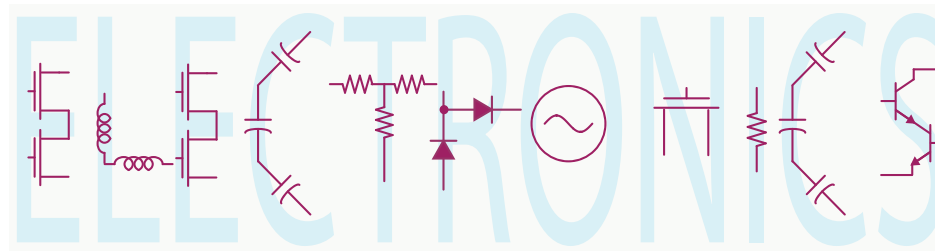


EE 42/100
Lecture 12: Capacitance



Rev B 2/22/2012 (9:59 PM)

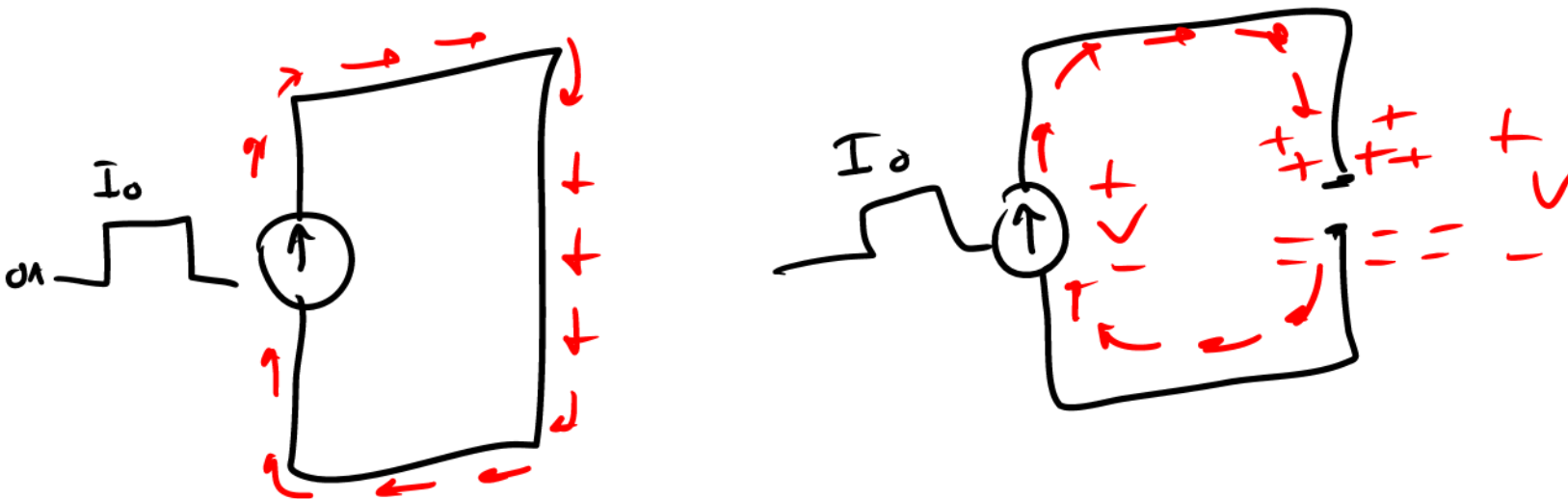
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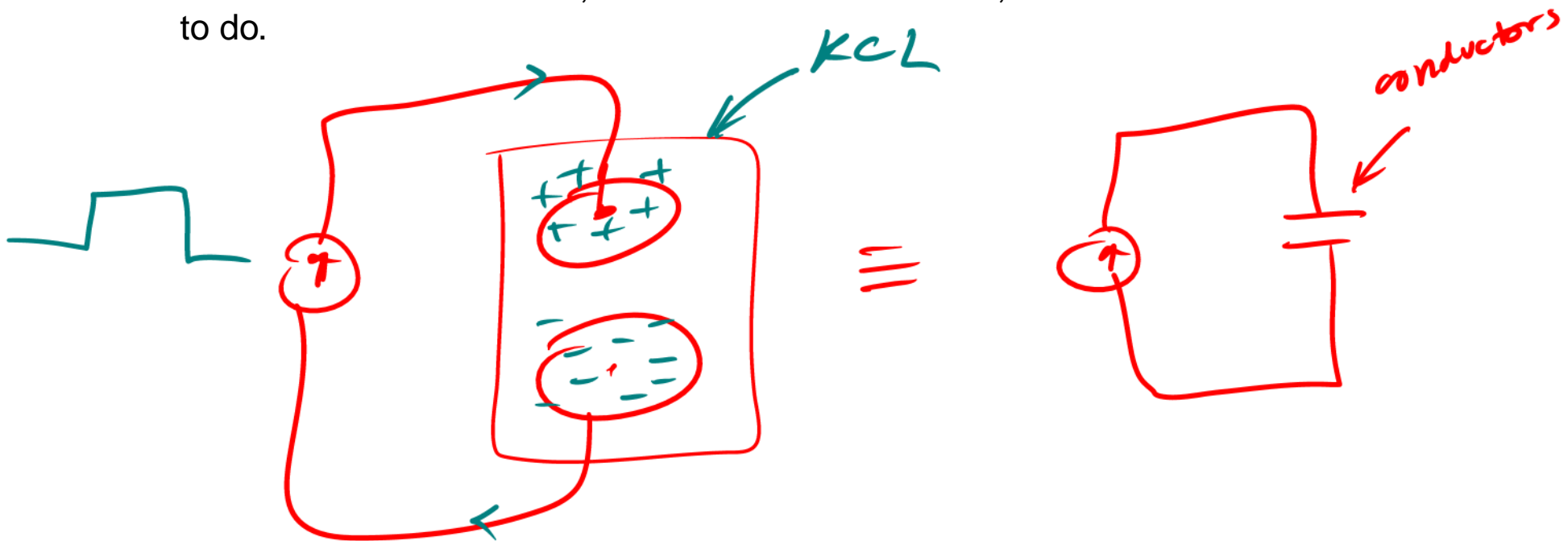
Thought Experiment

- Imagine a current source connected to an ideal short circuit as shown. The waveform for the current is shown. It's a constant current I_0 for a time T , so a total charge of $Q = I_0 \cdot T$ circulates around but no net work is done since $v(t) = 0$ (short circuit).
- Now imagine that we break the conductor in two and leave a gap between the conductors. Let's repeat the experiment by applying the same current waveform.
- What happens? The current flow is interrupted but the same amount of charge Q leaves the positive terminals of the current source. Where does it go?



