~R_2~~ R_3

$$\boxed{v_1^{i_{s1}} = \frac{R_2 R_1}{R_1+R_2}\, i_{s1}}$$

- Finally, we turn off all sources except i_{s2}. Now only R_1 and R_2 remain (R_3 is dangling)

$$\boxed{v_1^{i_{s2}} = \frac{R_2 R_1}{R_1+R_2}\, i_{s1}}$$



Superposition Example (3)

- By superposition, the node voltage v_1 is the sum of the three node voltages due to each source

$$v_1 = v_1^{i_{s1}} + v_1^{i_{s2}} + v_1^{v_{s1}} = \frac{R_2}{R_1 + R_2}(v_{s1} + R_1(i_{s1} + i_{s2}))$$

- We can verify the solution by performing KCL directly at node 1

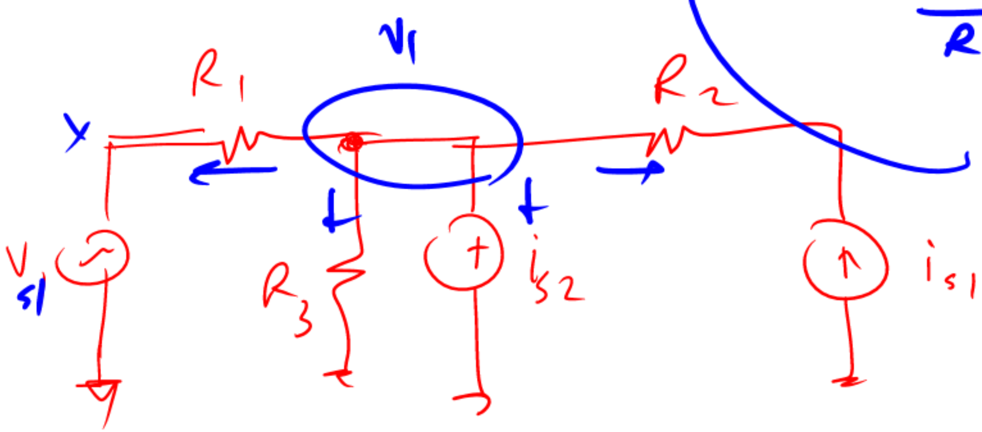
$$(v_1 - v_{s1})G_1 + v_1 G_2 - i_{s1} - i_{s2} = 0$$

$$v_1(G_1 + G_2) = v_{s1}G_1 + i_{s1} + i_{s2}$$

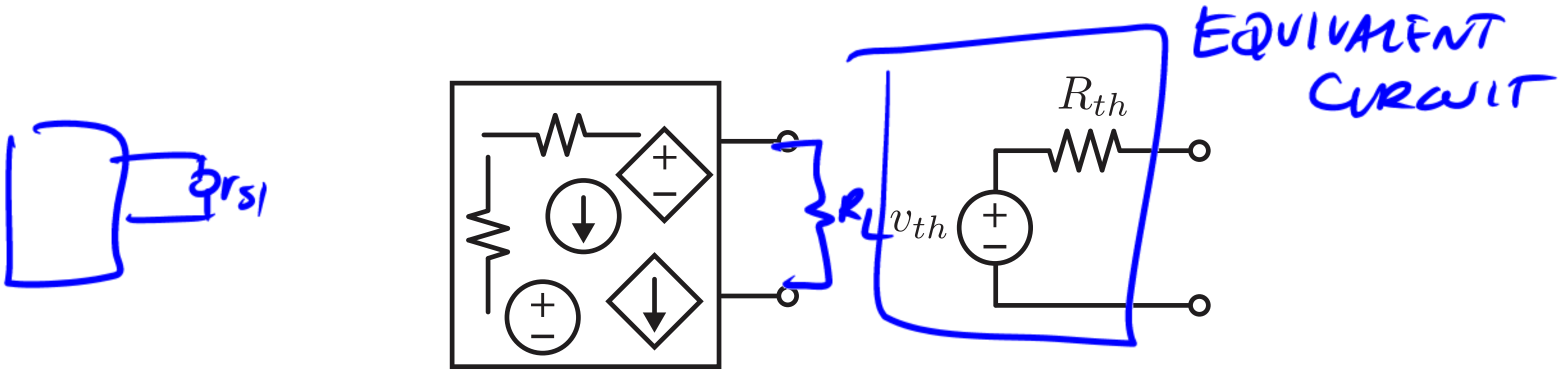
- The answer here is just as fast but we don't have any intuition about the operation of the circuit. We're perhaps more likely to make an algebraic error if it's all math without any thinking.

$$\frac{v_1 - v_{s1}}{R_1} + \frac{v_1}{R_3} - i_{s2} - i_{s1} = 0$$

$$v_1\left(\frac{1}{R_1} + \frac{1}{R_3}\right) = \frac{v_{s1}}{R_1} + i_{s1} + i_{s2}$$



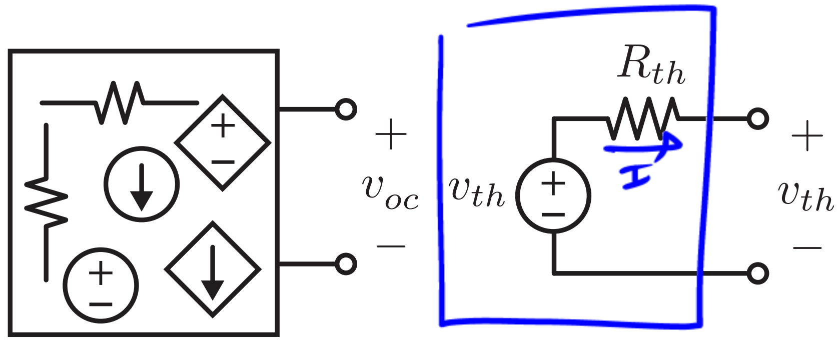
Thevenin Equivalent Circuit



“Black Box”

- A powerful theorem in circuit analysis is the Thevenin equivalent theorem, which let’s us replace a very complex circuit with a simple equivalent circuit model.
- In the black box there can be countless resistors, voltage sources (independent and dependent), current sources (independent and dependent), and yet the *terminal* behavior of the circuit is captured by two elements.
- How can this be? Well, there is a big assumption in that all the resistors are linear (follow Ohm’s Law) and all dependent sources are also linear.
- The equivalent circuit representation is often called a “black box", since the details of the circuitry are hidden.

