EE 42/100
Lecture 16: Sinusoidal Steady-State and Linear Systems

Rev B 3/13/2012 (2:23 PM)

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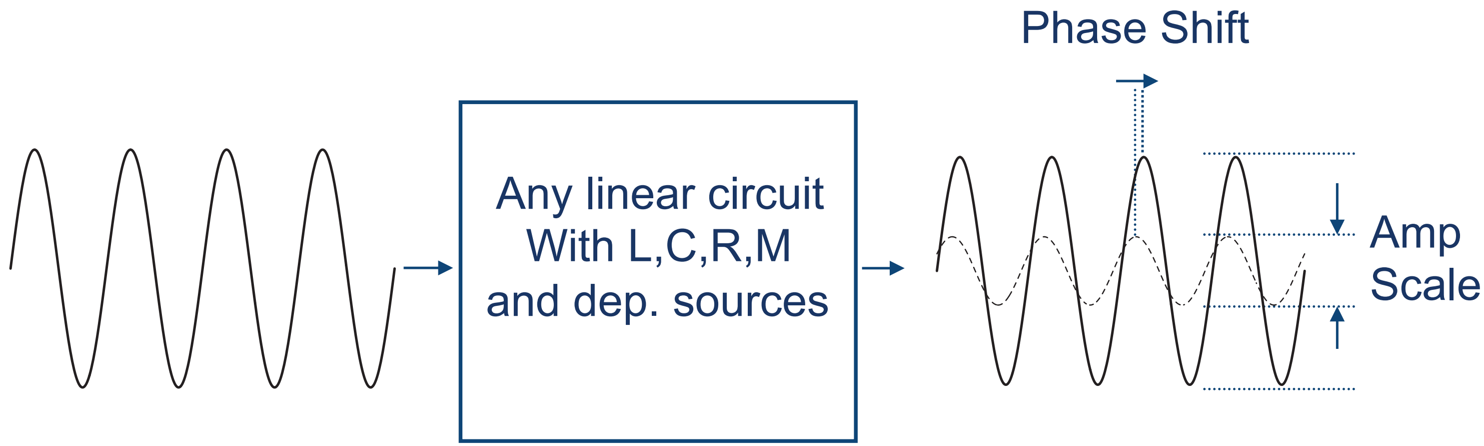
The Magic of Sinusoids



- When a linear, time invariant (LTI) circuit is excited by a sinusoid, it's output is a sinusoid at the *same* frequency. Only the magnitude and phase of the output differ from the input. Sinusoids are very special functions for LTI systems.
- The "Frequency Response" is a characterization of the input-output response for sinusoidal inputs at all frequencies.
- Since most periodic (non-periodic) signals can be decomposed into a summation (integration) of sinusoids via Fourier Series (Transform), the response of a LTI system to virtually any input is characterized by the frequency response of the system.

Example: Low Pass Filter (LPF)

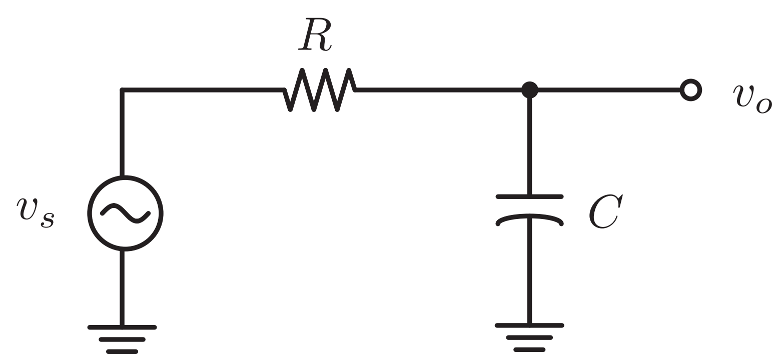
- Input signal: $v_s(t) = V_s \cos(\omega t)$
- We know that in SS the amplitude and phase will change:
  $v_o(t) = \underbrace{K \cdot V_s}_{V_o} \cos(\omega t + \phi)$
- The governing equations are:

$$v_o(t) = v_s(t) - i(t)R$$

$$i(t) = C\frac{dv_o(t)}{dt}$$

$$v_o(t) = v_s(t) - RC\frac{dv_o(t)}{dt}$$

$$v_o(t) = v_s(t) - \tau\frac{dv_o(t)}{dt}$$



LPF the “hard way"

- Plug the known form of the output into the equation and verify that it can satisfy KVL and KCL

$$V_s \cos(\omega t) = V_o \cos(\omega t + \phi) - \tau\omega V_o \sin(\omega t + \phi)$$

Use the following identities:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$V_s \cos(\omega t) = V_o \cos(\omega t)(\cos\phi - \tau\omega \sin\phi) - V_o \sin(\omega t)(\sin\phi + \tau\omega \cos\phi)$$

- Since sine and cosine are linearly independent functions:

$$a_1 \sin(\omega t) + a_2 \cos(\omega t) = 0$$

implies that $a_1 = 0$ and $a_2 = 0$.