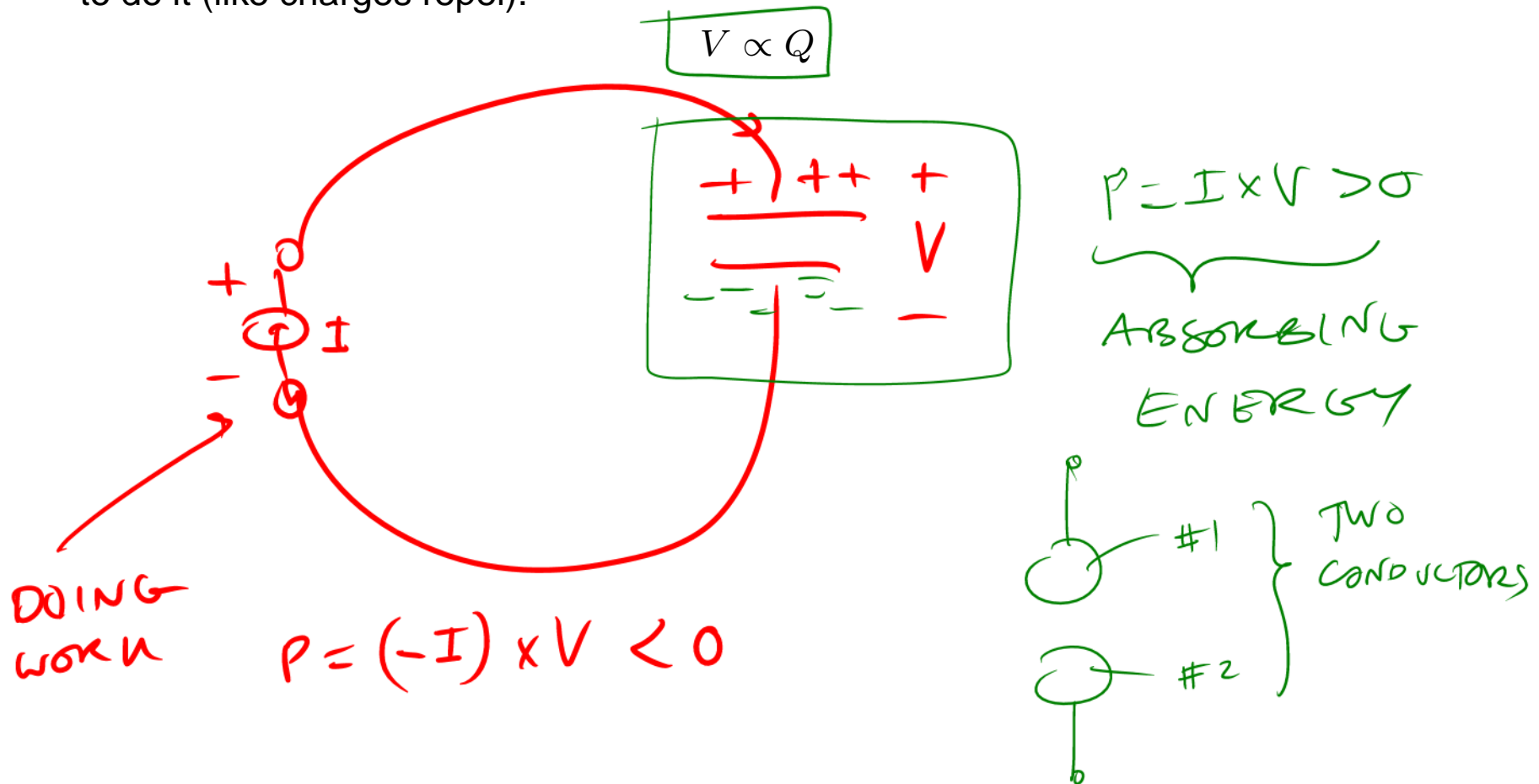
Capacitor Charge

- In addition to the positive charge Q leaving the positive terminal, the same amount of charge enters the negative terminal. Equivalently, a charge of $-Q$ leaves the negative terminal and goes into the conductors.
- We see that the charge cannot go anywhere but into the conductors, and therefore the charge is stored there.
- Since like charges repel, we have to do work to force the charges to accumulate on the conductors. In fact, the smaller the conductor, the more work that we have to do.



Capacitor Voltage

- By definition, the potential v across the capacitor represents the amount of work required to move a unit of charge onto the capacitor plates. This is the work done by the current source.
- For linear media, we observe that as we push more charge onto the capacitor with a fixed current, it's voltage increases linearly because it's more and more difficult to do it (like charges repel).



Definition of Capacitance

- The symbol for a capacitor is shown above. Sometimes a + label indicates that a capacitor should only be charged in a given direction. Most capacitors, though, are symmetric and positive or negative charge can be applied to either terminals.
- The proportionality constant between the charge and the voltage is defined as the capacitance of the two terminal element

$$q = Cv$$

✓ q ✓ v

- The units of capacitance are given by charge over voltage, or Farads (in honor of Michael Faraday)

$$[C] = \frac{[q]}{[v]} = \frac{C}{V} = F$$

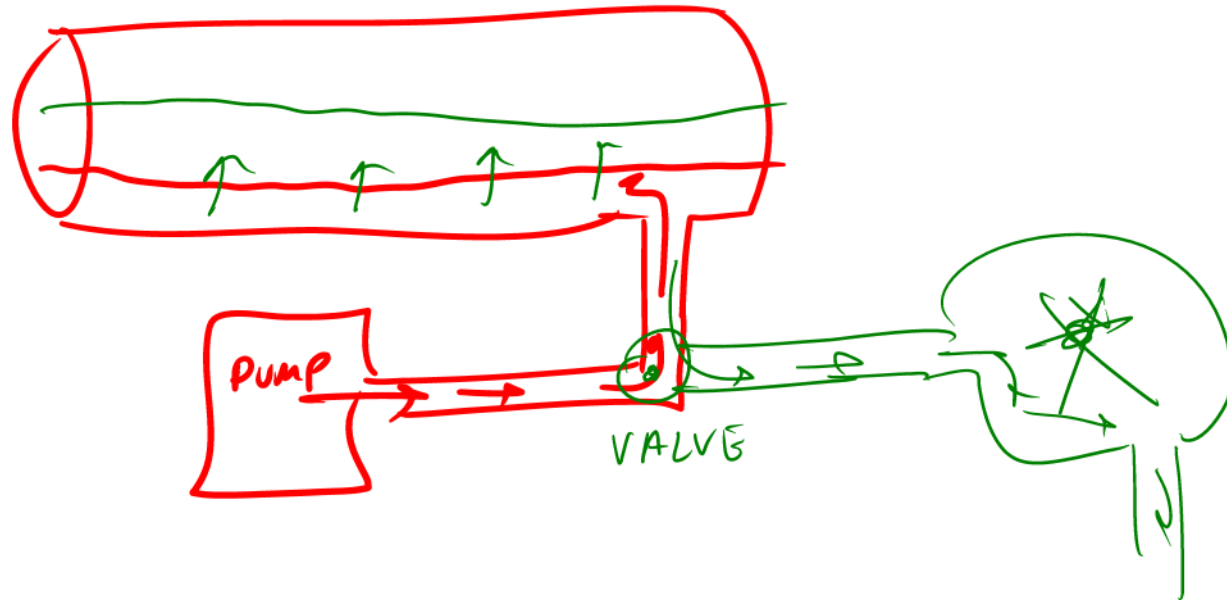
$$C = \frac{q}{V}$$

- We expect that a physically larger conductor should have a larger capacitance $C_1 > C_2$, because it has more surface area for the charges to reside. The average distance between the like charges determines how much energy you have to provide to push additional charges onto the capacitor.



Capacitor Analogy

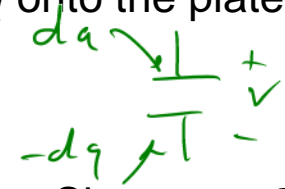
- Imagine a tank of water where we pump water into the tank from the bottom. As we initially pump water, there is no water and it takes virtually no work. But as the tank fills up, it takes more and more work since the liquid obtains gravitational potential energy.
- A smaller tank requires more work (it has less capacity) because the liquid column gets higher and higher.
- Notice that we can always recover the work by emptying the tank.



