EECS 142

Integrated Circuits for Communication

Oscillator Phase Noise

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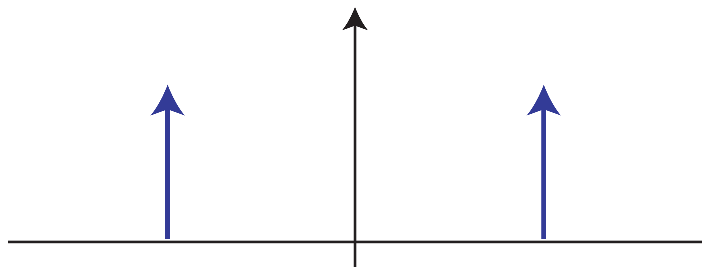
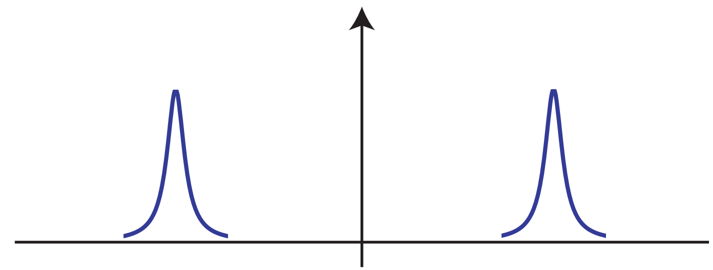
Oscillator Output Spectrum



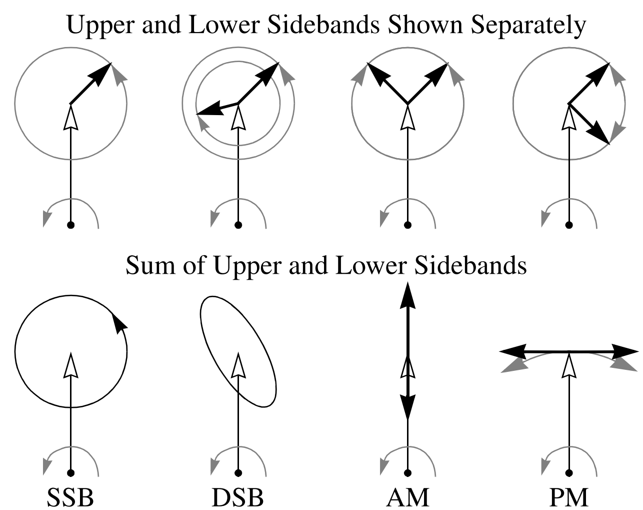
Ideal Oscillator Spectrum



Real Oscillator Spectrum

- The output spectrum of an oscillator is very peaked near the oscillation frequency but not infinitely so. It's not a pair of delta function! Why?
- If we ignore noise, the closed-loop gain of the system is infinite since $A_l = 1$. But in practice there is noise in any real oscillator.

Phase Noise versus Amplitude Noise



Source: The Designer's Guide Community (www.desingers-guide.org), *Noise in Mixers, Oscillators, Samplers, and Logic* by J. Philips and K. Kundert

- Notice that noise at offset frequency $\Delta\omega$ can be modeled as a phasor rotating around the rotating carrier phasor. It rotates because it's at a different frequency (offset).
- The upper side-band rotates in the same direction with frequency $\Delta\omega$ whereas the lower sideband rotates clockwise or with frequency $-\Delta\omega$.

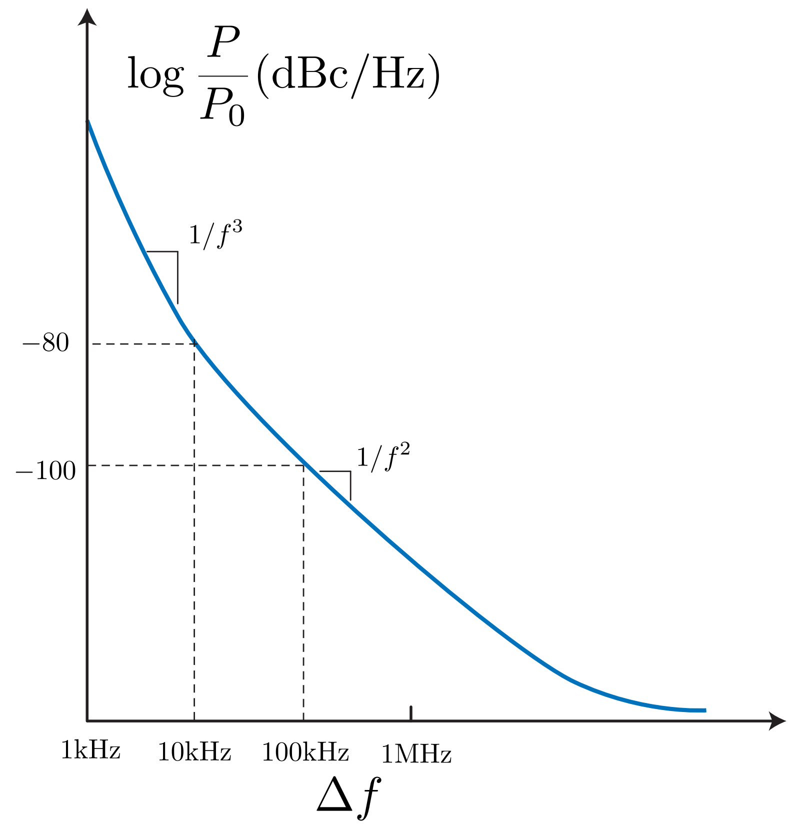
Graphical Picture of AM and PM

- In general the two side-bands are completely uncorrelated, meaning their amplitude and phase will vary randomly from one another. When summed together they trace an ellipse whose size and shape and orientation shifts randomly.
- If the noise is cyclostationary, there is correlation between the two sidebands, which reduces the random shifting of the shape and orienation. For perfect correlation, the shape and orienation will remain unchanged, and the size shifts randomly.
- Note that for a stationary noise source, the AM and PM components are equal.
- If we pass the signal through a limiting amplifier, the AM noise is rejected. This produces an output with only PM.
- We shall see that for an oscillator, the AM noise is rejected.
- Oscillators generate phase noise (AM component rejected), which traces a perpundicular line.

Phase Noise?

- Why do we say that the noise in the spectrum is due to “phase” noise rather than amplitude noise?
- An oscillator has a well defined amplitude which is controlled by the non-linearity of the circuit. If there is an amplitude perturbation, it is naturally rejected by the oscillator.
- This occurs because the oscillation occurs at a frequency when the loop gain is unity. If the amplitude grows, due to compressive characteristics of the non-linearity, the loop gain decreases and the oscillation amplitude dampens. Likewise, if the amplitude drops, the loop gain goes over unity due to expansive characterisitcs of the non-linearity, and the amplitude grows back.
- The phase of the oscillator, on the other hand, is “free running”. Any phase-shifted solution to the oscillator is a valid solution. So if a perturbation changes the phase of the oscillator, there is no “restoring force” and the phase error persists.