LPF: Solving for response

- Applying the linear independence gives us

$$-V_o \sin\phi - V_o \tau\omega \cos\phi = 0$$

this can be converted into

$$\tan\phi = -\tau\omega$$

- The phase response is therefore

$$\phi = -\tan^{-1}\tau\omega$$

Likewise we have

$$V_o \cos\phi - V_o \tau\omega \sin\phi - V_s = 0$$

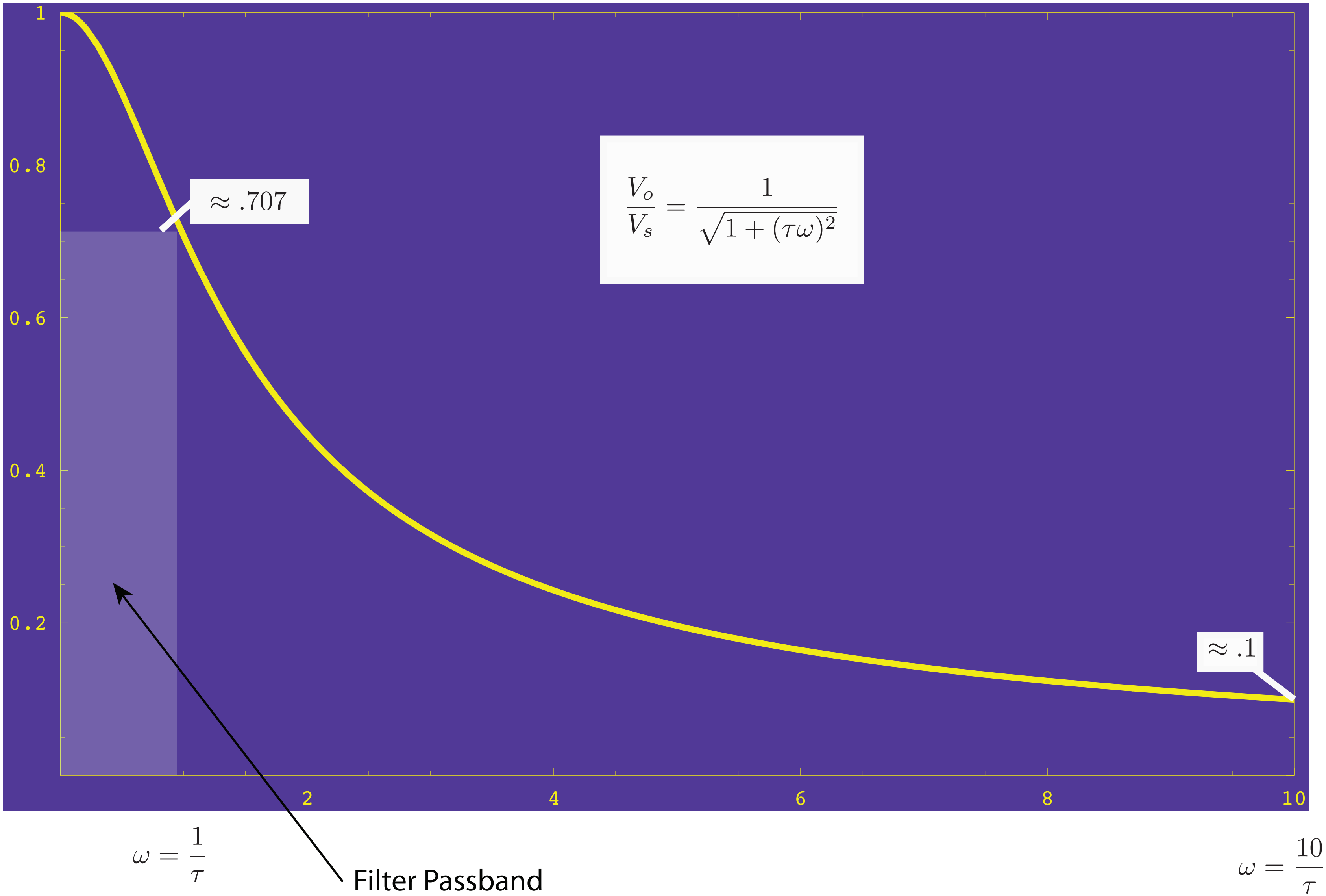
$$V_o(\cos\phi - \tau\omega\sin\phi) = V_s \qquad V_o \cos\phi(1 + (\tau\omega)^2) = V_s$$

$$V_o \cos\phi(1 - \tau\omega\tan\phi) = V_s \qquad V_o(1 + (\tau\omega)^2)^{1/2} = V_s$$