Capacitor Energy

- The incremental amount of work done to move a charge dq onto the plates of the capacitor is given by

$$dE = v dq$$



- where v is the potential energy of the capacitor in a given state. Since $q = Cv$, we have $dq = Cdv$, or

$$dE = Cvdv$$

$$q = Cv \\ dq = Cdv$$

- If we now integrate from zero potential (no charge) to some final voltage

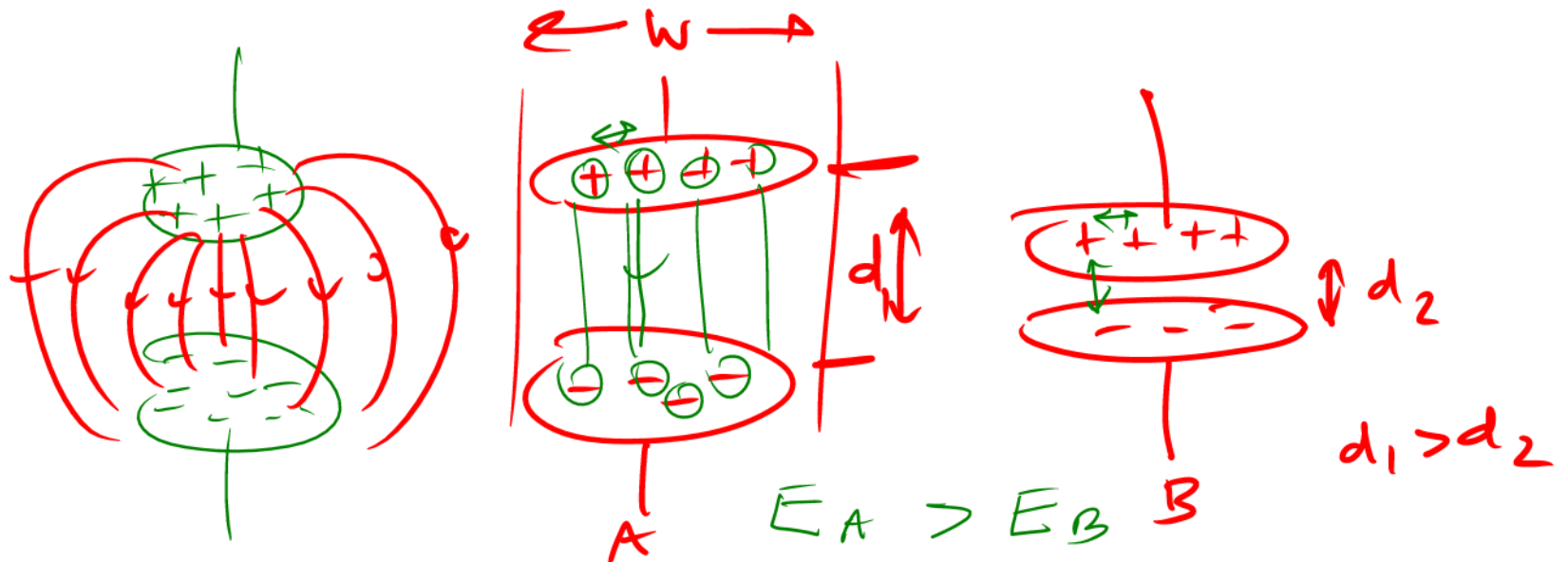
$$E = \int_0^{V_0} Cvdv = C \frac{v^2}{2} \Big|_0^{V_0} = \frac{1}{2} CV_0^2$$

Handwritten notes: The upper limit V_0 is circled in green, and an arrow labeled "FINAL" points to it. The lower limit 0 is also circled in green. The final result $\frac{1}{2} CV_0^2$ is enclosed in a green box.

- This is the energy stored in the capacitor. Just like the water tank, it's stored as potential energy that we can later recover.

Field Lines

- Since we get charge separation in a capacitor, we expect that field lines emanate from the positive charges to the negative charges. The energy of the capacitor is in fact stored in these field lines.
- So there are two competing charge mechanisms in a capacitor. Like charges are forced to reside on the same plate, which requires energy. On the other hand, unlike charges are placed in close proximity, which have attractive forces. So we can see that the charges should bunch up as close as possible to the charges of opposite sign.
- The smaller the gap spacing, the more capacity we have in a capacitor, because now the “unhappy” feelings of being cramped up to similar charge is somewhat alleviated by the “happy” feeling due to close proximity of unlike charges.