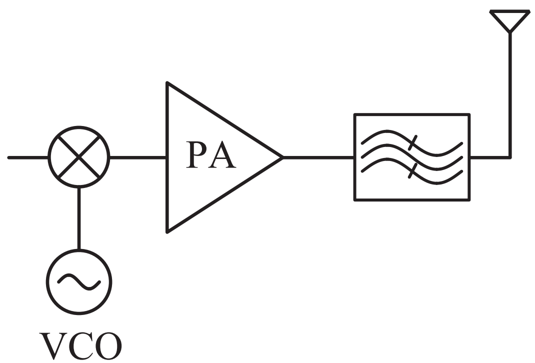
Phase Noise Measurement

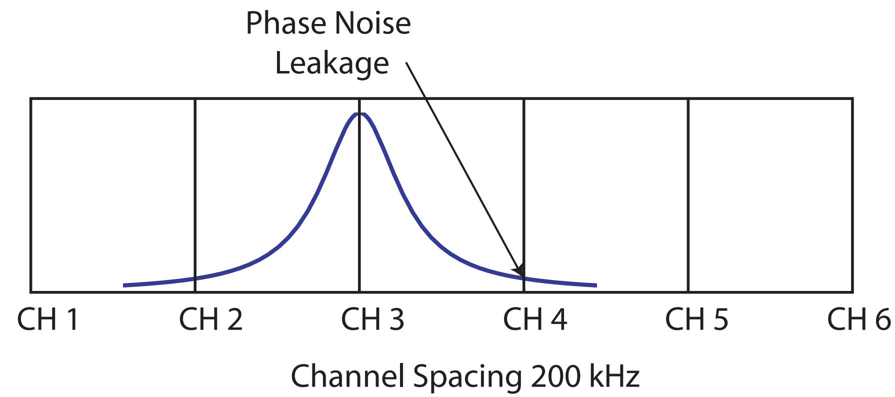


- If we zoom into the carrier on a log scale, the relative power at an offset frequency Δf from the carrier drops very rapidly. For the case shown above, at an offset of $100\mathrm{kHz}$, the power drops to $-100\mathrm{dBc}$.

- There is clearly a region where the slope is $20\mathrm{dB/dec}$. But this range only holds until the noise flattens out. Also, very near the carrier, the slope increases to approximately $30\mathrm{dB/dec}$.

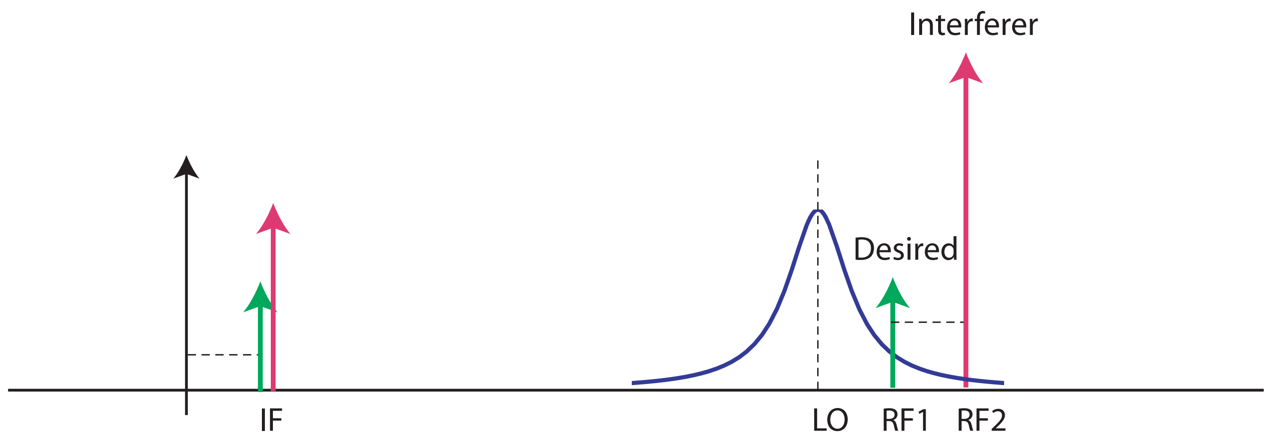
Phase Noise In TX Chain





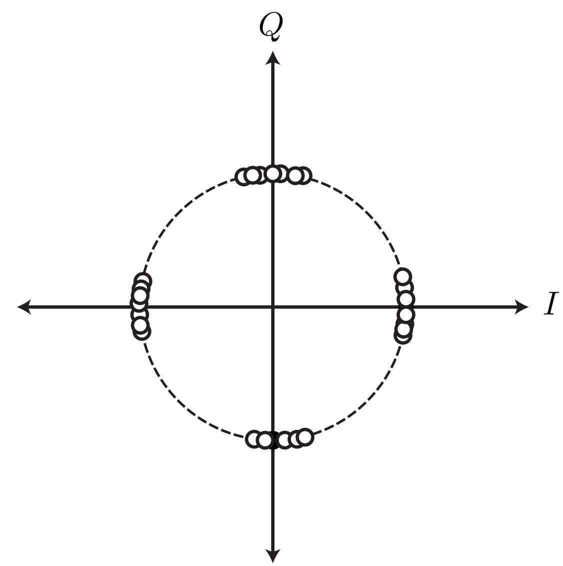
- Phase noise in a transmit chain will “leak” power into adjacent channels. Since the power transmitted is large, say about 30dBm, an adjacent channel in a narrowband system may only reside about 200kHz away (GSM), placing a stringent specification on the transmitter spectrum.

Phase Noise In RX Chain



- In a receive chain, the fact that the LO is not a perfect delta function means that there is a continuum of LO's that can mix with interfering signals and produce energy at the same IF. Here we observe an adjacent channel signal mixing with the "skirt" of the LO and falling on top of the a weak IF signal from the desired channel.

Phase Noise In Digital Communication



- In a digital communication system, phase noise can lead to a lower noise margin. Above, we see that the phase noise causes the constellation of a 4 PSK system to spread out.

- In OFDM systems, a wide bandwidth is split into sub-channels. The phase noise leads to inter carrier interference and a degradation in the digital communication BER.

Feedback Model of Phase Noise

- In a simple linear model for an oscillator, the closed-loop transfer function is given by

$$\frac{Y(f)}{X(f)} = \frac{H(f)}{H(f) - 1}$$

- This goes to infinity at oscillator since by definition $|H(f)| = 1$ for osicllation to occur (Barkhausen condition)
- At a frequency offset from the carrier, assuming the loop gain varies smoothly, we have

$$H(f) = H(f_0) + \frac{dH}{df}\Delta f$$

so that

$$\frac{Y(f+\Delta f)}{X(f+\Delta f)} = \frac{H(f_0) + \frac{dH}{df}\Delta f}{H(f_0) + \frac{dH}{df}\Delta f - 1}$$

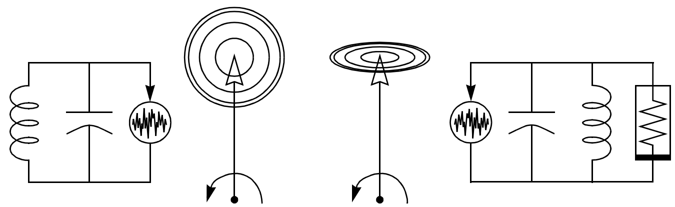
Feedback Model (cont)

- Since $H(f_0) = 1$ and assuming $dH/df \Delta f \ll 1$ for practical situations (near the carrier)

$$\frac{Y(f+\Delta f)}{X(f+\Delta f)} = \frac{1}{\frac{dH}{df}\Delta f}$$

- This shows that for circuits containing white noise sources, the noise voltage (current) is inversely proportional to Δf, while the noise power spectral density is proportional to Δf^2
- This simplistic picture already gives us some insight into the shape of the noise spectrum. But the noise does not “blow up” near the carrier. Also, why does all the noise go to phase noise and not amplitude noise? Clearly an LTI model is too simple.