Thevenin Derivation



“Black Box”

$$V_{oc} = V_{th} - \underbrace{I}_{=0} \cdot R_{th}$$

$$= V_{th}$$

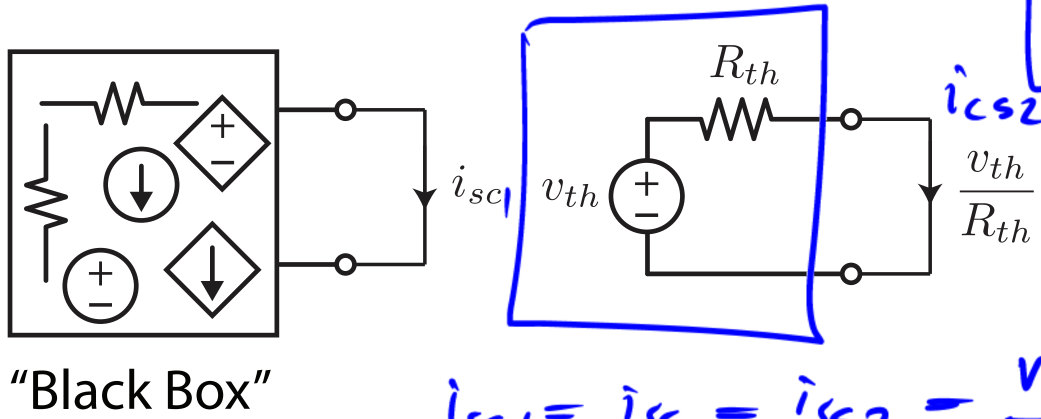
- Since a circuit is linear, then no matter how complicated it is, it’s response to a stimulus at some terminal pair must be linear. It can therefore be represented by a linear equivalent resistor and a a fixed constant source voltage due to the presence of independent sources in the circuit.
- To find the equivalent source value, called the Thevenin voltage source v_{th}, simply observe that the open-circuit voltage of both the “black box" and the original circuit must equal, which means

$$v_{th} = v_{oc}$$

- In other words, open-circuit the original circuit, find its equivalent output voltage at the terminals of interest, and that’s v_{th}

Thevenin Source Resistance

$$\boxed{i_{sc_1} = i_{sc_2}}$$



$$i_{sc_1} = i_{sc} = i_{sc_2} = \frac{v_{th}}{R_{th}}$$

- To find equivalent Thevenin source resistance R_{th}, notice that in order for the terminal behavior of the two circuits to match, the current flow into a load resistor has to be the same for any load value. In particular, take the load as a short circuit.
- The output current of the Thevenin equivalent under a short circuit is given by

$$R_{th} = \frac{v_{th}}{i_{sc}} = \frac{v_{oc}}{i_{sc}}$$

$$\frac{v_{th}}{R_t}$$

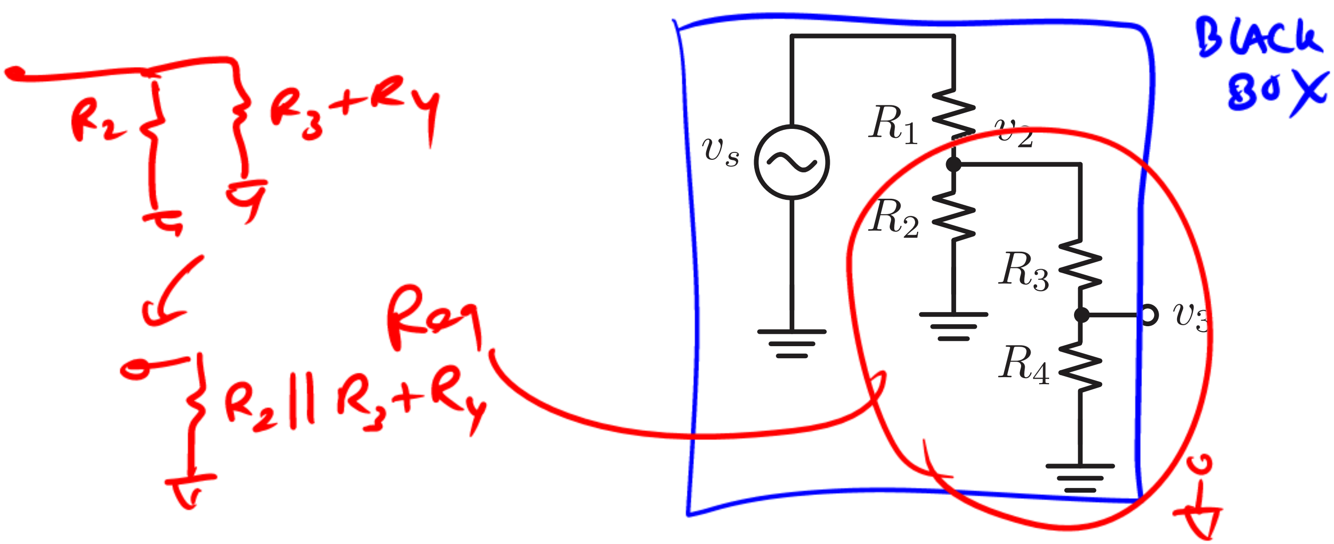
- Equating this to the short-circuit current of the original circuitry, we have

$$i_{sc} = \frac{v_{th}}{R_t}$$

or equivalently

$$\boxed{R_{th} = \frac{v_{th}}{i_{sc}} = \frac{v_{oc}}{i_{sc}}}$$

Thevenin Equivalent Example



V_{oc}, I_{sc} → V_{th}, R_{th}

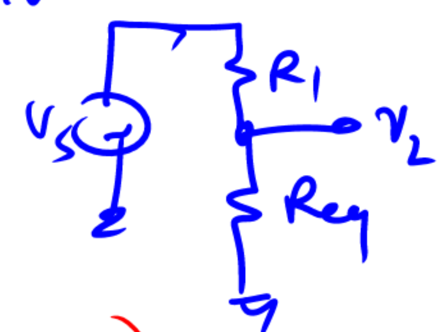
- In the above circuit, we will calculate the Thevenin equivalent circuit.
- We begin by finding the open-circuit voltage. In this case, it's a simple application of the voltage divider.