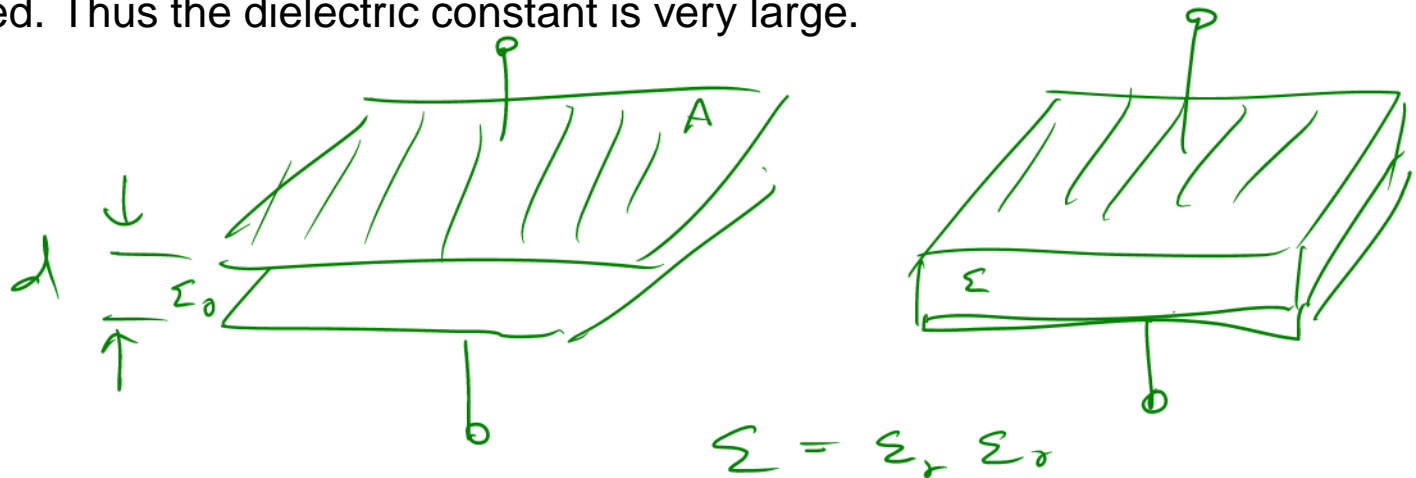


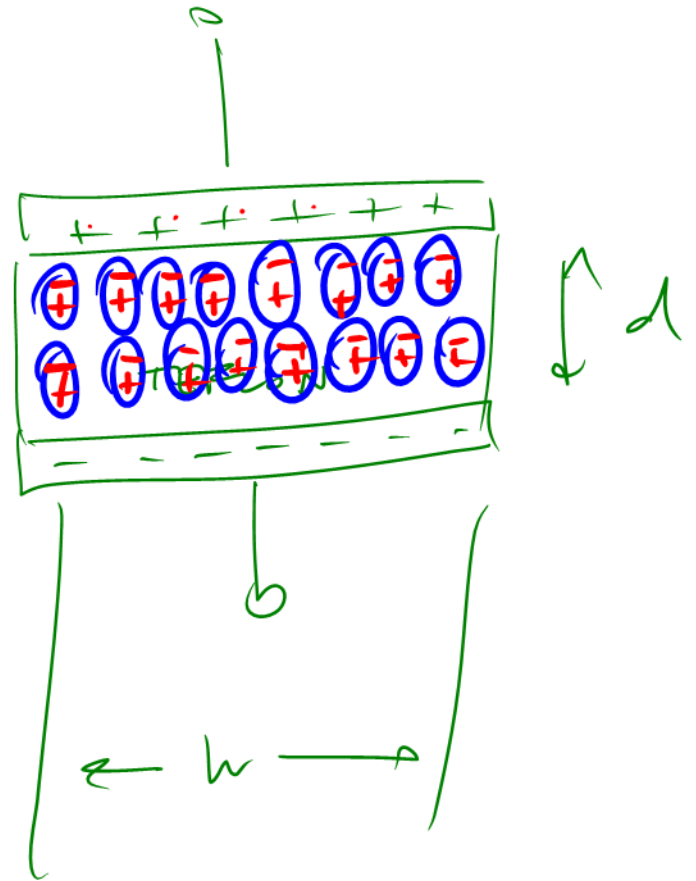
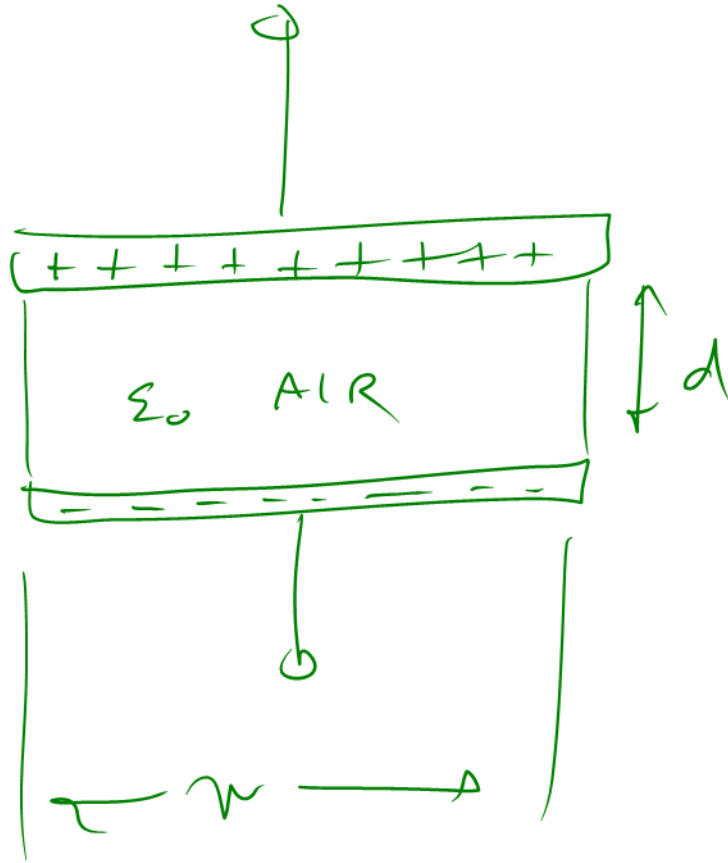
Parallel Plate Capacitor

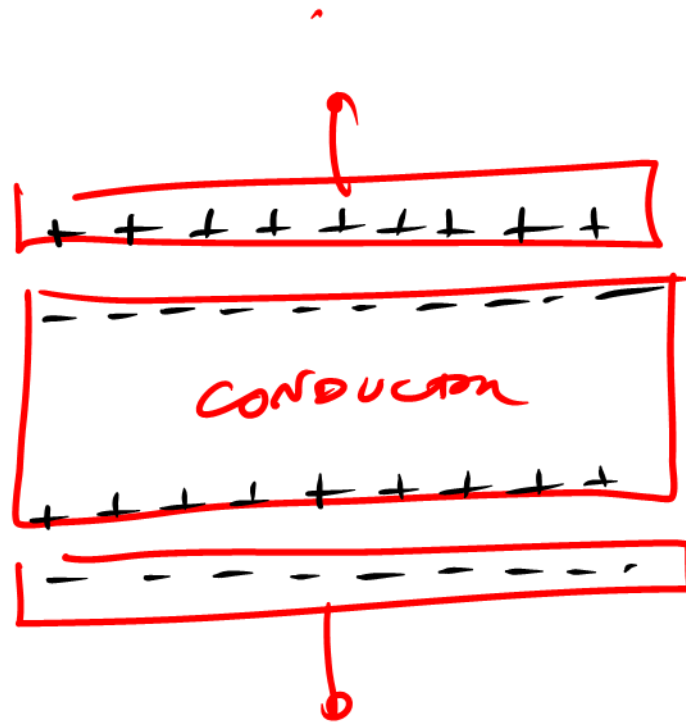
- From basic physics, it's easy to show that the capacitance of a parallel plate structure is given by

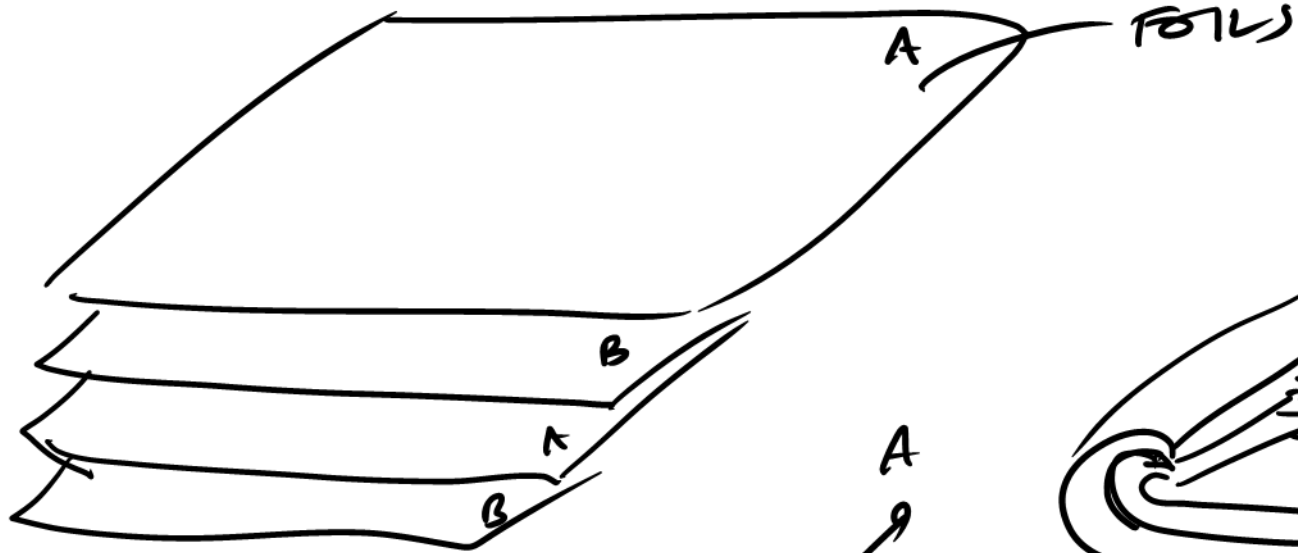
$$C = \frac{\epsilon A}{d}$$

- where A is the plate area, ϵ is the permittivity of the dielectric (also called the dielectric constant), and d is the gap spacing.
- The permittivity of free space is $\epsilon_0 = 8.854 \times 10^{-12}$ F/m. Most materials have a higher permittivity which is captured by the unitless relative permittivity $\epsilon_r = \epsilon/\epsilon_0$, with typical materials $\epsilon \sim 1 - 10$. For instance, air is mostly empty space and so $\epsilon_r \approx 1$.
- Some materials, such as water, have polar molecules that align when an electric field is applied. Thus the dielectric constant is very large.