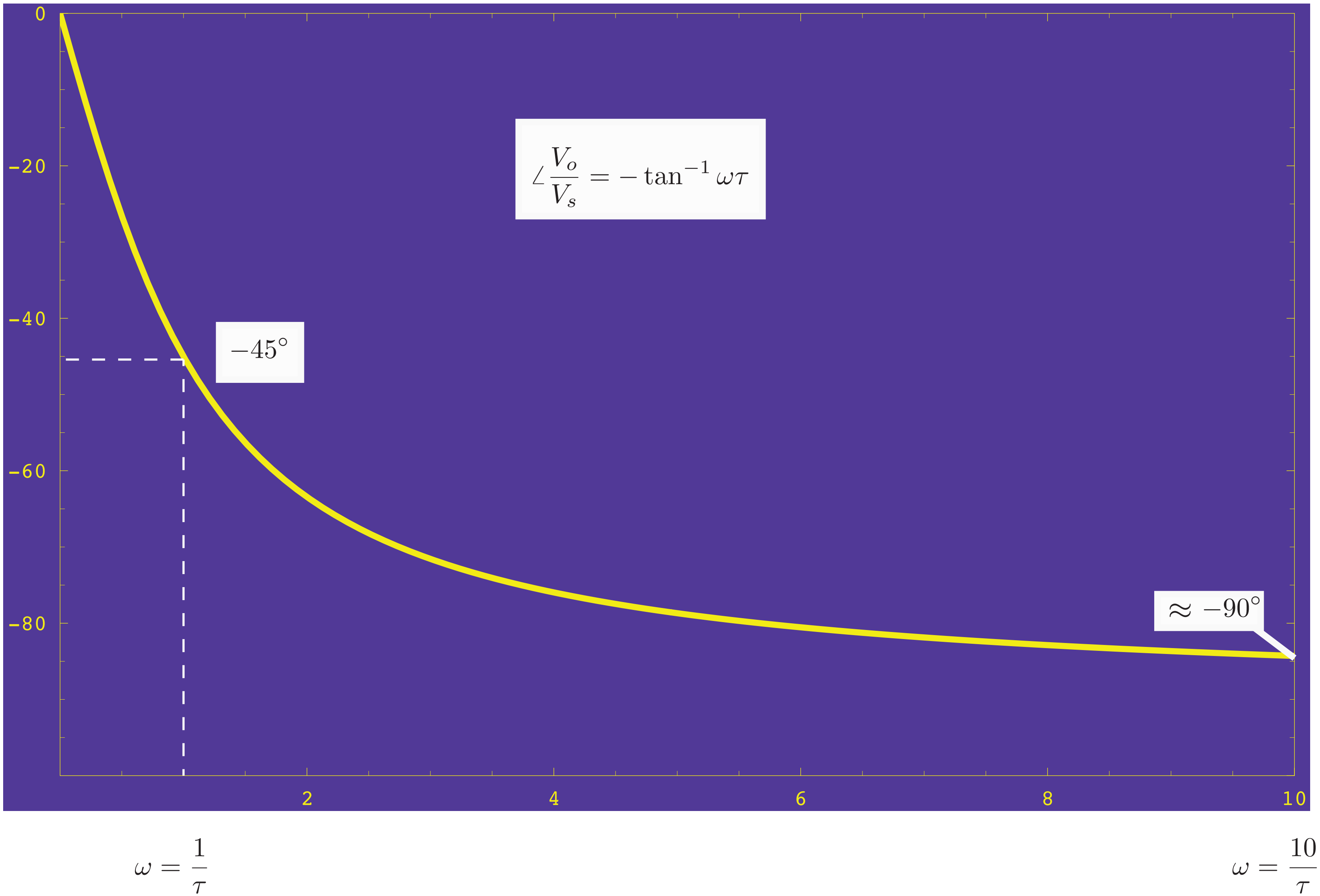
- The amplitude response is therefore given by