$$V_{oc} = v_3 = v_2 \frac{R_4}{R_3 + R_4}$$

V_{DIV}

$$v_2 = \frac{R_2||(R_3 + R_4)}{R_1 + R_2||(R_3 + R_4)} v_s$$

V_{DIV}

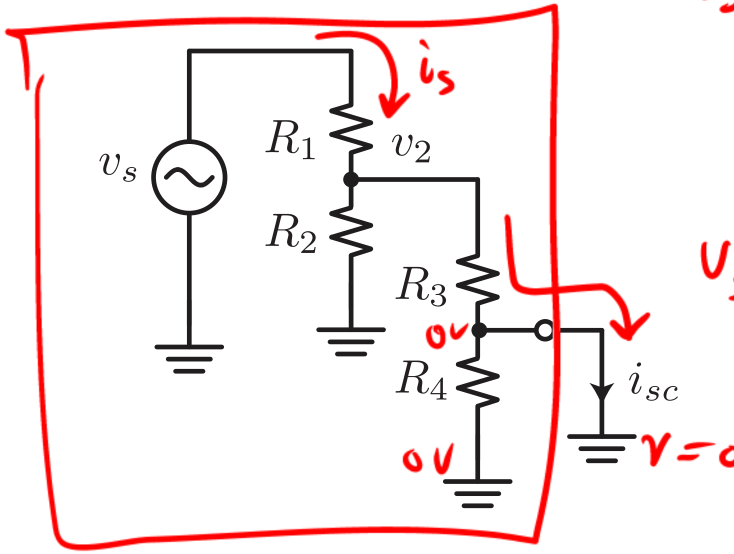


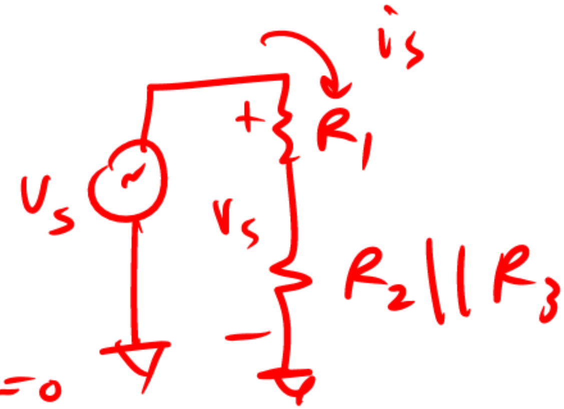
$$v_{oc} = v_3 = \frac{R_4}{R_3 + R_4} \frac{R_2||(R_3 + R_4)}{R_1 + R_2||(R_3 + R_4)} v_s$$

$$V_{oc} = \frac{R_4 R_2 (R_3 + R_4)}{(R_3 + R_4)(R_1(R_2 + R_3 + R_4) + R_2(R_3 + R_4))} V_s$$

Thevenin Example (2)

$i_{sc} = i_{R_3}$





- Next we find the short-circuit current in the original circuit. The resistance loading the source under this condition is given by

$$i_{sc} = i_s \frac{R_2}{R_2 + R_3}$$

CURRENT DIVIDER

$$i_s = \frac{v_s}{R_1 + (R_2||R_3)} = \frac{v_s(R_2 + R_3)}{(R_2 + R_3)R_1 + R_2 R_3}$$

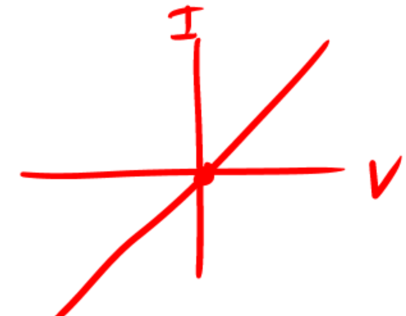
$$\boxed{i_{sc} = \frac{R_2 v_s}{R_1 R_2 + R_1 R_3 + R_2 R_3}}$$

$$\boxed{R_{th} = \frac{v_{oc}}{i_{sc}}}$$

Non-Linear Components and Sources

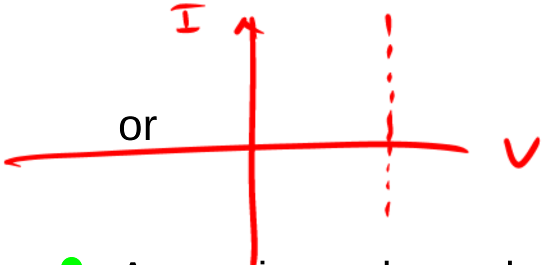
- A non-linear resistor has a non-linear I-V relation. For example

$V = I \cdot R$



$$V = r_1 I + r_2 I^2 + r_3 I^3$$

V-SOURCE

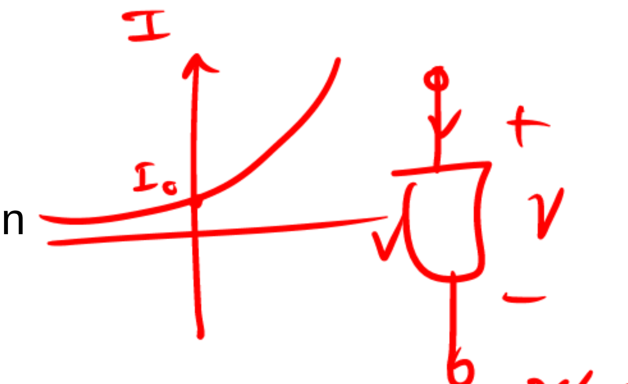


or

$$V = \cos(I \cdot R_x)$$

- A non-linear dependent source is a non-linear function of one or more independent currents/voltages in the circuit. Some examples of non-linear dependent sources are:

$v_k = K i_j^2$	A squarer
$i_k = K i_j^2 + M i_j$	A quadratic function
$v_k = K v_j \cdot v_m$	A multiplier
$v_k = A i_j + B$	Isn't this linear?