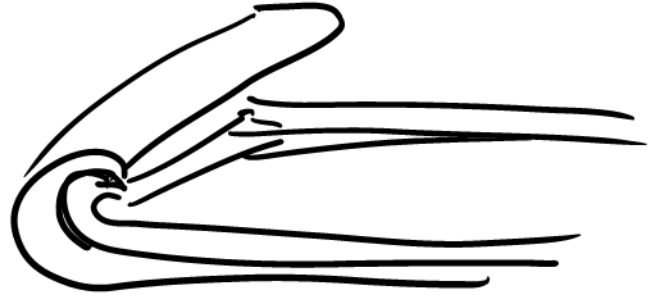
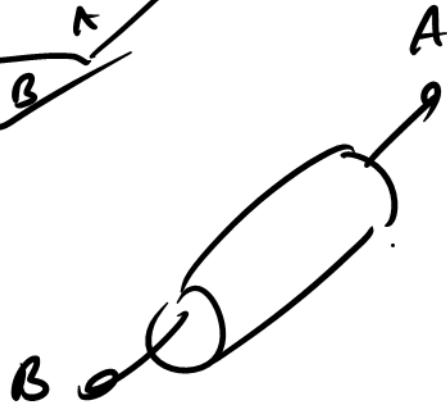








FOLDS



Practical Capacitors

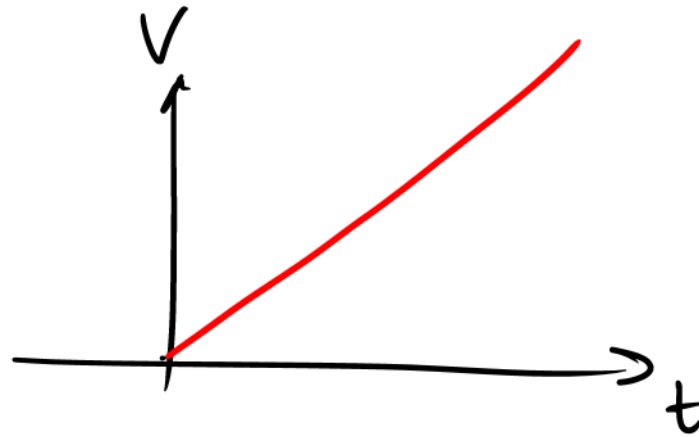
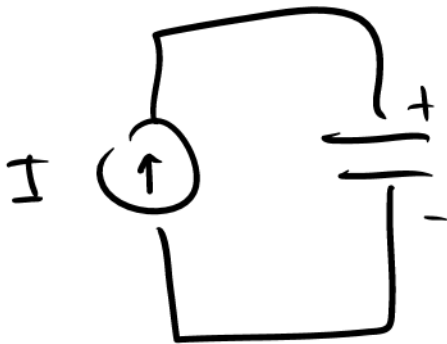
- Real capacitors are made of large sheets of conductors (to maximize surface area) and thin dielectric layers (to minimize the gap). A multi-layer sandwich structure can then be wrapped together to form a large capacitor.
- In integrated circuits, thin dielectrics and/or multiple metal fingers in close proximity form a high density capacitor.
- Super-capacitors use nanoparticles to boost the capacitance even higher.

$1\text{ nF} \leftarrow \text{-----} \rightarrow \text{ mF} \rightarrow \text{ F}$

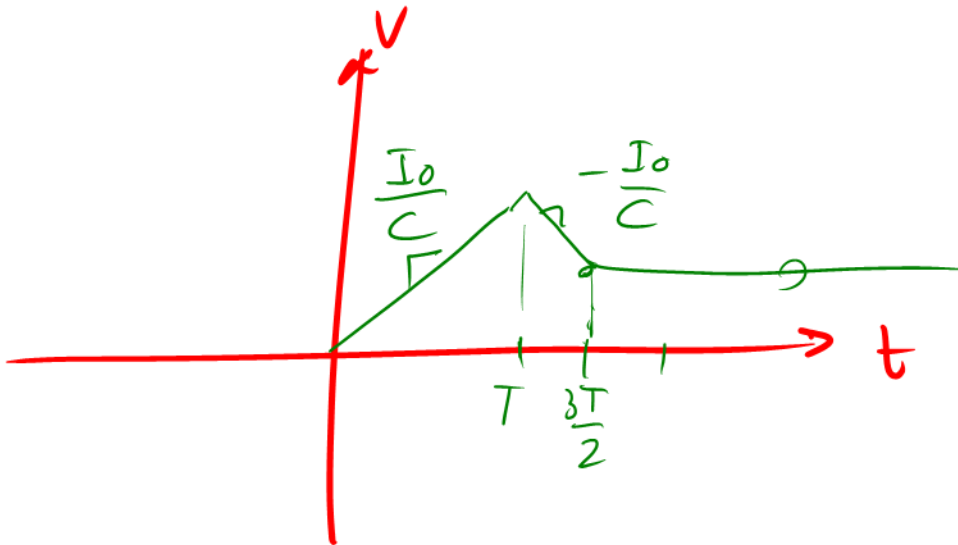
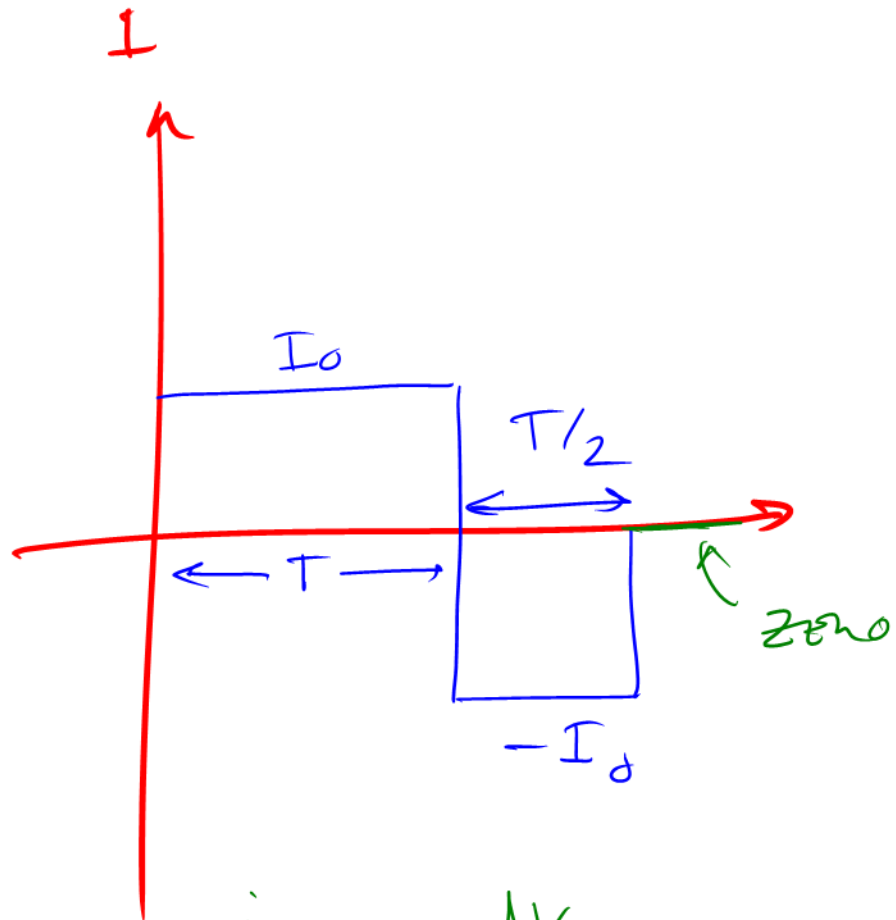
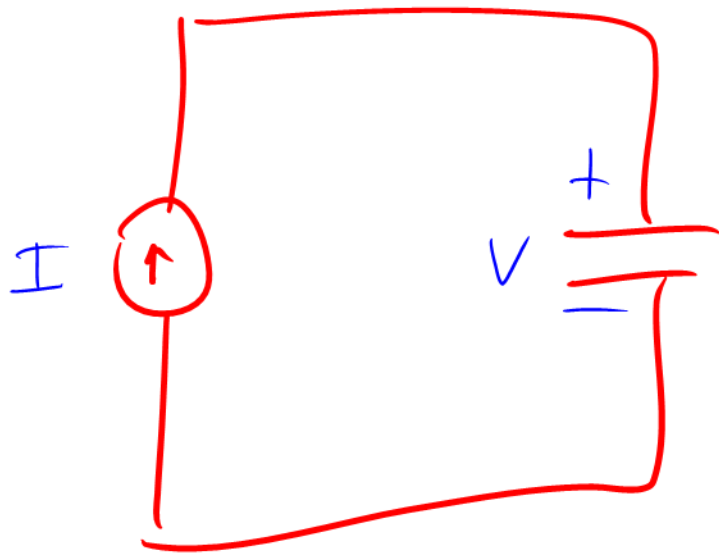
CHIP $\sim 1\text{ fF} (10^{-15})$
 $\sim 1\text{ aF} (10^{-18})$