$$\frac{V_o}{V_s} = \frac{1}{\sqrt{1 + (\tau\omega)^2}}$$

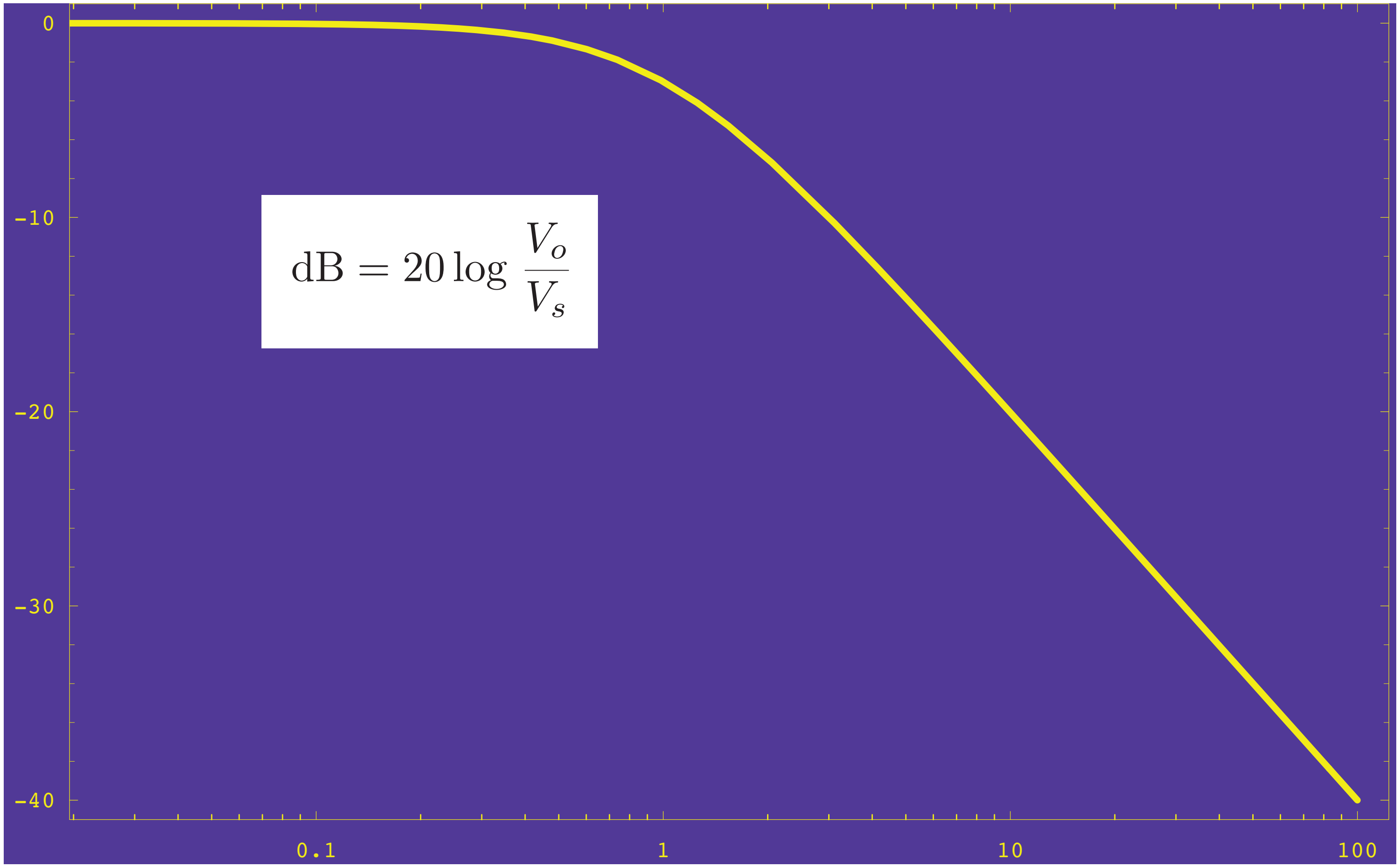
LPF Magnitude Response



LPF Phase Response



dB: Honor the inventor of the phone...



- The LPF response quickly decays to zero
- We can expand range by taking the log of the magnitude response
- dB = deciBel (deci = 10)

Why 20? Power!

- Why multiply log by “20” rather than “10”?
- Power is proportional to voltage squared:

$$\mathrm{dB} = 10\log\left(\frac{V_o}{V_s}\right)^2 = 20\log\left(\frac{V_o}{V_s}\right)$$

- At various frequencies we have:

$$\omega = 1/\tau \rightarrow \left(\frac{V_o}{V_s}\right)_{\mathrm{dB}} = -3\mathrm{dB}$$

$$\omega = 100/\tau \rightarrow \left(\frac{V_o}{V_s}\right)_{\mathrm{dB}} = -40\mathrm{dB}$$

$$\omega = 1000/\tau \rightarrow \left(\frac{V_o}{V_s}\right)_{\mathrm{dB}} = -60\mathrm{dB}$$

- Observe: slope of Signal attenuation is 20 dB/decade in frequency.
- Alternatively, if you double the frequency, the attenuation changes by 6 dB, or 6 dB/octave.

Complex Exponential

- Eulor's Theorem says that

$$e^{jx} = \cos x + j \sin x$$

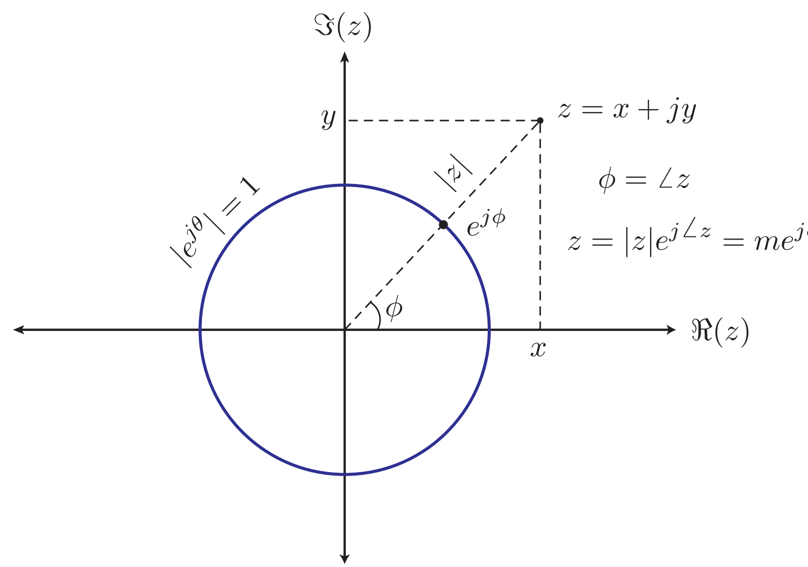
- This can be derived by expanding each term in a power series.
- If take the magnitude of this quantity, it's unity

$$|e^{jx}| = \sqrt{\cos^2 x + \sin^2 x} = 1$$

- That means that $e^{j\phi}$ is a point on the unit circle at an angle of ϕ from the x-axis.

Any complex number z, expressed as have a real and imaginary part $z = x + jy$, can also be interpreted as having a magnitude and a phase. The magnitude $|z| = \sqrt{x^2 + y^2}$ and the phase $\phi = \angle z = \tan^{-1} y/x$ can be combined using the complex exponential

$$x + jy = |z|e^{j\phi}$$



Eulor's Theorem and The Circle

- This implies that $e^{j\omega t}$ is nothing but a point rotating on a circle on the complex plane. The real part and imaginary parts are just projections of the circle, which by trigonometry we know equal the cosine and sine functions.
- We can also express $\cos$ and $\sin$ in terms of e as follows

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

- To see an animation of these equations, click below:

$e^{j\omega t}$ rotating around circle[a] $e^{-j\omega t}$ rotating around circle[b]
$e^{j\omega t} + e^{-j\omega t}$ oscillates on the real axis[c]

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[a] http://rfic.eecs.berkeley.edu/ee100/pdf/exp1.gif

[b] http://rfic.eecs.berkeley.edu/ee100/pdf/exp2.gif

[c] http://rfic.eecs.berkeley.edu/ee100/pdf/exp3.gif

Why introduce complex numbers?