Limit Cycle Model of Phase Noise

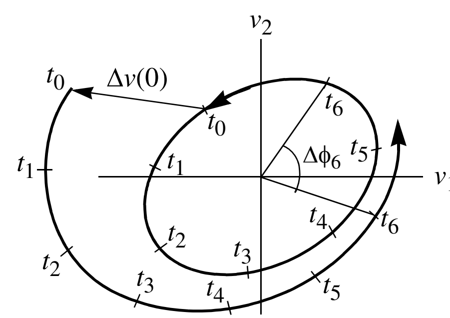


- The figure above shows that the non-linearity in the oscillator tends to reject AM noise.
- The noise is very small, so why is a linear model not valid?
- There are two reasons for this. First, the transfer function that the noise sees is a periodically time-varyng function (similar to a mixer).
- Second, we must be careful and correctly model the noise transfer characteristics from the source of noise (say thermal or flicker noise) top the phase of the oscillator (not amplitude), which is a non-linear process.

Linearized Phase Model

- The voltage perturbation to noise can be written as

$$\Delta v(t) = (1+\alpha(t))v(t+\frac{\phi(t)}{2\pi f_0})-v(t)$$



- The voltage $v(t)$ is the unperturbed oscillator voltage and α is the amplitude noise, and ϕ is the phase noise.
- The oscillator is able to reject the amplitude noise ($\alpha(t) \to 0$ as $t \to \infty$.
- On the other hand, a perturbation causes a permanent shift in the oscillator phase $\phi(t) \to \Delta\phi$ as $t \to \infty$.

Phase Noise Power Spectral Density

- This can be modeled as a step function impulse repsonse for a disturbance $n(t)$

$$\phi(t) = \int_{-\infty}^{\infty} u(t-\tau)n(\tau)d\tau = \int_{-\infty}^{t} n(\tau)d\tau$$

  or the power spectral density of the noise is

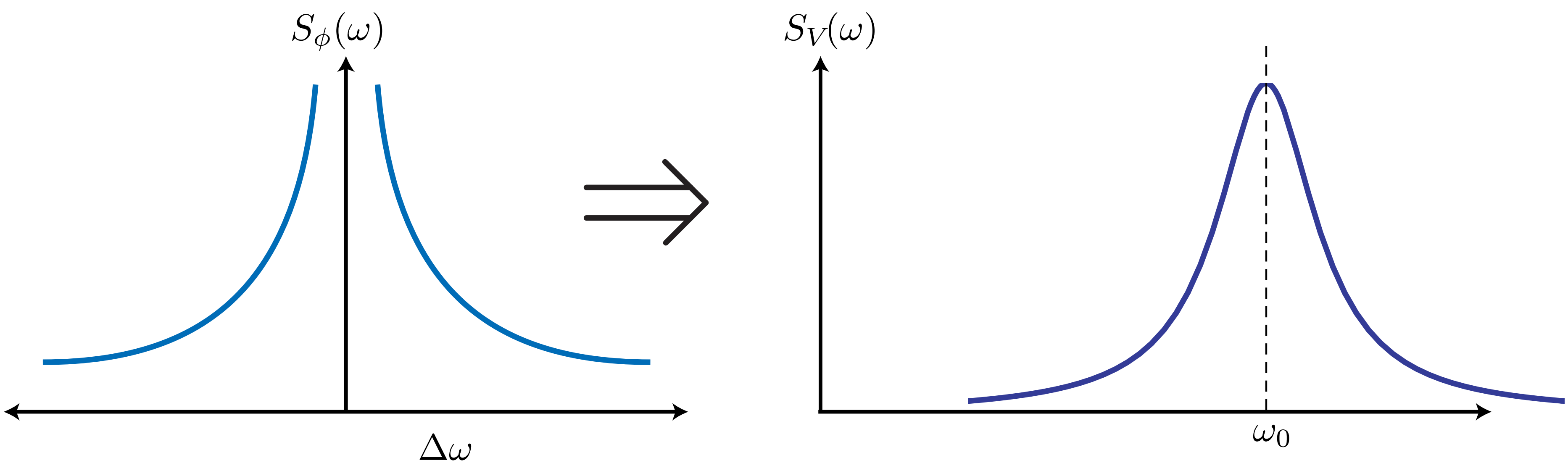
$$S_\phi(\Delta f) = \frac{S_n(\Delta f)}{(2\pi\Delta f)^2}$$

- If we linearize the equation for $v(t)$ for observation times that are short, then

$$\Delta v = \frac{dv}{dt}\frac{\Delta\phi(t)}{2\pi f_0}$$

- This says simply that phase perturbations that are tangential to the oscillator “limit cycle” cuase a deviation. A complete understanding of this requires Floquet theory to obtain the correct “tangential” and “perpundicular” directions.

Phase Noise versus Voltage Noise



- While the phase noise is *unbounded*, the output voltage is bounded. This is because the sinusoid is a bounded function and so the output voltage spectrum flattens around the carrier. In fact, if we assume that the phase is a Brownian noise process, the spectrum is computed to be a Lorentzian.

Oscillator Ideal Model

- Consider a simple LCR tank and a noiseless “energy restorer” circuit in parallel which sustains the oscillation.
- The energy stored in the tank is given by the peak voltage across the capacitor

$$E_{\text{stored}} = \frac{1}{2} C V_{pk}^2 = C \overline{v_{rms}^2}$$

Resonator Noise Power

- The total mean-square noise voltage is found by integrating the resistor's thermal noise density over the noise bandwidth of the RLC resonator

$$\overline{v_n^2} = 4kTR\int_0^\infty \left|\frac{|Z(f)|}{R}\right|^2 df = 4kTR \cdot \frac{1}{4RC} = \frac{kT}{C}$$

- The noise-to-signal ratio is therefore given by

$$\frac{N}{S} = R\frac{\overline{v_n^2}}{\overline{v_{rms}^2}} = \frac{kT}{E_{\mathrm{stored}}}$$

- By definition, the quality factor of the tank is given by

$$Q = \frac{\omega E_{\mathrm{stored}}}{P_{diss}}$$

Phase Noise of Oscillator

- We can now write the noise-to-signal power as

$$\frac{N}{S} = \frac{\omega k T}{Q P_{diss}}$$

- Even from this simple relation, we see that increasing the Q directly benefits the SNR.
- This applies to an ideal oscillator where the only source of noise is the tank and the tank losses are compensated by a noiseless generator (which does not load the circuit).