Capacitor Current

- A steady DC current into a capacitor means that the voltage ramps linearly. In theory it would ramp to infinity but in practice dielectric breakdown (arcing) would short out the capacitor.
- Now consider an AC current. Since a steady positive current increases the voltage linearly, a steady negative current does the opposite. Thus if we pass a time varying current that has positive and negative polarity, the voltage across the capacitor would increase and decrease. For a fixed current, we can infer the voltage. What about a general relation?



$$q \propto V \Rightarrow V \propto q$$



$$i = C \frac{dV}{dt}$$

Capacitor Current-Voltage Relationship

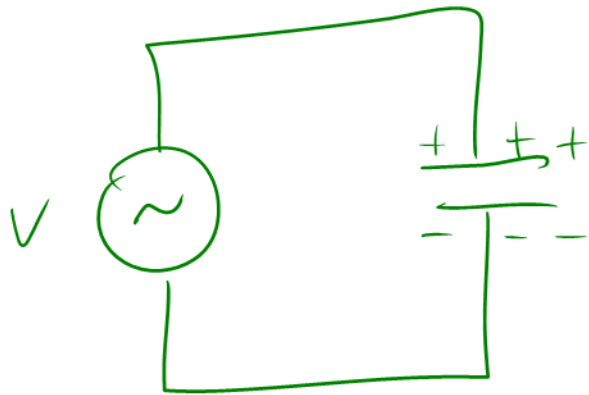
- Since charge flow through time results in current, the relationship between the current and voltage in a capacitor is easily derived

$$q = Cv$$

$$i = \frac{dq}{dt} = C \frac{dv}{dt}$$

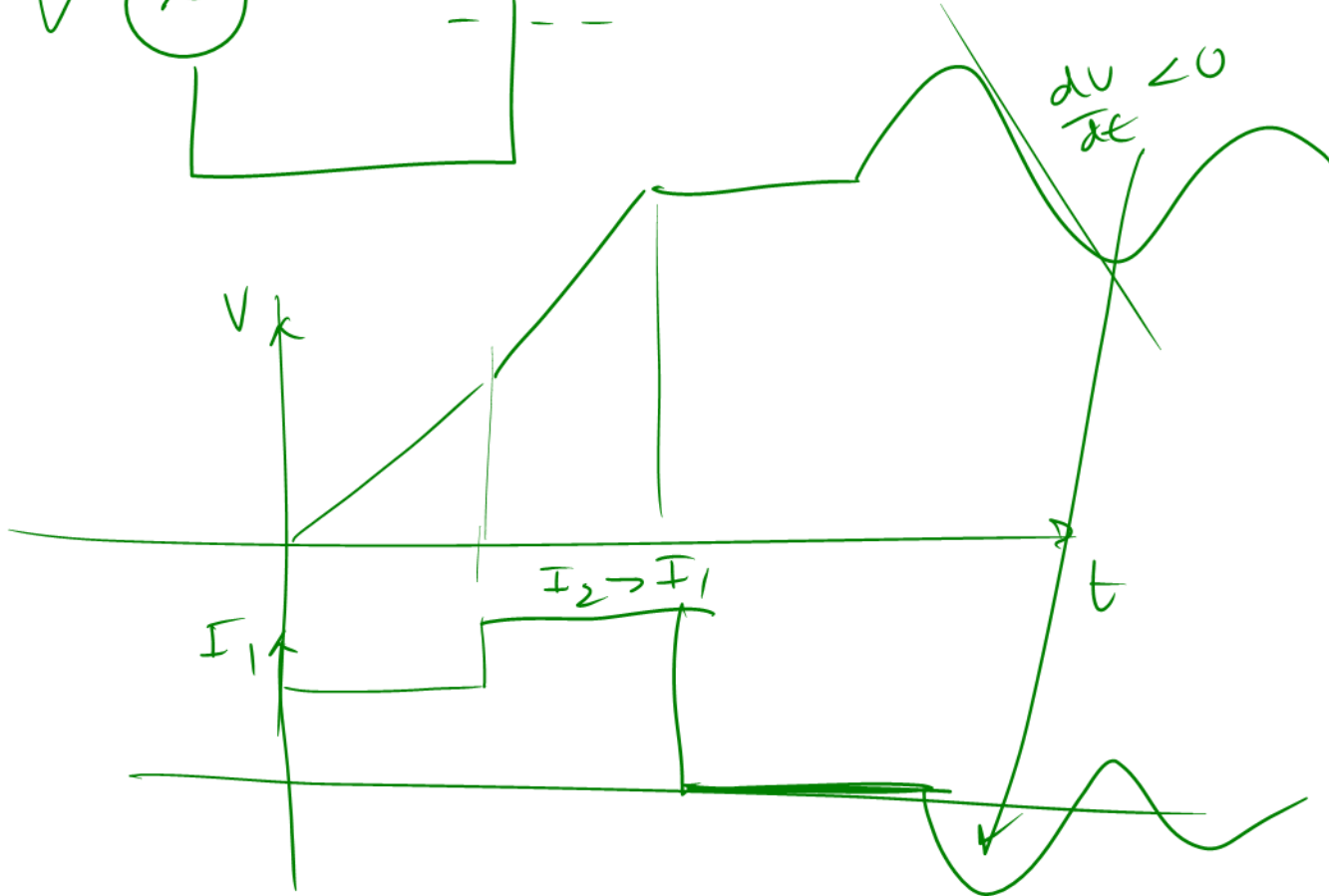
$$i = Gv$$

- Recall that for a resistor we had $i = Gv$. The capacitor has a “conductance” which is proportional to its capacitance but also it depends on how quickly the voltage is varied. If a fixed voltage is applied to a capacitor, no current flows (if the capacitor is uncharged, or charged to a different voltage, then in theory an enormous (infinite) current would instantaneously flow to charge the capacitor to the right voltage).
- When a slowly varying voltage is applied, the current flow is slow as well since only enough charge needs to flow to change the potential. When the potential is varied very quickly, the current flow is larger because it has to keep up with the voltage changes. Only changes in the voltage require new charge to flow!



$$q = CV$$

$$V = \frac{q}{C}$$



Capacitor Voltage-Current Relationship

- We can re-write the capacitor "I-V" relationship as

$$\frac{dv}{dt} = \frac{i}{C}$$

- If we integrate the above relation, we have

$$\int_{t_0}^t \frac{dv}{d\tau} d\tau = \int_{t_0}^t \frac{i}{C} d\tau$$
$$v(t) - v(t_0) = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau$$
$$v(t) = v(t_0) + \frac{1}{C} \int_{t_0}^t i(\tau) d\tau$$

- The integral of current over time is nothing but the *net charge* flow into the capacitor

$$Q = \int_{t_0}^t i(\tau) d\tau$$

- Let's break this down term-by-term

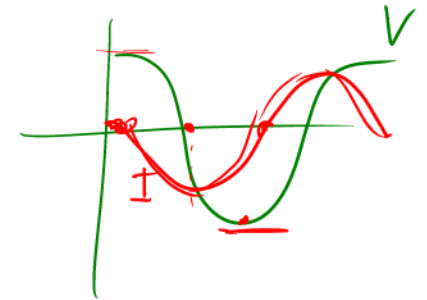
$$\underbrace{v(t)}_{\text{voltage now}} = \underbrace{v(t_0)}_{\text{voltage then}} + \underbrace{Q/C}_{\text{net charge flow over capacitance}}$$

Sinusoidal Drive

- If we drive the capacitor with an AC sinusoidal voltage, we can calculate the current

$$v(t) = V_0 \cos \omega_0 t$$

$$i(t) = C \frac{dv}{dt} = -CV_0 \omega_0 \sin \omega_0 t$$



- Current is out of phase with the voltage. It is exactly 90° out of phase, which is very important! The power flow onto the capacitor is given by