- They actually make things easier!
- Integration and differentiation are trivial with complex exponentials:

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}$$

$$\int e^{j\omega x}dx = \frac{1}{j\omega}e^{j\omega t}$$

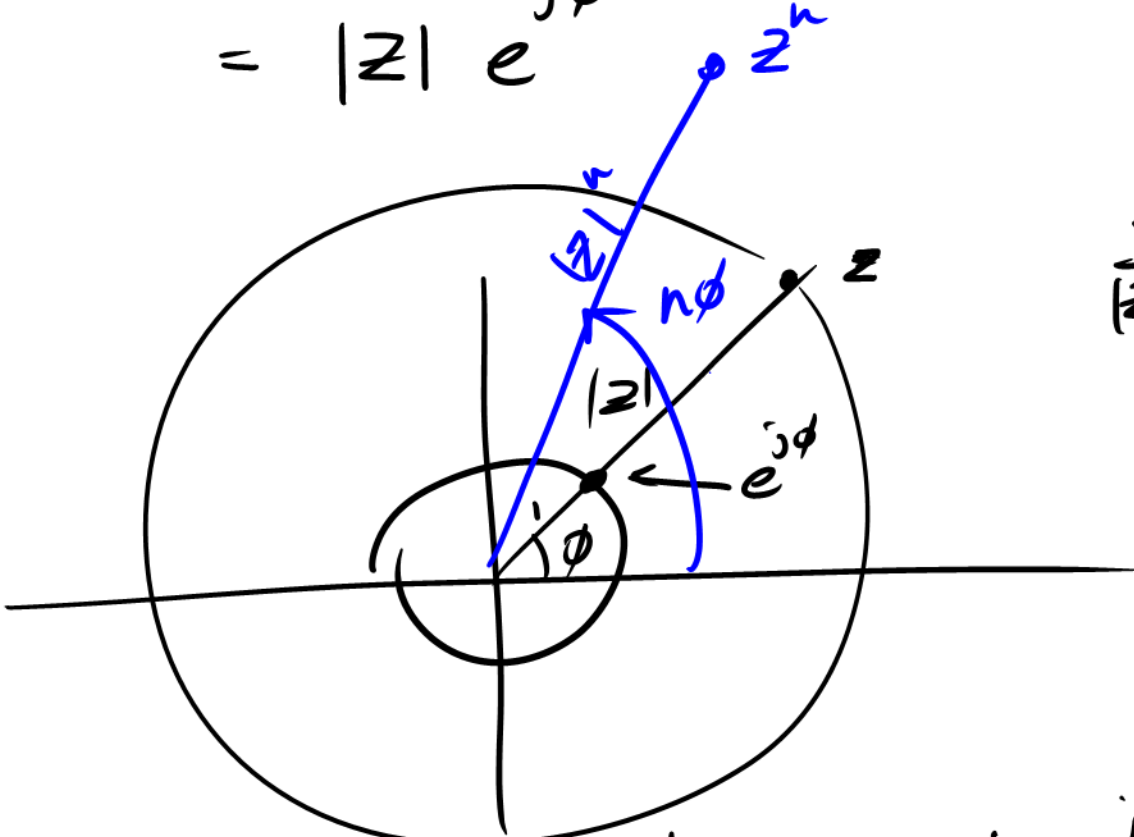
- Any ODE is now trivial algebraic manipulations ... in fact, we'll show that you don't even need to directly derive the ODE by using phasors (phasor is essentially a shorthand notation for a complex number)
- The key is to observe that the current/voltage relation for any element can be derived for complex exponential excitation