Phase Noise of "Real" Oscillator

- In a real oscillator, even if the energy restoring element is noiseless, it still presents negative resistance to the circuit which must be taken into account. At a small offset from resonance, the impedance of the LC tank is given by

$$Z(\omega_0 + \Delta\omega) \approx j \cdot \frac{\omega_0 L}{2\frac{\Delta\omega}{\omega_0}}$$

- Substituting that $Q = 1/(G\omega_0 L)$

$$|Z(\omega_0 + \Delta\omega)| = \frac{1}{G} \cdot \frac{\omega_0}{2Q\Delta\omega}$$

$$\frac{\overline{v_n^2}}{\Delta f} = \frac{\overline{i_n^2}}{\Delta f} \cdot |Z|^2 = 4kTR\left(\frac{\omega_0}{2Q\Delta\omega}\right)^2$$

Phase Noise Expression

$$L(\Delta\omega) = 10\log\left[\frac{2kT}{P_{sig}} \cdot \left(\frac{\omega_0}{2Q\Delta\omega}\right)^2\right]$$

- Notice that only half of the noise is attributed to phase noise. This is due to a non-rigorous argument that the noise partitions to FM and AM noise and therefore only half of the noise contributes to the phase noise.
- The right hand side shows that the phase noise drops like $1/f^2$, an experimentally observed fact in a region of the spectrum. It's also clear that the Q factor is the key factor in determining the phase noise level.

Leeson's Phase Noise Model

- Leeson modified the above noise model to account for several experimentally observed phenomena, including a $1/f^3$ region and a flat region in the phase noise as shown above.

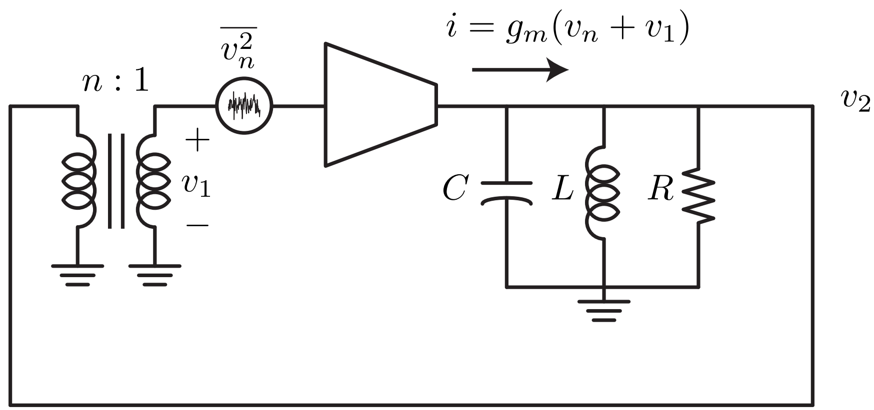
$$L(\Delta\omega) = 10\log\left[\frac{2FkT}{P_{sig}} \cdot \left(1 + \left(\frac{\omega_0}{2Q\Delta\omega}\right)^2\right)\left(1 + \frac{\Delta\omega_{1/f^3}}{|\Delta\omega|}\right)\right]$$

Leeson's Model Discussion

- In Leeson's model, the factor F is a fitting parameter rather than arising from any physical concepts. It's tempting to call this the oscillator "noise figure", but this is misleading.
- Leeson also assumed that the $1/f^3$ and $1/f^2$ corner occurred precisely at the $1/f$ corner of the device. In measurements, this is not always the case.
- Although this equation is very intuitive and simple to use, it's difficult to gain insight beyond increase P_{sig} and increase Q! This equation is the foundation for the oscillator FOM:

$$FOM = L(f_m) - 20\log\left[\frac{f_0}{f_m}\right] + 10\log(P_{diss})$$

LTI Analysis of Oscillator



- Consider a simple LTI analysis of the oscillator with a noise voltage v_n. An active device is assumed to pump energy into the tank through positive feedback. We have

$$v_2 = g_m(v_1 + v_n)Z_T = g_m\left(\frac{v_2}{n} + v_n\right)Z_T$$

$$v_2\left(1 - \frac{g_m Z_T}{n}\right) = g_m Z_T v_n$$

Noise Analysis

- Continuing to simplify the above results

$$v_2 = \frac{g_m Z_T v_n}{1 - \frac{g_m Z_T}{n}} = \frac{g_m R v_n}{\frac{R}{Z_T} - \frac{g_m R}{n}}$$

or

$$v_2 = v_n \frac{g_m R}{1 - \frac{g_m R}{n} + jBR}$$

- The reactive term B can be simplified at a small offset $\delta\omega$ from the resonance ω_0

$$B = \frac{1}{j(\omega_0 + \delta\omega)L} + j(\omega_0 + \delta\omega)C$$

Simplification Near Resonance

- Now comes the approximation

$$B \approx \frac{1}{j\omega_0 L}\left(1 - \frac{\delta\omega}{\omega_0}\right) + j(\omega_0 + \delta\omega)C$$

$$= j\delta\omega C - \frac{\delta\omega/\omega_0}{j\omega_0 L} = 2j\delta\omega C$$

- where ${\omega_0}^2 = 1/(LC)$. Using the notation $A_\ell = g_m R/n$

$$v_2 = v_n \frac{nA_\ell}{(1 - A_\ell) + j2\delta\omega RC}$$

$$v_{2,rms}^2 = \overline{v_n^2}\frac{n^2 {A_\ell}^2}{(1 - A_\ell)^2 + 4\delta{\omega}^2 R^2 C^2}$$

Oscillator Power

- If we now observe that the total power of the oscillator is fixed we have

$$P = \frac{v_{2,rms}^2}{R} = \frac{1}{R}\overline{v_n^2}\int_{-\infty}^{\infty}\frac{n^2 {A_\ell}^2}{(1-A_\ell)^2 + 4\delta\omega^2 R^2 C^2} d(\delta\omega)$$

- This integral is closed since it's in the known form

$$\int_{-\infty}^{\infty}\frac{dx}{1+a^2x^2} = \frac{\pi}{a}$$

$$P = \frac{\overline{v_n^2}}{R}\frac{{A_\ell}^2}{(1-A_\ell)^2}\frac{\pi(1-A_\ell)n^2}{2RC} = \frac{\overline{v_n^2}n^2}{R}\frac{\pi}{2}\frac{1}{RC}\frac{{A_\ell}^2}{(1-A_\ell)}$$

- Since $P = P_{osc}$, we can solve for A_ℓ.