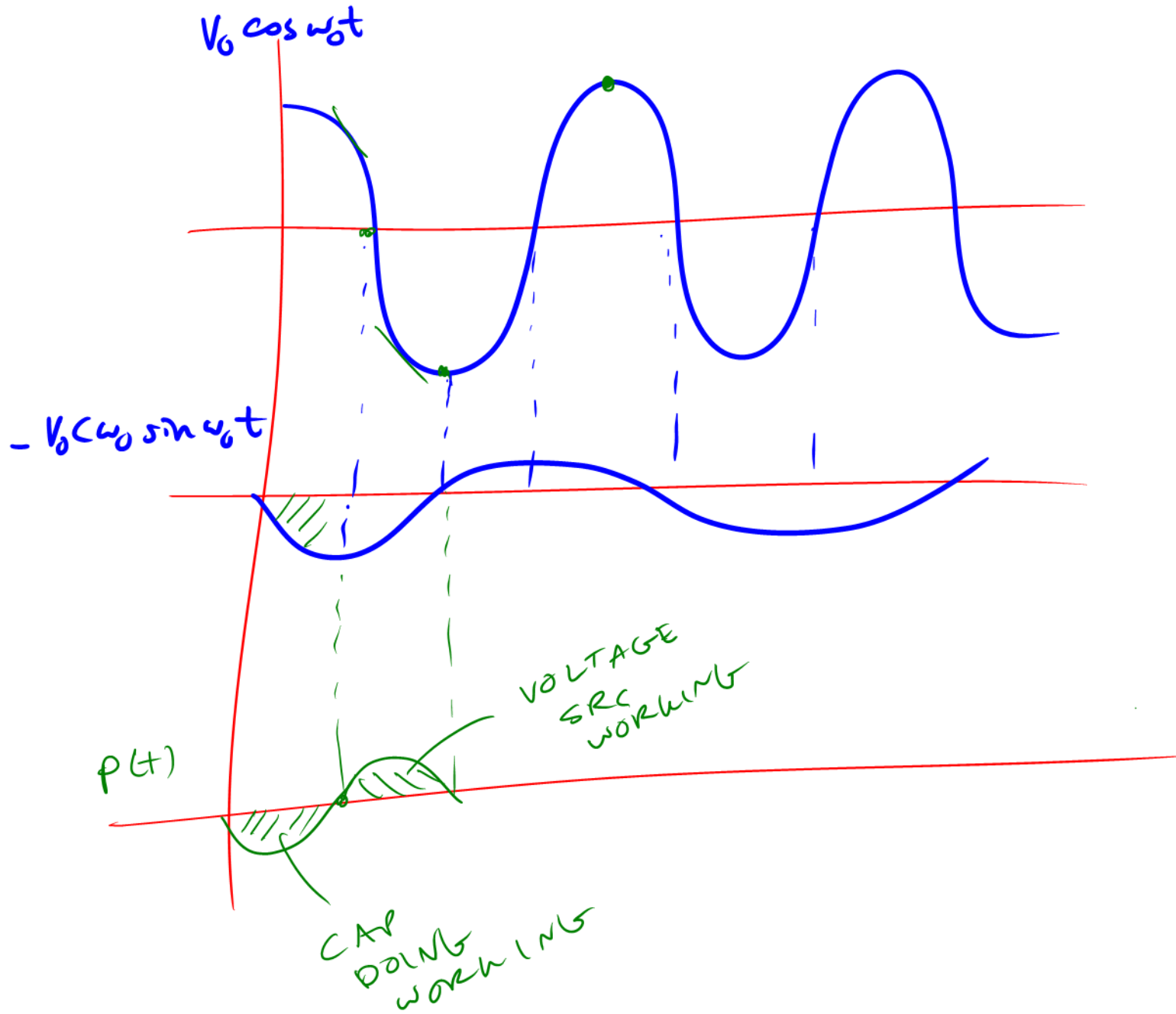
$$p(t) = v(t)i(t) = -CV_0^2 \omega_0 \cos \omega_0 t \sin \omega_0 t$$

$$= -\frac{CV_0^2}{2} \omega_0 \sin 2\omega_0 t$$



- The instantaneous power flow switches signs twice per cycle, as in each cycle energy is first delivered onto the capacitor, but then the energy is returned back to the source. In other words, there is no net energy flow into the capacitor

$$\int_0^T p(\tau) d\tau = -\frac{CV_0^2}{2} \omega_0 \int_0^T \sin 2\omega_0 \tau d\tau \equiv 0$$



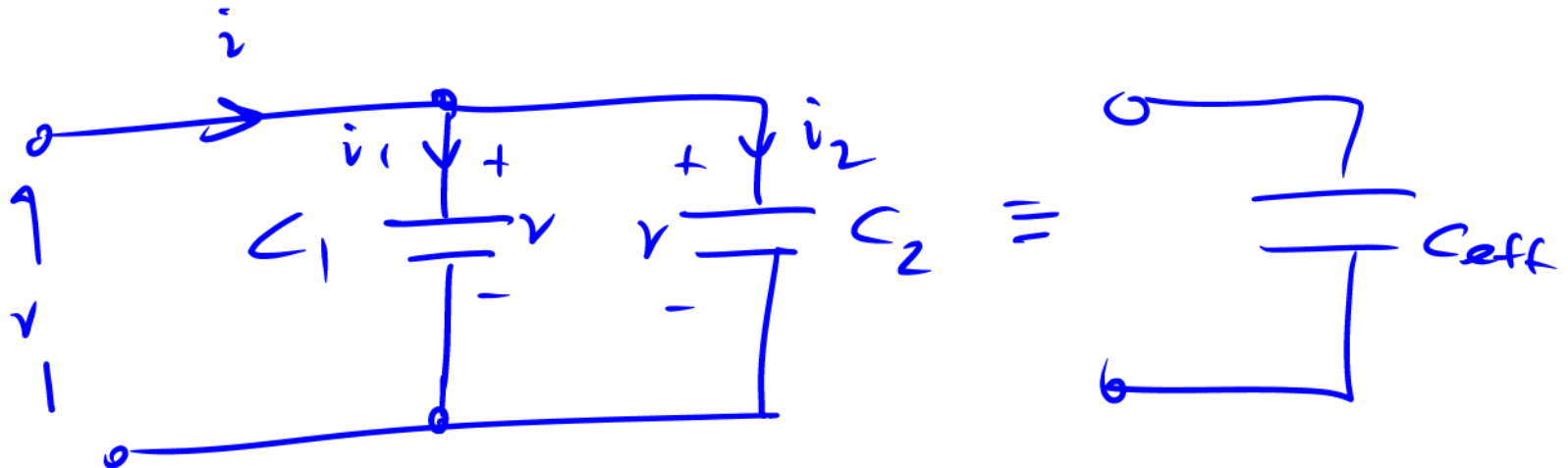
Circuits with Capacitors: Shunt Connection

- When capacitors are connected in parallel, they have the same voltage. The total current drawn from the supply is given by

$$\underline{i} = \underline{i_1} + \underline{i_2} = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} \quad \text{KCL}$$

$$\underline{i} = (C_1 + C_2) \frac{dv}{dt} = C_{eff} \frac{dv}{dt}$$

- In other words, two parallel capacitors act like an effectively larger capacitor $C_{eff} = C_1 + C_2$. This is easily generalized to N capacitors in parallel.