COMPLEX NUMBER CONVENT

$$z = x + jy = |z| \angle \phi$$

$$\phi = \angle z$$

$$= |z| e^{j\phi}$$

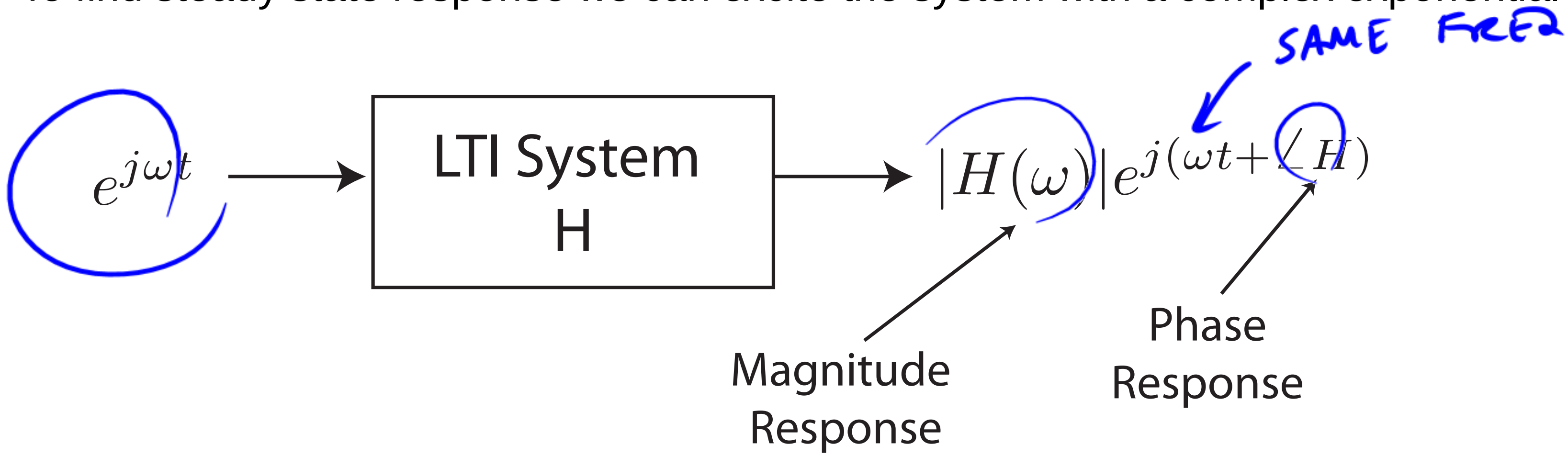


$$\frac{z}{|z|} = e^{j\phi}$$

$$z^n = \left(|z| e^{j\phi}\right)^n = |z|^n e^{jn\phi}$$

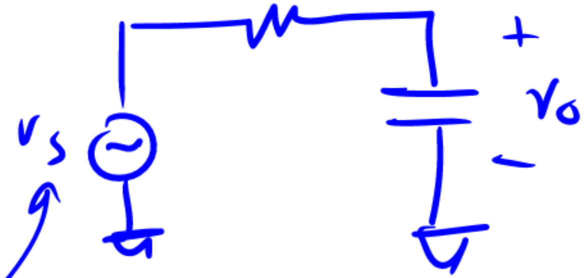
Complex Exponential is Powerful

- To find steady state response we can excite the system with a complex exponential



- At any frequency, the system response is characterized by a single complex number H:
  - The magnitude response is given by $|H(\omega)|$
  - The phase response is given by $\angle H$
- We see that the complex exponential is an “eigenfunction" of the system. It is used to probe the system.
- Since a sinusoid is a sum of complex exponentials (and because of linearity!), we can also probe a system by applying a real sinusoidal input.

LPF Example: The "soft way"



- Let's excite the system with a complex exponential

$$v_s(t) = v_o(t) + \tau \frac{dv_o}{dt}$$

$$v_s(t) = V_s e^{j\omega t}$$

$$v_o(t) = |V_o| e^{j(\omega t + \phi)} = V_o e^{j\omega t} = \underbrace{|V_o| e^{j\phi}}_{V_o} \cdot e^{j\omega t}$$

V_o complex number

- Now substitute into the original equation

$$V_s \cancel{e^{j\omega t}} = \underbrace{V_o \cancel{e^{j\omega t}}}_{v_o(t)} + \underbrace{\tau \cdot j\omega \cdot V_o \cancel{e^{j\omega t}}}_{\tau \frac{dv_o}{dt}}$$

divide out the non-zero common factors

$$\boxed{V_s = V_o(1 + j\omega \cdot \tau)}$$

$$\boxed{\frac{V_o}{V_s} = \frac{1}{1 + j\omega \cdot \tau}}$$

Magnitude and Phase Response

- The system is characterized by the complex function

$$H(\omega) = \frac{V_o}{V_s} = \frac{1}{1 + j\omega \cdot \tau}$$

- The magnitude and phase response match our previous calculation

$$|H(\omega)| = \left|\frac{V_o}{V_s}\right| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle H(\omega) = -\tan^{-1}\omega\tau$$

$$|H(\omega)| = \left|\frac{V_o}{V_s}\right| = \left|\frac{1}{1 + j\omega\tau}\right| = \frac{|1|}{|1 + j\omega\tau|} = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

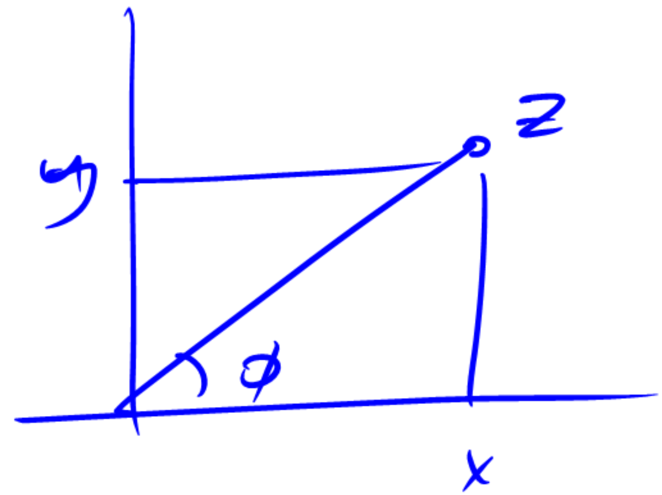
$$\angle H = \angle \frac{1}{1 + j\omega\tau} = \angle 1 - \angle(1 + j\omega\tau)$$

$$= 0 - \tan^{-1}\omega\tau = -\tan^{-1}\omega\tau$$

$$Z = x + jy$$

$$|Z| = \sqrt{x^2 + y^2}$$

$$\phi = \angle Z = \tan^{-1} \frac{y}{x}$$



$$Z = \frac{a}{b}$$

$$|Z| = \frac{|a|}{|b|}$$

$$\angle Z = \angle a - \angle b$$

$$Z = \frac{a}{b} = \frac{|a|\, e^{j\phi_a}}{|b|\, e^{j\phi_b}} = \frac{|a|}{|b|} e^{j(\phi_a - \phi_b)}$$

$$|Z| = \frac{|a|}{|b|}$$

$$\angle Z = \phi_a - \phi_b$$

Why did it work?