$i = I_o e^{v/V_o}$

Linear Relations

- By definition, a linear relation means that if two inputs are applied, then the output of the sum is the sum of the individual outputs. Suppose

$$y = Kx$$

then

$$y_1 = Kx_1 \qquad\qquad y_2 = Kx_2$$

$$y = K(x_1 + x_2) = Kx_1 + Kx_2 = y_1 + y_2$$

- Now suppose $y = Kx + z$. Note that

$$y_1 = Kx_1 + z$$

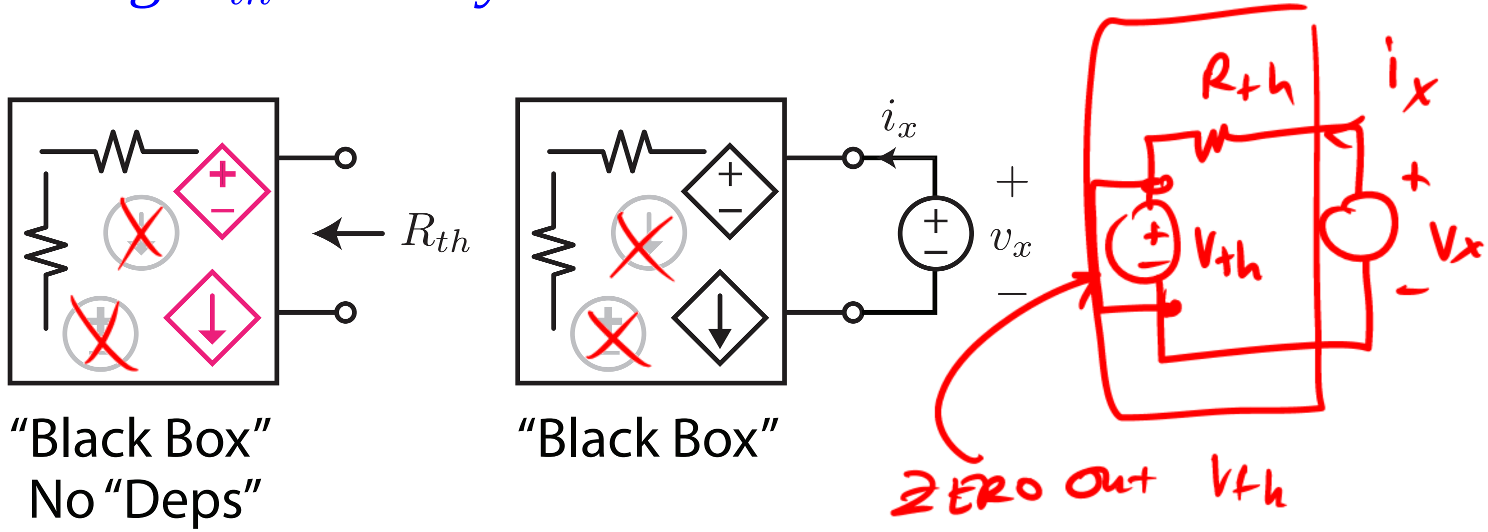
$$y_2 = Kx_2 + z$$

$$y = K(x_1 + x_2) + z = Kx_1 + Kx_2 + z \neq y_1 + y_2$$

Calculating R_{th} Directly

$i_x = \frac{V_x}{R_{th}}$

$R_{th} = \frac{V_x}{i_x}$



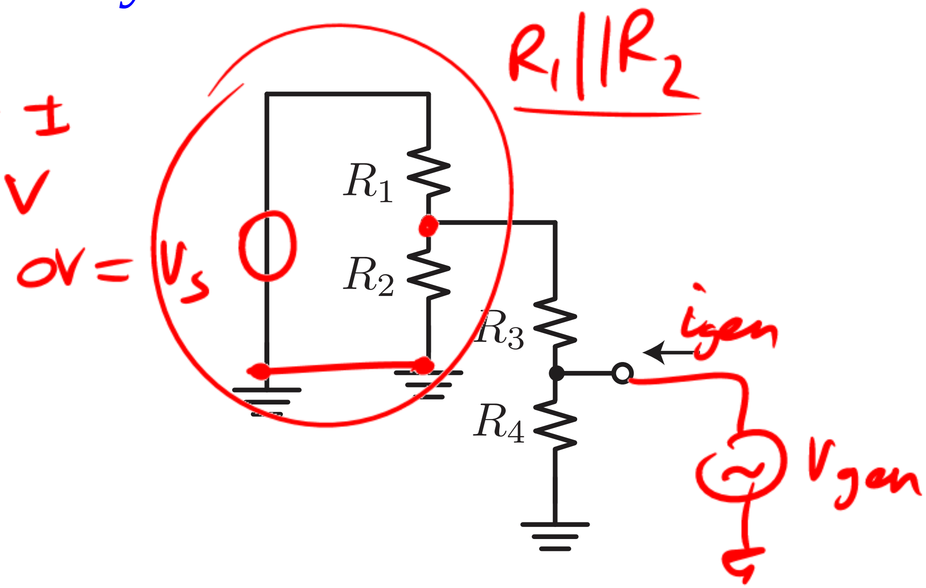
- If there are no dependent sources in the "black box", then you can calculate the Thevenin equivalent circuit *directly* by setting the independent source values to zero. As before, zeroing out sources means shorting voltage sources (zero volts) and open circuiting current sources (zero current). Now just "inspect" to find the equivalent Thevenin resistance.
- For the general linear circuit, another approach to find R_{th} is to probe the circuit with an independent voltage/ current source while zeroing out all internal sources. The current / voltage is monitored and the ratio of the test voltage to test current is the equivalent R_{th}

$$R_{th} = \frac{v_x}{i_x}$$

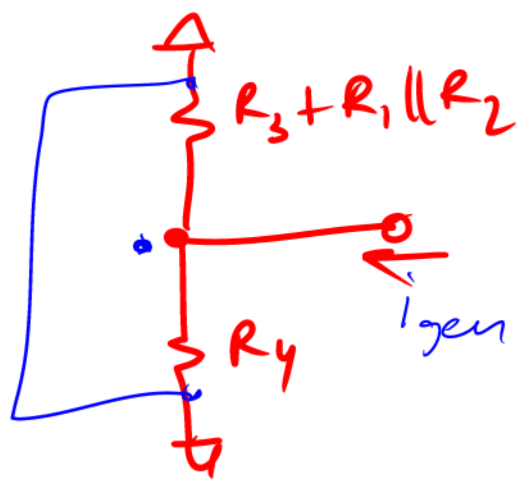
Example: By Direct Method

SERIES = SAME I

SHUNT = SAME V



- Let's redo the same example but now we zero out the voltage source and redraw the circuit. Now we can readily find R_{th} by simply observing that the resistors in of the original circuit are in series/parallel:

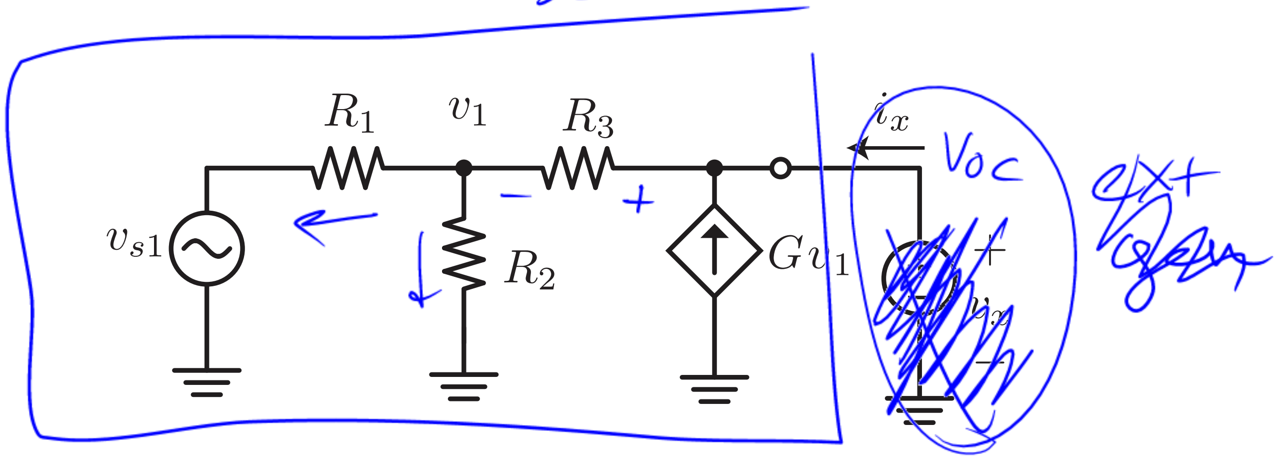


$$R_{th} = R_4||(R_3 + R_1||R_2)$$

$$= \frac{R_4R_3 + R_4R_1||R_2}{R_4 + R_3 + R_1||R_2}$$

$$= \frac{R_4R_3(R_1 + R_2) + R_4R_1R_2}{(R_4 + R_3)(R_1 + R_2) + R_1R_2}$$

Thevenin Example (2)



- Consider the above circuit with two sources. We find the equivalent open circuit voltage by writing KCL at the intermediate node.

$$v_{oc} = v_1 + v_{R_3} = v_1 + (G v_1) \cdot R_3$$

$$= v_1 (1 + G R_3)$$

KCL @ v_1:

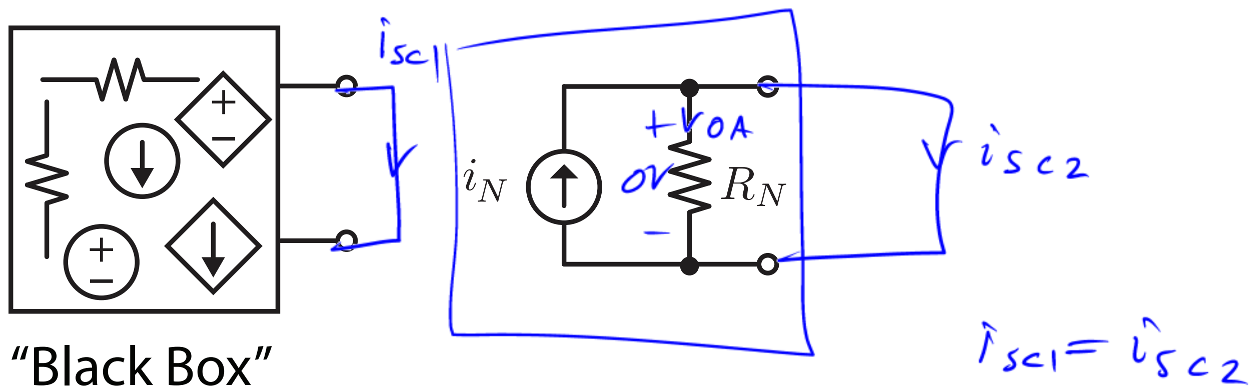
$$\frac{v_1 - v_{s1}}{R_1} + \frac{v_1}{R_2} + (-G v_1) = 0$$

$$v_1 (G_1 + G_2 - G) = v_{s1} \cdot G_1$$

$$v_1 = v_{s1} \frac{G_1}{G_1 + G_2 - G}$$

$$\boxed{V_{th} = V_{s1} \frac{G_1 (1 + G R_3)}{G_1 + G_2 - G}}$$

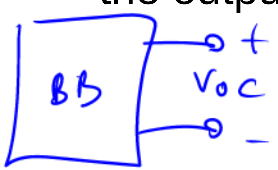
Norton Equivalent Circuit



- In a similar vein, we can also replace a complex circuit with a Norton equivalent circuit, which contains a current source and a shunt source resistance.
- To find the equivalent source value, we find the short circuit current for both the model and the original circuit and note that

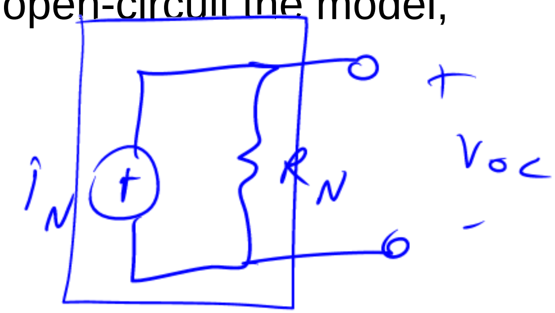
$$\boxed{i_N = i_{sc}}$$

- Similarly, to find the Norton resistance, we note that if we open-circuit the model, the output voltage is given by