Circuits with Capacitors: Series Connection

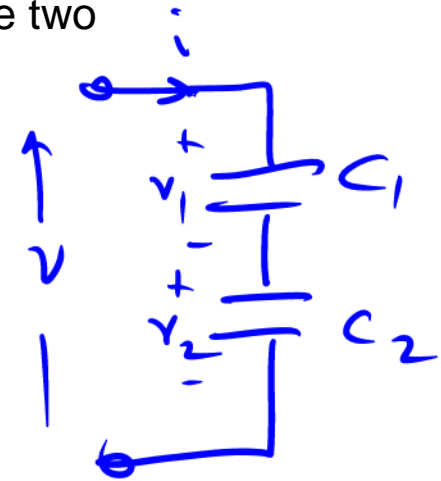
- When capacitors are placed in series, the same current flows through both capacitors whereas the voltage drop is shared across the two

$$i = C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt}$$

$$v(t) = v_1(t) + v_2(t)$$

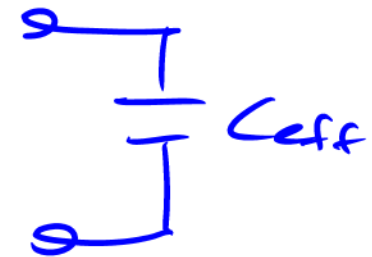
$$\frac{dv}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt}$$

$$= \frac{1}{C_1} i + \frac{1}{C_2} i$$



- Re-arranging the above equation

$$i = \left(\frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \right) \frac{dv}{dt}$$



- If we treat “||” as an operator, then for capacitors in series we have

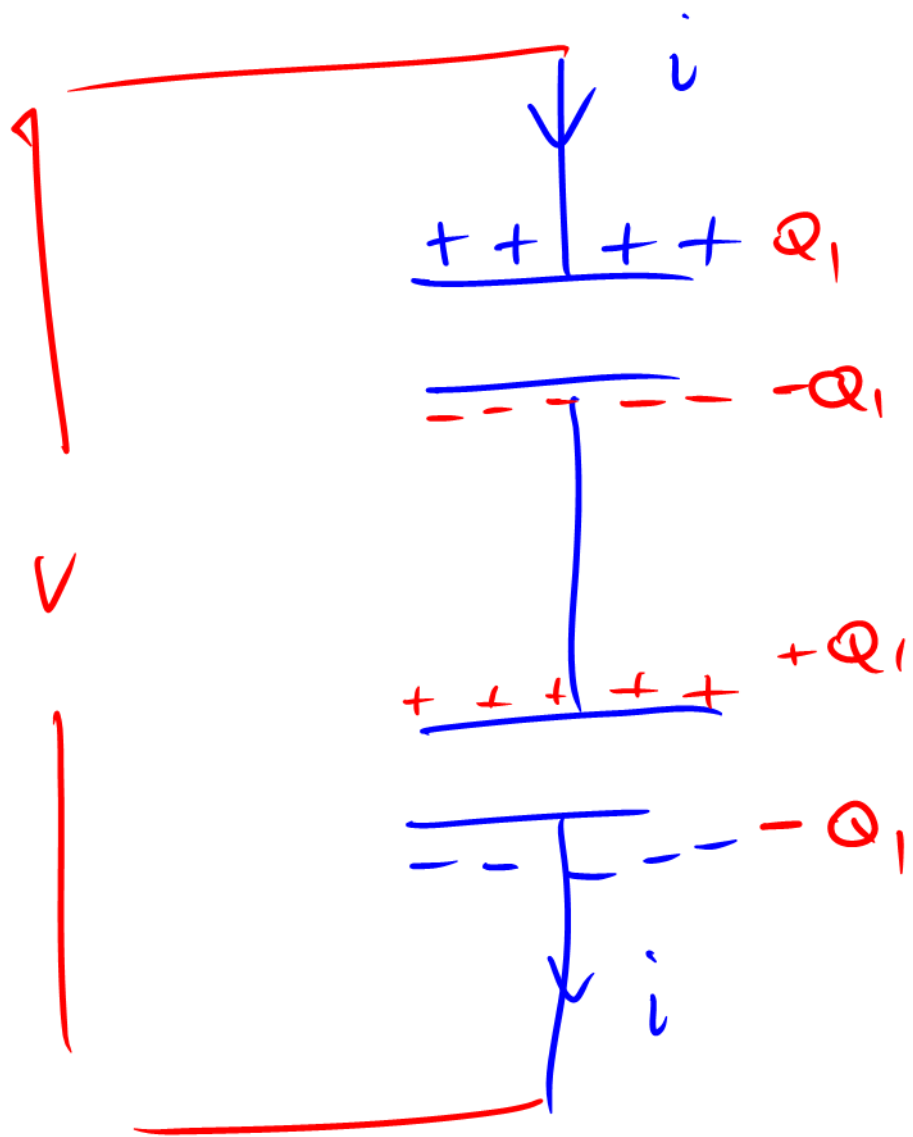
$$C_{eff} = C_1 || C_2$$

Series Capacitors

- It follows that the smaller capacitor dominates. That's because the smaller capacitor has a much higher voltage drop across it. Why is that?
- The key observation is that both capacitors have equal charge storage. This follows from conservation of charge at the "floating" node. If a charge of $+q_1$ appears on the top plate of C_1 , then a charge of $-q_1$ appears on the bottom plate. By the same token, a charge of $+q_2$ and $-q_2$ appears on the plates of the second capacitor.
- But the top plate and bottom plate of the second and first capacitors are shorted together. Also, if the system is initially uncharged ($v = 0$), then this node is at ground potential and there is no net charge at this node. This means that

$$-q_1 + q_2 \equiv 0$$

- Or in other words, $q_1 = q_2$. So if both capacitors hold the same amount of charge, the one with the lower capacitance will have a higher voltage drop.



$$V_1 = \frac{Q_1}{C_1}$$

$$Q_1 - Q_1 = 0$$

$$V_2 = \frac{Q_1}{C_2}$$

$$C_2 < C_1 \Rightarrow V_2 > V_1$$

KCL with Capacitors

- We can continue to write KCL and KVL equations with capacitors.
- Unfortunately, instead of linear equations we are now dealing with linear differential equations.

Non-Linear Capacitors

- For some elements the charge/voltage relationship is non-linear. A good example is a semi-conductor diode. We can then write

$$q(v) = f(v)$$

- where f is an arbitrary function. The current through the device is given by

$$i = \frac{dq}{dt} = \frac{dq}{dv} \frac{dv}{dt} = C(v) \frac{dv}{dt}$$

- which allows us to define a voltage-dependent incremental capacitance

$$C(v) = \frac{dq}{dv}$$

Capacitors Everywhere!

- A typical circuit may employ a lot of capacitors. Many of them are simply connected to the power supply. Why is that?
- Capacitors can act as small batteries with very small internal resistance, so they can deliver a lot of power.
- The current available from a capacitor is related to the *Effective Series Resistance* (ESR). This is partly why we use a bank of capacitors, from small values to large values, in parallel on important power supply nodes. The small capacitors can deliver current much faster, but only a limited amount. They take care of high frequency transients. The big capacitors have more charge but they respond more slowly. They respond to slow transients. Finally, the battery, which is “far away” (due to inductance), is by far the slowest and keeps these capacitors charged.

Can Capacitors Replace Batteries?

- A small AA battery has a cell voltage of 1.5V and a capacity of about 3000mA-hr. The total energy is therefore:

$$E = Q \cdot V = 3000 \text{ mA} \cdot \text{hr} \cdot \frac{3600 \text{ s}}{1 \text{ hr}} \cdot 1.5 \text{ V} = 16.2 \text{ kJ}$$

- Compare that to a small 1-oz steak, which has about 65 kCal or 273 kJ ! Now using about the same volume, you can buy a “super capacitor” with a capacitance of 100 F. The energy you can pack into this depends not the voltage rating, which is 2.5 V in this case

$$E = \frac{1}{2} CV^2 = \frac{1}{2} 100 \cdot 2.5^2 = 313 \text{ J}$$

- This energy is considerably less than a battery and it costs about \$25!
- Here’s an interesting comparison:

Technology	Volume (mL)	Weight (g)	Energy (kJ)	Cost (\$)
Meat	29	28	273	.25
AA Battery	30	23	16	.25
Super Cap	61	22	.3	25
Gas	30	23	1050	.03