- The system is linear:

$$\Re[y] = \mathbf{L}(\Re[x]) = \Re[\mathbf{L}(x)]$$

- If we excite system with a sinusoid:

$$v_s(t) = V_s \cos(\omega t) = V_s \Re[e^{j\omega t}]$$

- If we push the complex exponential through the system first and take the real part of the output, then that's the "real" sinusoidal response

$$v_o(t) = V_o \cos(\omega t + \phi) = V_o \Re[e^{j(\omega t+\phi)}]$$

$$y = \mathcal{L}\, x$$

$$Re(y) = Re(\mathcal{L}\, x) = \mathcal{L} \cdot Re(x)$$

$$Re(e^{j\omega t+\phi} \cdot |V_o|) = \mathcal{L}\, Re(e^{j\omega t} V_s)$$

$$|V_o| \cos(\omega t + \phi) = \mathcal{L}(\cos \omega t \cdot V_s)$$

And yet another perspective

$H = |H| e^{j\angle H}$

$H \cdot e^{j\omega t}$

$e^{j\omega t} \longrightarrow$ [LTI System H] $\longrightarrow |H(\omega)|e^{j(\omega t + \angle H)}$

positive freq

$e^{-j\omega t} \longrightarrow$ [LTI System H] $\longrightarrow |H(-\omega)|e^{j(-\omega t + \angle H(-\omega))}$

negative freq

$\boxed{cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}} \longrightarrow$ [LTI System H] $\longrightarrow \dfrac{H(\omega)e^{j\omega t} + H(-\omega)e^{-j\omega t}}{2}$

complex number

- Another way to see this is to observe the system is linear so that

$$y = \mathbf{L}(x_1 + x_2) = \mathbf{L}(x_1) + \mathbf{L}(x_2)$$

- To find the response to a sinusoid, we can find the response to $e^{j\omega t}$ and $e^{-j\omega t}$ and sum the results.

Another perspective (cont.)

$1+jx$

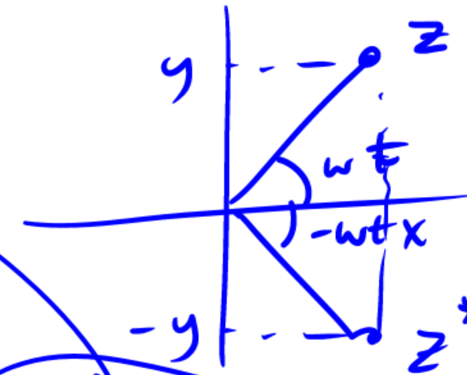
$2+jy = 3$

$x=-y$

- Since the input is real, the output has to be real:

$(e^{j\omega t})^*$

$$y(t) = \frac{H(\omega)e^{j\omega t} + H(-\omega)e^{-j\omega t}}{2}$$



- That means the second term is the conjugate of the first:

$$|H(-\omega)| = |H(\omega)|$$

(even function)

$$\angle H(-\omega) = -\angle H(\omega) = -\phi$$

(odd function)

- Therefore the output is:

$$y(t) = \frac{|H(\omega)|e^{j\phi}e^{j\omega t} + |H(\omega)|e^{-j\phi}e^{-j\omega t}}{2}$$

$e^{j\theta} + e^{-j\theta}$

$$y(t) = \frac{|H(\omega)|}{2}\left(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right)$$

$$= |H(\omega)|\cos(\omega t + \phi)$$

"Proof" for Linear Systems

- For an arbitrary linear circuit (L,C,R,M, and linear dependent sources), decompose it into linear sub-operators, like multiplication by constants, time derivatives, or integrals:

$$y = \mathbf{L}(x) = ax + b_1 \frac{d}{dt} x + b_2 \frac{d^2}{dt^2} x + \cdots + \int x + \iint x + \cdots$$

- For a complex exponential input this simplifies to:

$$y = \mathbf{L}(e^{j\omega t}) = ae^{j\omega t} + b_1 \frac{d}{dt} e^{j\omega t} + b_2 \frac{d^2}{dt^2} e^{j\omega t} + \cdots + \int e^{j\omega t} + \iint e^{j\omega t} + \cdots$$

$$y = ae^{j\omega t} + b_1 j\omega e^{j\omega t} + b_2 (j\omega)^2 e^{j\omega t} + \cdots + \frac{e^{j\omega t}}{j\omega} + \frac{e^{j\omega t}}{(j\omega)^2} + + \cdots$$

- Note that every term is of the form $e^{j\omega t}$ times a constant, which when grouped together gives

$$y = e^{j\omega t} \underbrace{\left(a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{1}{j\omega} + \frac{1}{(j\omega)^2} + + \cdots \right)}_{H(\omega)}$$

"Proof" (cont.)

- The amplitude of the output is the magnitude of the complex number and the phase of the output is the phase of the complex number

$$y = e^{j\omega t}\underbrace{\left(a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{1}{j\omega} + \frac{1}{(j\omega)^2} + + \cdots\right)}_{H}$$

$$H = |H| e^{j\angle H}$$

or

$$y = e^{j\omega t}|H(\omega)|e^{j\angle H(\omega)} = |H| e^{j(\omega t + \angle H)}$$

$$\Re[y] = |H(\omega)| \cos(\omega t + \angle H(\omega))$$

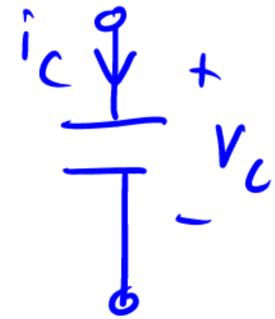
Phasors

- With our new confidence in complex numbers, we go full steam ahead and work directly with them ... we can even drop the time factor $e^{j\omega t}$ since it will cancel out of the equations.
- When a circuit is excited by a sinusoidal input, we assign the source a “phasor” with magnitude and phase equal to the source. Then we analyze the circuit assuming all voltages/currents are sinusoidal, which can be represented by the shorthand phasor form.
- Excite system with a phasor: $\widetilde{V_1} = V_1 e^{j\phi_1} = V_1 \angle \phi_1$
- Response will also be phasor: $\widetilde{V_2} = V_2 e^{j\phi_2} = V_2 \angle \phi_2$
- We see that a phasor is nothing more than a complex number which represents the complex exponential form of the voltage/current where we divide out the time dependence.