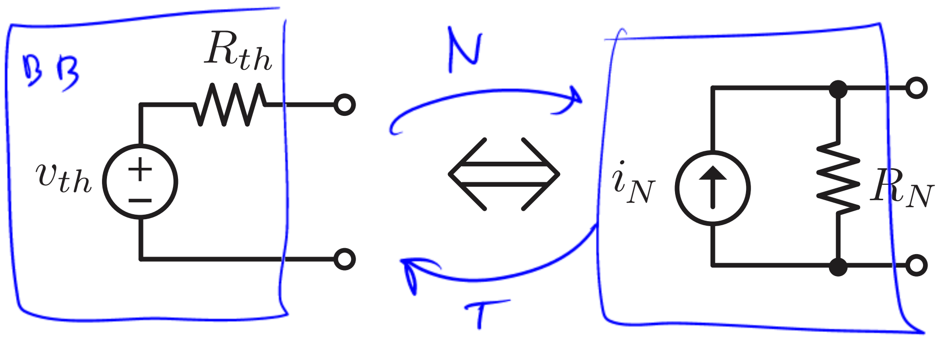
$$v_{oc} = i_N R_N \qquad \text{or} \qquad R_N = \frac{v_{oc}}{i_{sc}}$$

i_N



- This is the same exact equation as before.

Source Transformations



$v_{oc} = i_N \cdot R_N$

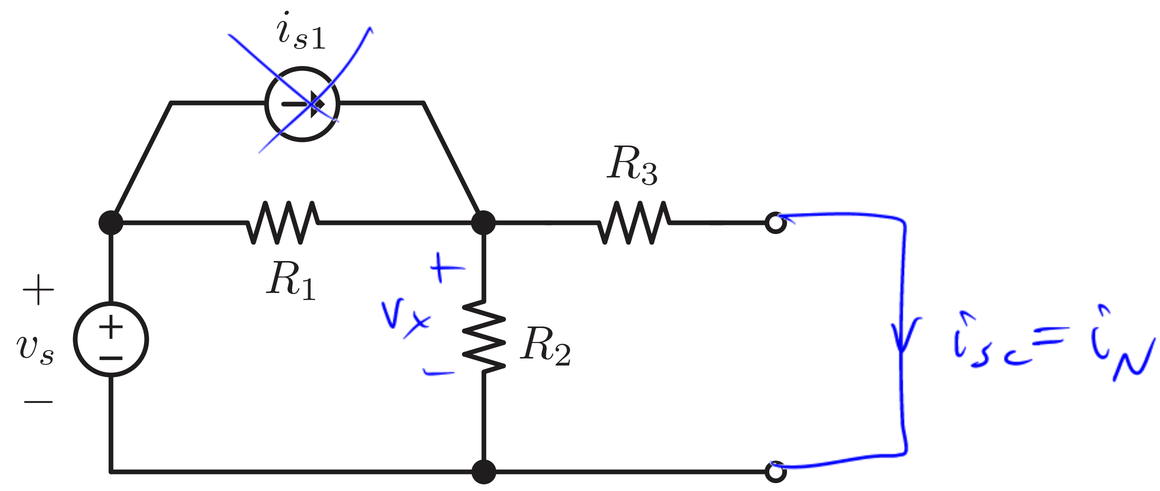
$i_{sc} = i_N = \frac{V_{th}}{R_{th}}$

- A trivial application of Norton or Thevenin's Theorem shows us that we can transform from one representation to the other. For instance, starting from the Thevenin, let's find the Norton. Short circuit the Thevenin to find

$$i_n = i_{sc} = \frac{V_{th}}{R_{th}}$$

and it's trivial to see that $R_N = R_{th}$.

Norton Equivalent Example



- For the above circuit, find the Norton equivalent circuit. We first start out by finding the short-circuit current.

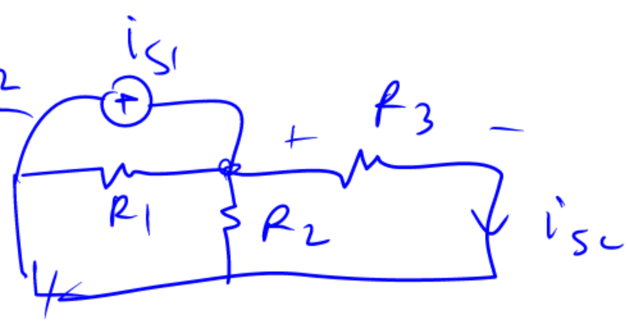
SUPERPOSITION:

SRC #1

$$v_x = \frac{R_2 \| R_3}{R_2 \| R_3 + R_1} \cdot V_s$$

$$i_{sc}^{(1)} = \frac{V_x}{R_3} = \frac{R_2 \| R_3}{R_2 \| R_3 + R_1} \times \frac{V_s}{R_3}$$