Non-unity Loop Gain

- Since P_{osc} is finite, $A_\ell \neq 1$ but it's really close to unity

$$P_{osc}(1 - A_\ell) = \frac{\overline{v_n^2} n^2}{R} \frac{\pi}{2} \frac{1}{RC} {A_\ell}^2$$

- Since $A_\ell \approx 1$

$$(1 - A_\ell) = \frac{\frac{\overline{v_n^2}}{R}}{P_{osc}} \underbrace{\frac{\pi}{2} \frac{1}{RC}}_{\Delta f_{RC}}$$

- Since we integrated over negative frequencies, the noise voltage is given by

$$\overline{v_n^2} = 2kTR_{eff}$$

Magnitude of A_ℓ

- But since $\overline{v_n^2}/R$ over the equivalent bandwidth Δf_{RC} is much small than P_{osc}, we expect that

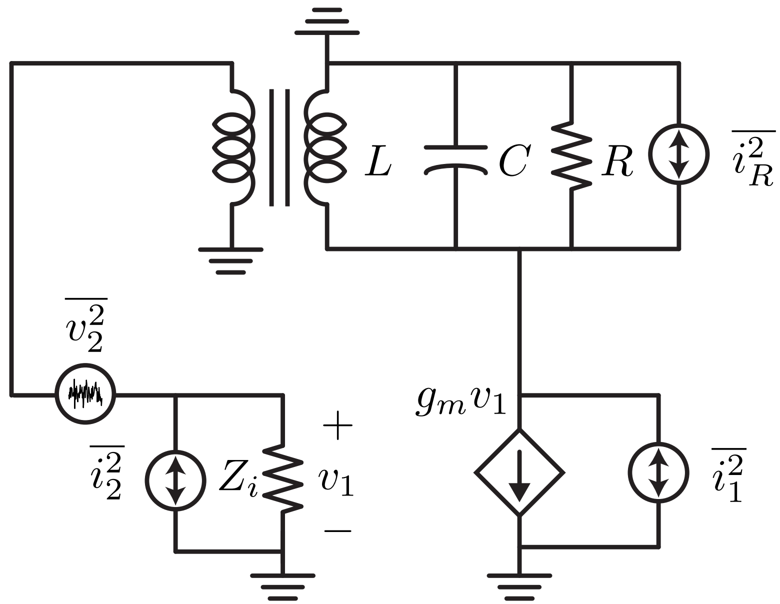
$$(1 - A_\ell) = \frac{\frac{\overline{v_n^2}}{R}}{P_{osc}} \Delta f_{RC} = \epsilon$$

  or

$$A_\ell = 1 - \epsilon$$

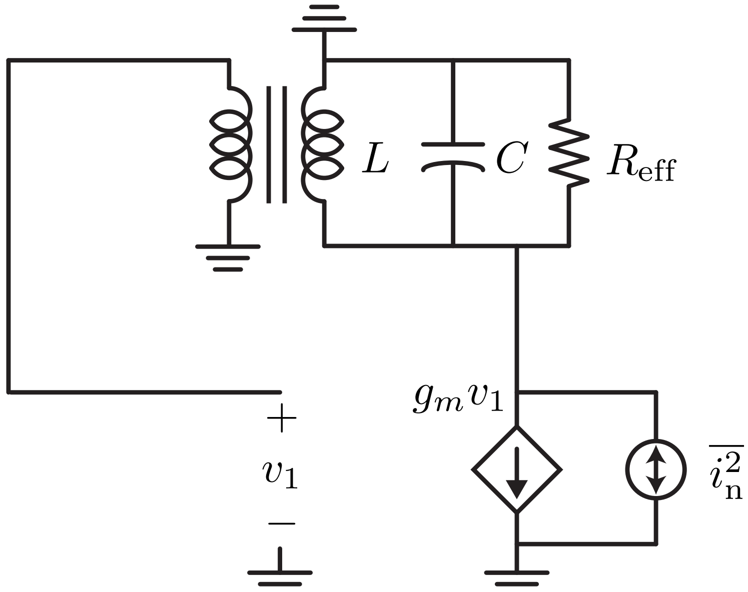
- The LTI interpretation is that the amplifier has positive feedback and it limits on it's own noise. The loop gain is nearly unity but just below so it's "stable".

Standard Oscillator LTI Analysis



- The above equivalent circuit includes the “drain” noise $\overline{i_1^2}$, the load noise $\overline{i_R^2}$, and an input voltage/current noise $\overline{v_2^2}$ and $\overline{i_2^2}$.

Equivalent Noise Model



- All the noise sources can be moved to the output by an appropriate transformation

$$\overline{i_n^2} = \overline{i_1^2} + \frac{\overline{i_2^2}}{n^2} + \overline{v_n^2}\left(g_m - \frac{1}{Z_i}\right)^2 + \overline{i_R^2}$$

LTI Noise Analysis

- The output voltage is given by

$$v_o = -(g_m v_1 + i_n) Z_T$$

since

$$v_1 = \frac{-v_o}{n}$$

we have

$$v_1 = \frac{g_m Z_T}{n} v_o - i_n Z_T$$

$$v_o = \frac{-i_n Z_T}{1 - \frac{g_m Z_T}{n}}$$

Tank Near Resonance

- The tank impedance can be put into this form

$$Z_T = \frac{1}{\frac{1}{R_1} + j\omega C + \frac{1}{j\omega L}} = \frac{R_1}{1 + j\frac{\omega}{\omega_0}Q + \frac{1}{j\omega}\omega_0 Q}$$

- Where the loaded tank $Q = R_1/(\omega_0 L) = \omega_0 R_1 C$

$$Z_T = \frac{R_1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

- If $\omega = \omega_0 + \delta\omega$ and $\delta\omega \ll \omega_0$

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \approx \frac{2\delta\omega}{\omega_0}$$

Transfer Near Resonance

- We now have that

$$Z_T(\omega_0 + \delta\omega) \approx \frac{R_1}{1 + j2Q\frac{\delta\omega}{\omega_0}}$$

- This allows to write the output voltage as

$$v_o = -i_n \frac{Z_T}{1 - \frac{g_m R_1}{n} \frac{1}{1+j2Q\frac{\delta\omega}{\omega_0}}} = -i_n \frac{R_1}{\left(1 - \frac{g_m R_1}{n}\right) + j2Q\frac{\delta\omega}{\omega_0}}$$

- Now it's time to observe that $A_\ell = \frac{g_m R_1}{n}$ is the initial loop gain. If we assume that $A_\ell \leq 1$, then the circuit is a high gain positive feedback amplifier.

Lorentzian Spectrum

- The power spectrum of v_o is given by

$$v_o^2 = \overline{i_n^2} \frac{R_1^2}{\left(1 - \frac{g_m R_1}{n}\right)^2 + 4Q^2 \frac{\delta\omega^2}{{\omega_0}^2}}$$

- This has a Lorentzian shape for white noise. For offsets frequencies of interest

$$4Q^2 \frac{\delta\omega^2}{{\omega_0}^2} \gg \left(1 - \frac{g_m R_1}{n}\right)^2$$

- Thus a characteristic $\delta\omega^2$ roll-off with offset.

Noise at Offsets

- The spectrum normalized to the peak is given by

$$\left(\frac{v_o}{V_o}\right)^2 \approx \frac{\overline{i_n^2} R_1^2}{V_o^2} \left(\frac{\omega_0}{\delta\omega}\right)^2 \frac{1}{4Q^2}$$

- The above equation is in the form of Leeson's Equation. It compactly expresses that the oscillator noise is expressed as noise power over signal power (N/S), divided by Q^2 and dropping like $1/\delta\omega^2$.