$$v(t) = V_0 \cos(\omega t + \phi) = \mathrm{Re}\left(V_0 e^{j(\omega t + \phi)} \right)$$

$$= \mathrm{Re}\Big(\underbrace{V_0 e^{j\phi}}_{\text{PHASOR}} \cdot e^{j\omega t} \Big)$$

Capacitor I-V Phasor Relation



- Find the phasor relation for current and voltage in a capacitor:

$$i_c = C\frac{dv_c(t)}{dt}$$

- Assume the current/voltage can be written in a complex exponential form

$$i_c(t) = I_c e^{j\omega t}$$

$$v_c(t) = V_c e^{j\omega t}$$

- Substitute in the governing equation

$$I_c e^{j\omega t} = C\frac{d}{dt}V_c e^{j\omega t} = j\omega C V_c e^{j\omega t}$$

const

$$I_c \cancel{e^{j\omega t}} = j\omega C V_c \cancel{e^{j\omega t}}$$

or directly in Phasor form

$$\boxed{I_c = j\omega C V_c}$$

$V = I \cdot R \longrightarrow V = I \cdot Z$

$\Rightarrow$

$$\frac{1}{j\omega C} = Z$$

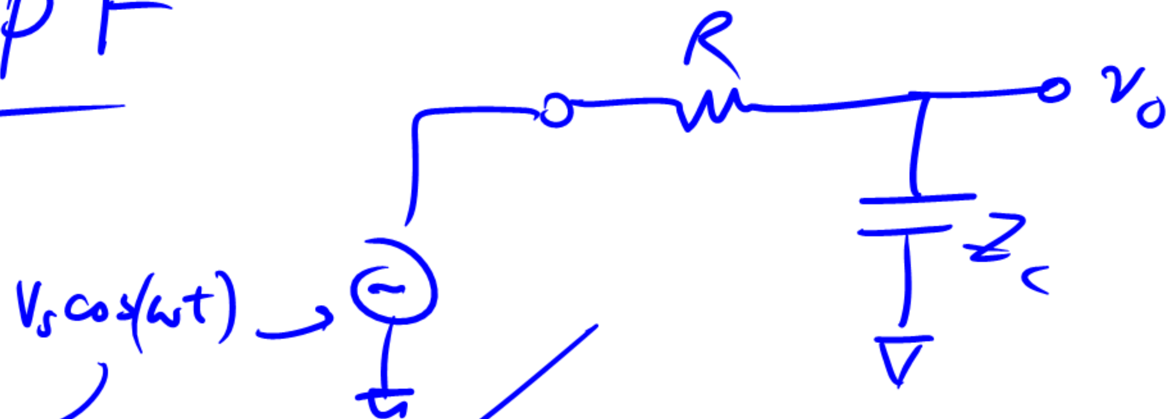
$$R = Z$$

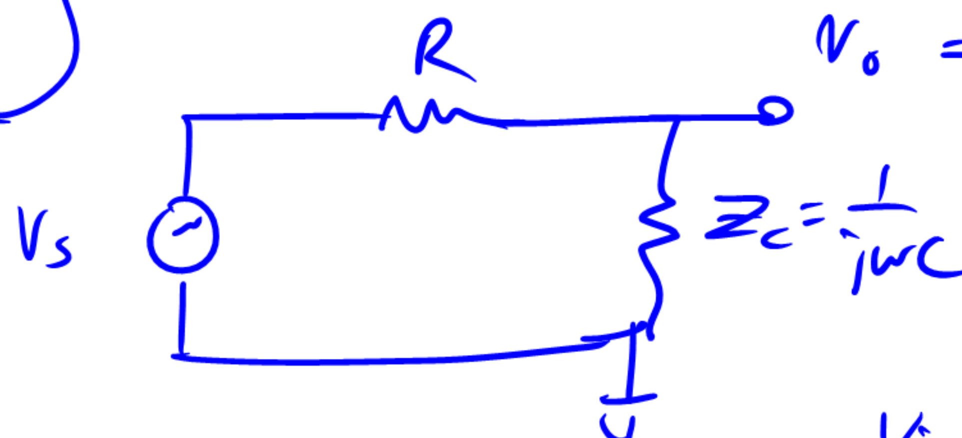
$$I_R = \frac{V}{R}$$

$$I_C = \frac{V}{Z_c} = \frac{V}{\left(\frac{1}{j\omega C}\right)}$$

$$I_C = j\omega C \cdot V$$

LPF





$$V_o = \frac{Z_c}{Z_c + R} \cdot V_s$$

$$\frac{V_o}{V_s} = \frac{Z_c}{Z_c + R} = \frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1 + j\omega RC}$$

PARAMETER

$$V_o(t) = ?$$

$$V_o = H(i\omega) \cdot V_s$$

$$Re(V_o) = Re\left(\frac{1}{1+i\omega RC} V_s \times e^{i\omega t}\right)$$

$$V_o(t) = \left| H(i\omega) \cdot V_s \right| \cos\left(\omega t + \angle H(i\omega) V_s\right)$$

$$= \frac{V_s}{\sqrt{1+(\omega RC)^2}} \cdot \cos\left(\omega t - \tan^{-1} \omega RC\right)$$