SRC #2

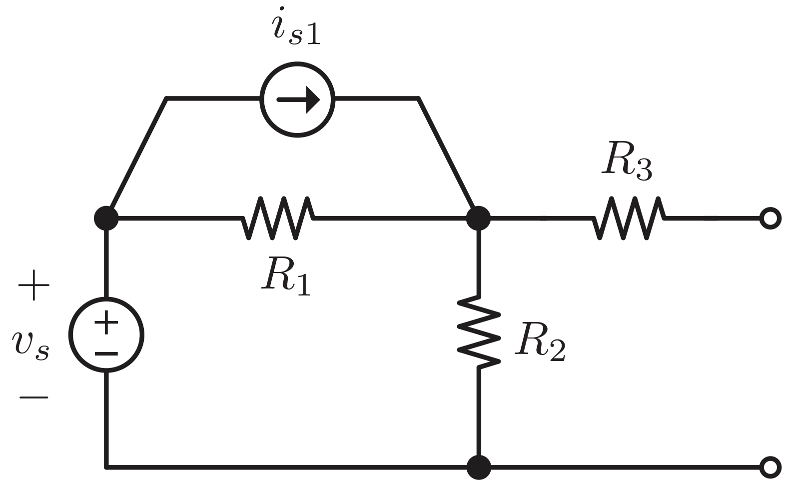


$$v_{R_3}^{(2)} = i_{s1} \times R_1 \| R_2 \| R_3$$

$$i_{R_3} = i_{sc}^{(2)} = i_{s1} \frac{R_1 \| R_2 \| R_3}{R_3}$$

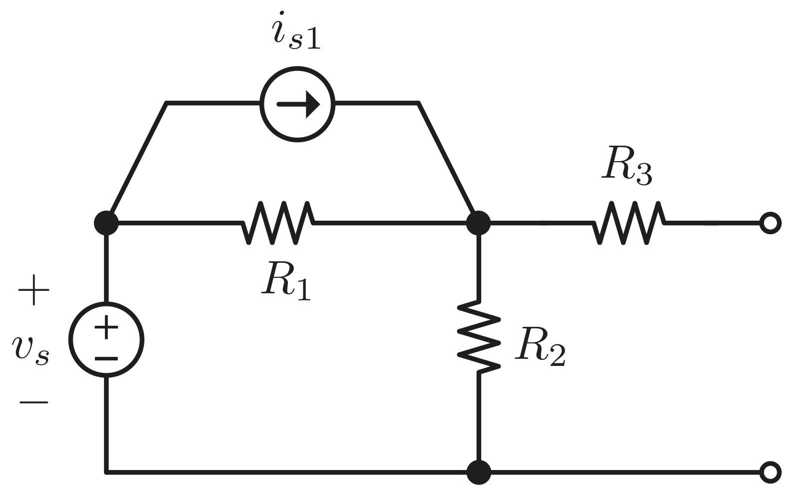
$$i_N = i_{sc}^{(1)} + i_{sc}^{(2)} = \bigcirc + \bigcirc$$

Norton Example (2)



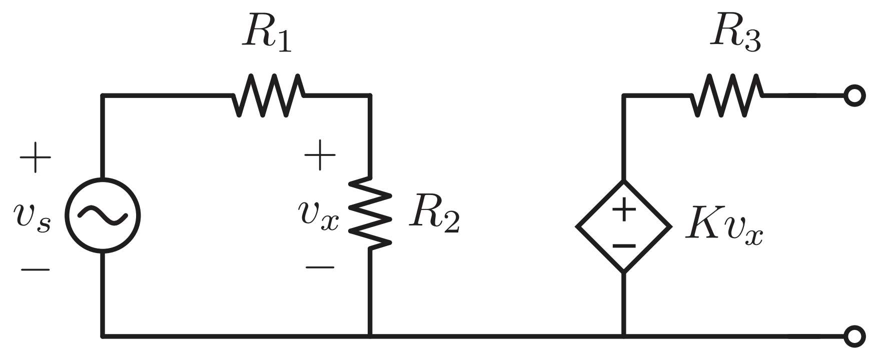
- Next we find the open-circuit voltage.

Norton Example (3)

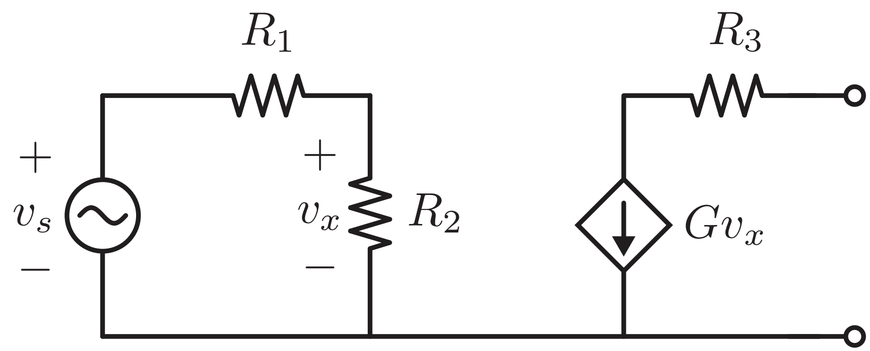


- The same calculation can be done using superposition.

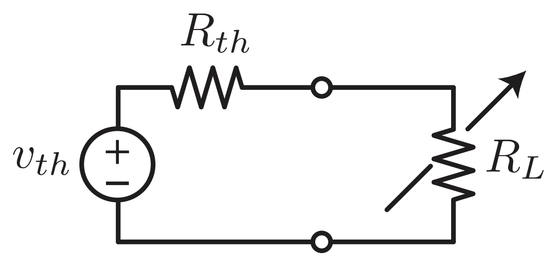
Voltage Amplifier Example



Transconductance Amplifier Example

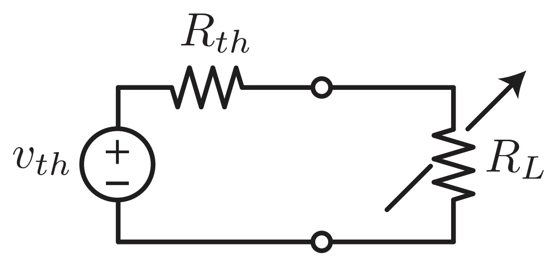


Maximum Power Transfer



- An important question arises in many electrical circuits when we wish to interface one component to another while maximizing the power transfer to the second component.
- A good example is a battery with internal resistance and a motor. What's the best "load" resistance to choose in order to maximize the power transfer?
- Interestingly, if we maximize the current or voltage transfer, the power transfer is exactly zero.

Maximum Power Transfer (cont)



- No matter how complicated the black box source, we can represent it as a Thevenin equivalent circuit, v_{th} and R_{th}.
- The general procedure is to find the power through the load and then to find the optimal load value. We can take the derivative of the load power with respect to the load resistance and set it equal to zero (occurs at only a maximum or minimum). A second derivative test confirms that it's a peak.

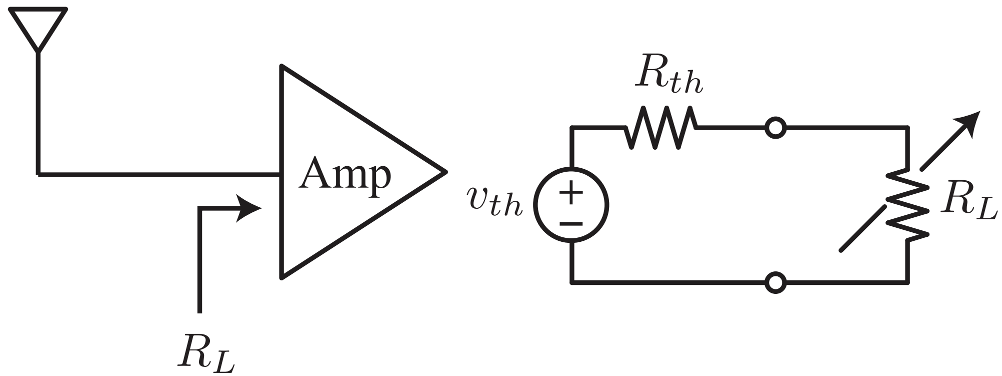
$$P_L = I_L^2 R_L = \left(\frac{v_{th}}{R_L + R_{th}}\right)^2 R_L$$