Total Noise Power

- We can express the total noise power similar to before

$$V_o^2 = \int_{-\infty}^{\infty} v_o^2 d(\delta\omega)$$

$$= \frac{\overline{i_n^2} R_1^2}{(1 - A_\ell)^2} \int_{-\infty}^{\infty} \frac{d(\delta\omega)}{1 + 4Q^2 \left(\frac{\delta\omega}{\omega_0}\right)^2 \frac{1}{(1-A_\ell)^2}}$$

$$V_o^2 = \frac{\overline{i_n^2} R_1^2}{(1 - A_\ell)} \frac{\pi}{2} \frac{f_o}{Q}$$

Lorentzian Bandwidth

- We again interpret the amplitude of oscillation as the the noise power $\overline{i_n^2} R_1^2$ gained up by the positive feedback

$$(1 - A_\ell) = \frac{\overline{i_n^2} R_1^2}{V_o^2} \frac{\pi f_o}{2Q}$$

- The $3\,\mathrm{dB}$ bandwidth of the Lorentzian is found by

$$(1 - A_\ell) = 2Q \frac{f - f_o}{f_o} = \frac{2Q \Delta f}{f_o}$$

$$\Delta f = \frac{f_o}{2Q}(1 - A_\ell) = \frac{\overline{i_n^2} R_1^2}{V_o^2} \frac{\pi}{4Q^2} f_o$$

Example Bandwidth

- For example, take $\overline{i_n^2} = 10^{-22}\mathrm{A}^2/\mathrm{Hz}$, $f_o = 1\mathrm{GHz}$, $R_1 = 300\Omega$, $Q = 10$ and $V_o = 1\mathrm{V}$. This gives a $\Delta f = 0.07\mathrm{Hz}$.
- This is an extremely low bandwidth. This is why on the spectrum analyzer we don't see the peak of the waveform. For even modest offsets of $100 - 1000\mathrm{Hz}$, the $1/\delta\omega^2$ behavior dominates. But we do observe a $1/\delta\omega^3$ region.

Noise Corner Frequency

- Because the oscillator is really a time-varying system, we should consider the effects of noise folding. For instance, consider any low frequency noise in the system. Due to the pumping action of the oscillator, it will up-convert to the carrier frequency.
- In reality the pumping is not perfectly periodic due to the noise. But we assume that the process is *cyclostationary* to simplify the analysis.
- Since there is always $1/f$ noise in the system, we now see the origin of the $1/f^3$ region in the spectrum.

Noise due to Non-Linear Caps

- Another noise upconversion occurs through non-linear capacitors. This is particularly important on the VCO control line.
- Assume that $C_j = C_0 + K_C \Delta V_c$. Since the frequency is given by

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

- We see that $f_o = f_{oQ} + K_f \Delta V_c$. $K_f \approx 10 - 100\mathrm{MHz/V}$. The oscillation waveform is given by

$$V(t) = V_o \cos\left(\int 2\pi f_o dt\right)$$

Noise Sidebands

- For ΔV_c a tone at some offset frequency ω_m, we have

$$\Delta V_c = V_m \cos \omega_m t$$

- where $V_m = \sqrt{4kTR_c}\sqrt{2}\mathrm{V}/\sqrt{\mathrm{Hz}}$ due to noise. This produces noise sidebands

$$V(t) = V_o \cos\left(\omega_0 t + \sqrt{2}\sqrt{4kTR_c}\frac{K_f 2\pi}{\omega_m}\sin\omega_m t\right)$$

- For small noise

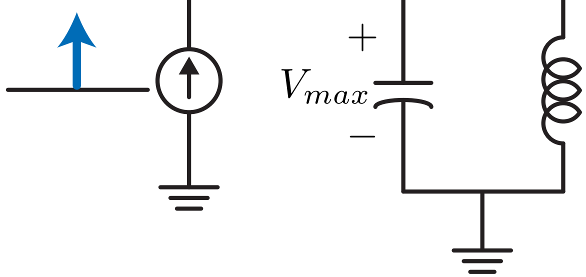
$$V(t) \approx V_o \cos(\omega_0 t) - V_o \sin(\omega_0 t)\sqrt{8kTR_c}\frac{K_f}{\omega_m}\sin(\omega_m t)$$

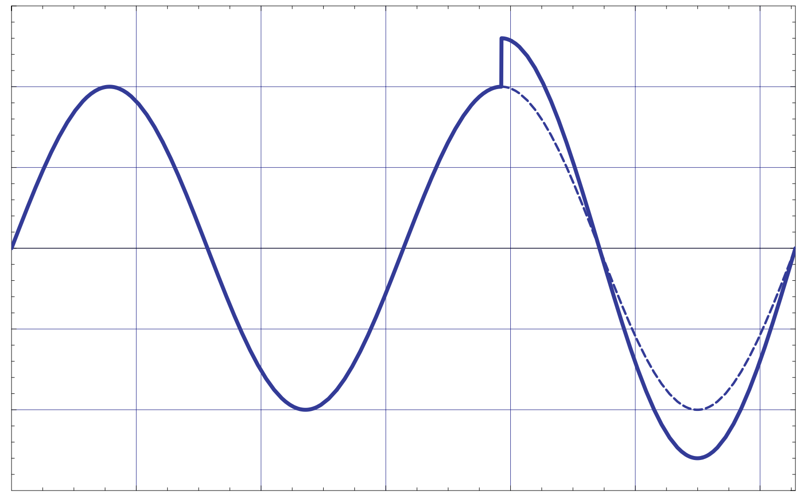
$$V(t) \approx \frac{V_o}{\omega_m} K_f \sqrt{8kTR_c}\frac{1}{2}\cos(\omega_o \pm \omega_m)$$

LTV Phase Noise Model

- The noise analysis thus far makes some very bad assumptions. Most importantly, we neglect the time-varying nature of the process. Every oscillator is a quasi-periodic system and the noise analysis should take this into account.
- The following noise model is due to Hajimiri/Lee. It begins with a simple thought experiment.
- Imagine injecting a current impulse into an LC tank at different times. We assume the LC tank is oscillating at the natural frequency.