$$\frac{dP_L}{dR_L} = \left(\frac{v_{th}}{R_L + R_{th}}\right)^2 - 2R_L v_{th}^2 \left(\frac{1}{R_L + R_{th}}\right)^3 = 0$$

$$(R_L + R_{th}) = 2R_L \rightarrow R_L = R_{th}$$

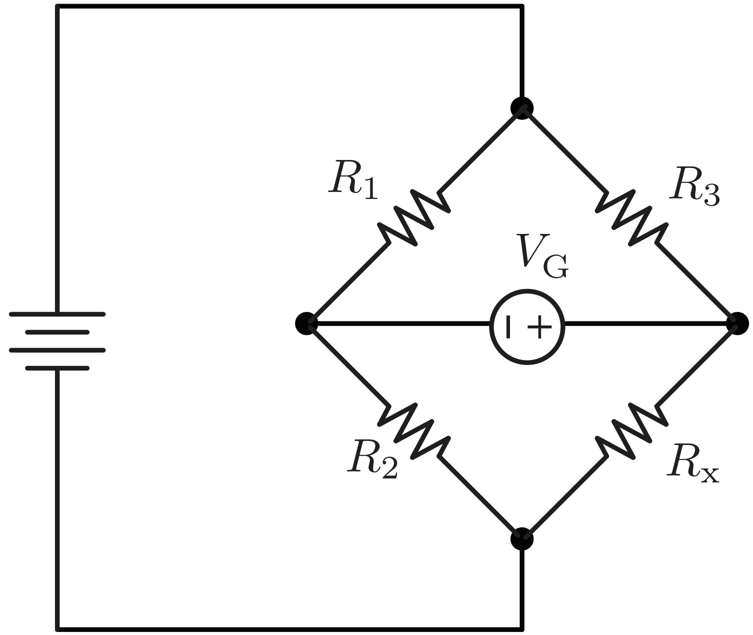
The optimum load resistance is equal to the Thevenin Equivalent Value, or it's *Matched*.

Example of Maximum Power Transfer



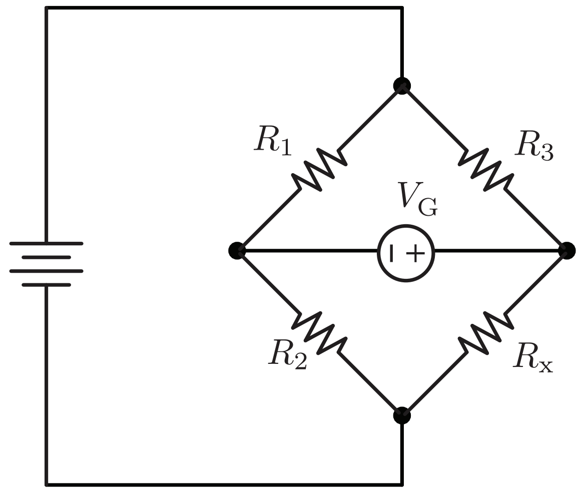
- A common example occurs when designing amplifiers in RF applications. For a receiver, we wish to extract the maximum possible power from the antenna (since the received signal can be very weak). We can represent the antenna by its Thevenin equivalent source resistance R_{th}.
- The load resistance, which is the input of the amplifier in this case, should present a value that is *matched* to the antenna impedance.

The Wheatstone Bridge



- The Wheatstone Bridge (originally invented by Samuel Hunter Christie in 1833 and then popularized by Sir Charles Wheatstone in 1843) is used to measure an unknown resistance. It is highly accurate and only requires an adjustable resistor (or set of well known calibrated resistors) and a method of measuring zero current, such as a galvanometer.
- Since we only need to measure if the current is zero, we can do this very precisely with a galvanometer.
- The Wheatstone bridge is often used with strain gauges, thermocouples, and other transducers.

Wheatstone (2)



- The operation of the Wheatstone Bridge is as follows. One leg of the bridge contains an unknown resistance which we would like to find. The other leg contains an adjustable resistor R_2 (of known value). The goal is to adjust the resistor R_2 until the circuit is "balanced", in other words until no current flows through the galvanometer.
- Under the balanced condition, there is no current I_g, so the current in R_1 and R_2 is the same, say I_1, and the current through R_3 and R_x is also the same, I_3. By KVL, under the balanced condition $V_g = 0$, we have

$$I_3 R_3 = I_1 R_1$$

$$I_3 R_x = I_1 R_2$$

Wheatstone (3)

- Taking the ratio of these two currents, we have

$$\frac{R_3}{R_x} = \frac{R_1}{R_2}$$

- Which means that the unknown resistance if found

$$R_x = \frac{R_3}{R_1} R_2$$

- In practice, we vary R_2 until we achieve balance. In some commercial units, the scale factor R_3/R_1 can be changed as well.

Wheatstone Example

Suppose that $R_1 = 100\Omega$, R_2 can be adjusted from 1Ω steps from 0 to 100Ω, and R_3 can be selected to be 100Ω, $1\,\mathrm{k}\Omega$, $10\,\mathrm{k}\Omega$, or $100\,\mathrm{k}\Omega$. (a) If the bridge is balanced with $R_2 = 36\Omega$ and $R_3 = 10\,\mathrm{k}\Omega$, find R_x. (b) What's the largest R_x that can be measured? (c) For $R_3 = 10\,\mathrm{k}\Omega$, what is the accuracy of the measurement?

- (a) This is a trivial application of the equations

$$R_x = \frac{R_3}{R_1} R_2 = \frac{10,000}{100} 36 = 3600\Omega$$

- (b) The largest R_x is given by

$$R_{x,max} = \frac{R_{3,max}}{R_1} R_{2,max} = \frac{100,000}{100} 100 = 100\,\mathrm{k}\Omega$$

- (c) The accuracy of the measurement is set by the scale factor. As R_2 changes by 1Ω, the value of R_x changes by $\frac{R_3}{R_1}$, which is equal to 100Ω.