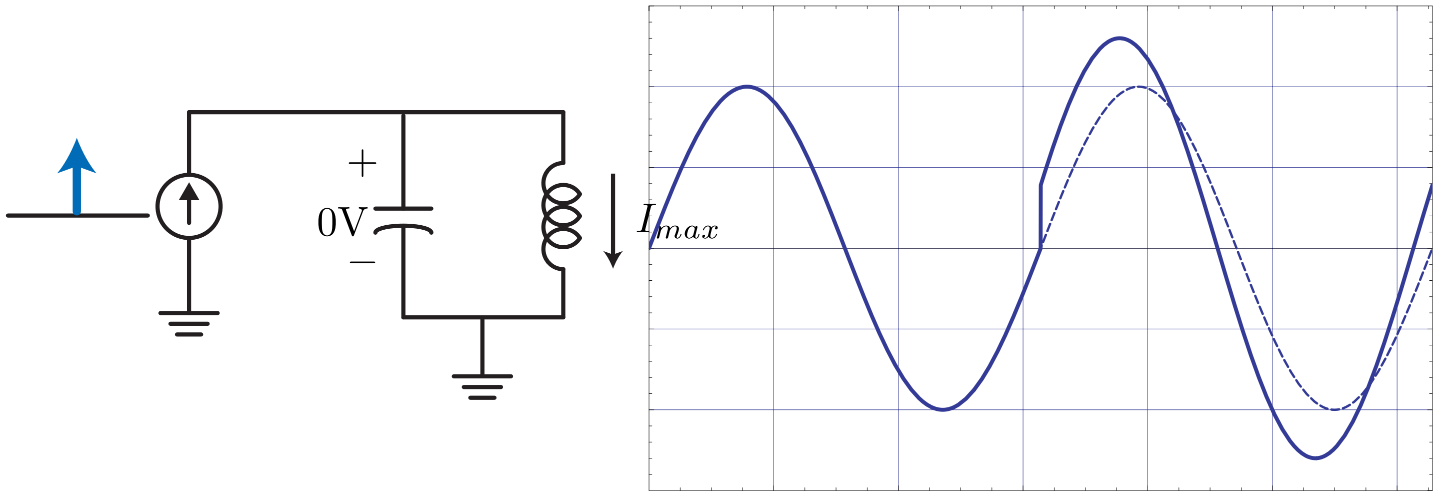
Injection at Peak Amplitude





- Since the impulse of current "sees" an open circuit across the inductor but a short circuit across the capacitor, all the current will flow into the capacitor, dumping a charge δq onto the capacitor plates.
- Note that if the injection occurs at the peak voltage amplitude, it will change the amplitude of oscillation.
- The phase of oscillation, though, is unaltered.

Injection at Zero-Crossing



- If the injection occurs at the waveform crossing, though, the change in amplitude also changes the phase of the oscillator.
- So we see the *sensitivity* of the oscillator to noise injection is a periodic function of time. There are points of zero sensitivity and points of peak sensitivity.

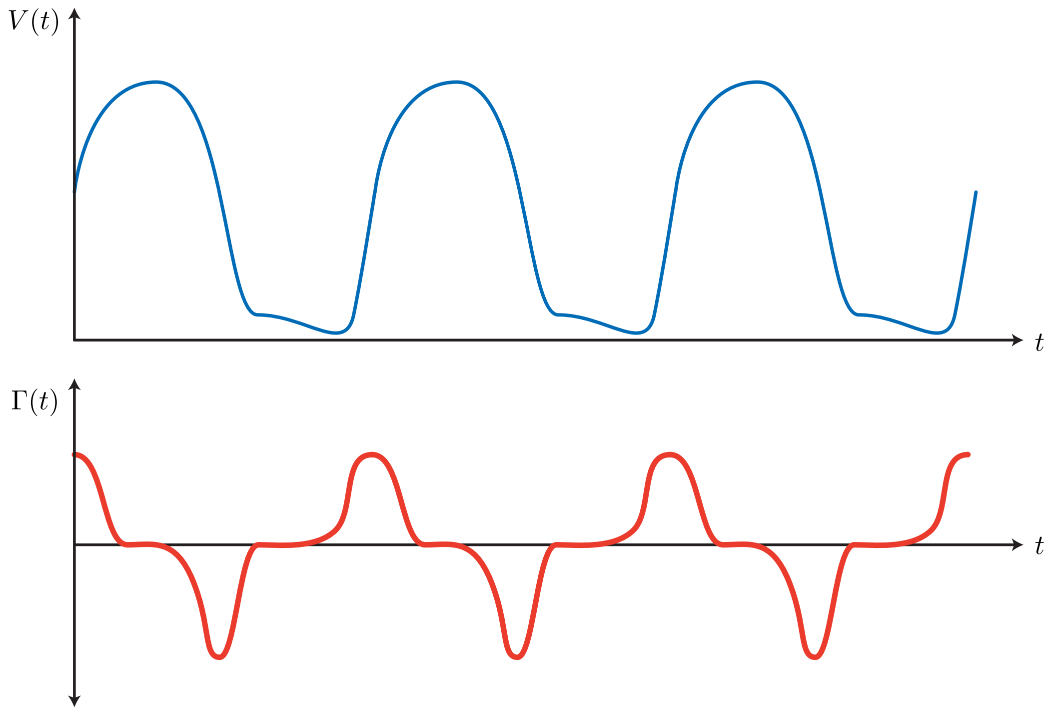
ISF Model

- The key observation (experimentally confirmed) is that the phase change is a linear function of the disturbance injection (for small injections). Therefore we write the impulse response in the following normalized form

$$h_\phi(t,\tau) = \frac{\Gamma(\omega_0\tau)}{q_{max}} u(t-\tau)$$

- The constant $q_{max} = CV_{peak}$ is simply a normalization constant, the peak charge in the oscillator. The response is zero until the system experiences the input (causality), but then it is assumed to occur instantaneously, leading the the step function response. The function $\Gamma(\omega_0\tau)$, the Impulse Sensitivity Function (ISF), is a periodic function of time, capturing the time varying periodic nature of the system.

Example Waveforms



This ISF can be estimated analytically or calculated from simulation

- Note a hypothetical system with output voltage waveform and ISF. As expected, the ISF peaks during “zero” crossings and is nearly zero at the peak of the waveform.

General Response

- For any deterministic input, we have the convolution integral

$$\phi(t) = \int_{-\infty}^{\infty} h_\phi(t,\tau) i(\tau) d\tau$$

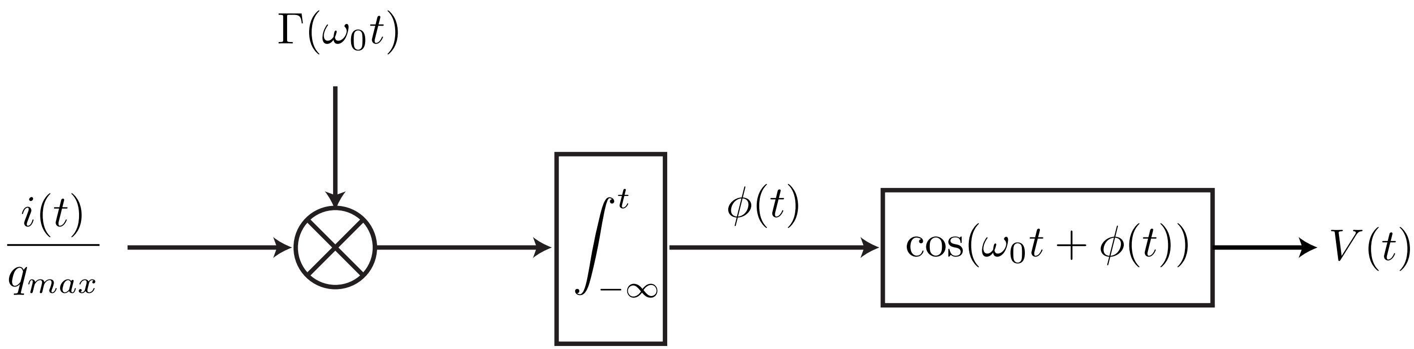
$$= \frac{1}{q_{max}} \int_{-\infty}^{t} \Gamma(\omega_0 \tau) i(\tau) d\tau$$

- Since the ISF function Γ is periodic

$$\Gamma(\omega_0 \tau) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$$

$$\phi(t) = \frac{1}{q_{max}} \left(\frac{c_0}{2} \int_{-\infty}^{t} i(\tau) d\tau + \sum_{n=1}^{\infty} c_n \int_{-\infty}^{t} \cos(n\omega_0 t) i(\tau) d\tau \right)$$

Phase Deviation to Output



- Graphically, we see that a noise disturbance creates a phase disturbance as shown above, and this in turn modulates the phase of the carrier. This last step is a non-linear process.
- The phase function $\phi(t)$ appears in an oscillator as a phase modulation of the carrier. Note that the phase itself is not (easily) observed directly.

Noise Sidebands

- The noise sidebands due to current noise at an offset $\Delta\omega$ from the m'th harmonic (including DC) is now calculated.

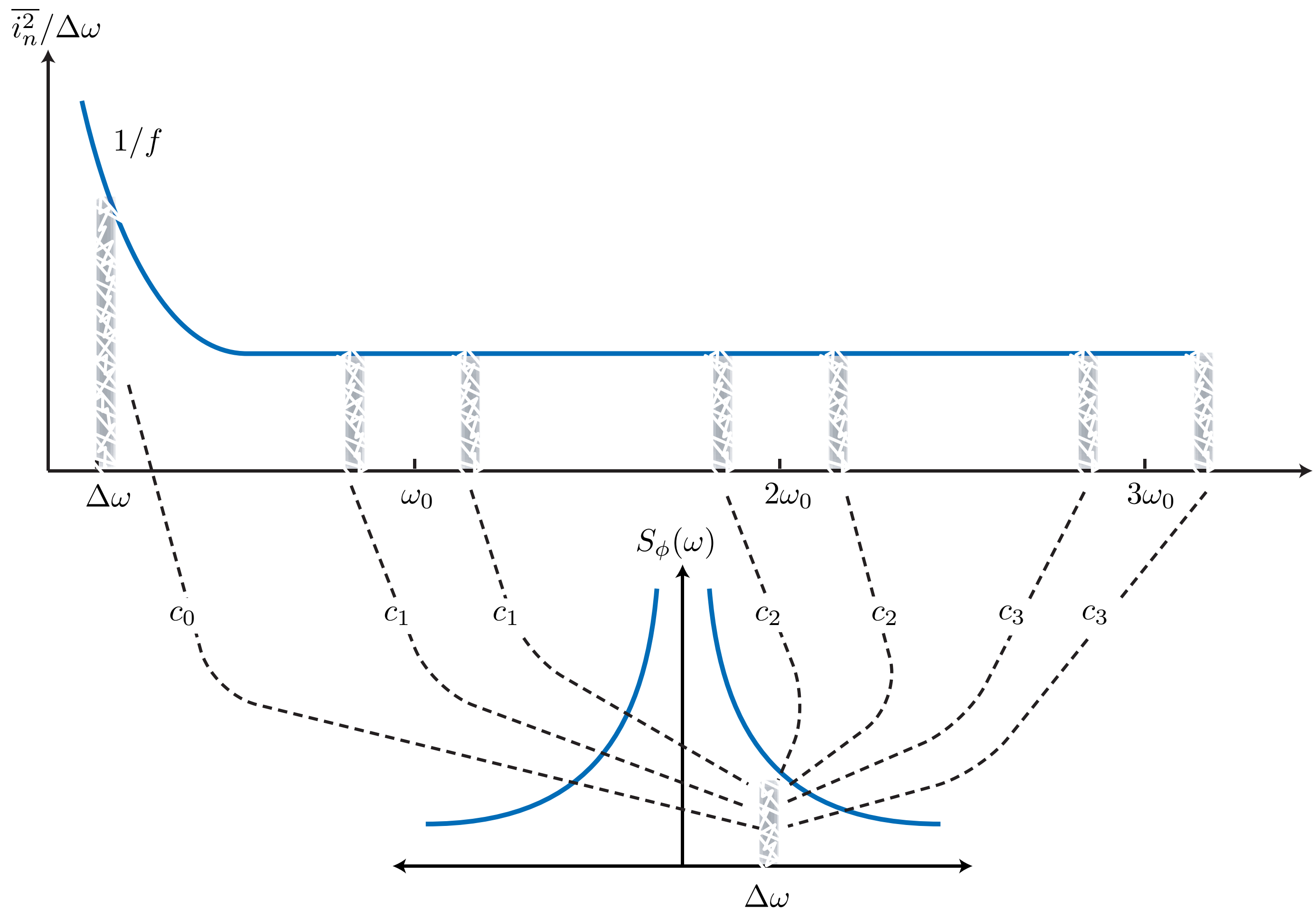
$$i(t) = I_m \cos(m\omega_0 + \Delta\omega)t$$

$$= I_m(\cos m\omega_0 t \cos \Delta\omega t - \sin m\omega_0 t \sin \Delta\omega t)$$

- If we insert this into the above integration, for small offsets $\Delta\omega \ll \omega_0$, we find (approximately) that all terms are orthogonal and integrate to zero except when $n = m$.
- The non-zero term integrates to give

$$\phi(t) \approx \frac{1}{2} c_m \frac{I_m \sin \Delta\omega t}{q_{max} \Delta\omega}$$

Graphical Interpretation



- We see that all noise a distance $\Delta\omega$ around all the harmonics, including DC, contributes to the phase noise. DC $1/f$ noise contributes to the $1/f^3$ region.

White Noise Expression

- We see that the noise power at offset $\Delta\omega$ is given by

$$P_{SBC}(\Delta\omega) \approx 10 \cdot \log \left(\frac{I_m c_m}{2 q_{max} \Delta\omega} \right)^2$$

- If the noise is white, then we get equal contribution from all sidebands

$$P_{SBC}(\Delta\omega) \approx 10 \cdot \log \left(\frac{\overline{i_n^2} \sum_{m=0}^{\infty} c_m^2}{4 q_{max}^2 \Delta\omega^2} \right)$$

Final Expression

- Parseval taught us that

$$\sum_{m=0}^{\infty} c_m^2 = \frac{1}{\pi}\int_0^{2\pi} |\Gamma(x)|^2 dx = 2\Gamma_{rms}^2$$

- This allows us to write the phase noise in the following form

$$P_{SBC}(\Delta\omega) \approx 10 \cdot \log\left(\frac{\overline{i_n^2}\Gamma_{rms}^2}{2q_{max}^2\Delta\omega^2}\right)$$

- Thus to minimize the phase noise we must minimize the RMS value of the ISF.

Cyclostationary Noise

- If we assume that the noise sources of the active devices can be modeled as a stationary noise multiplied by a periodic function

$$i_n(t) = i_{n0}(t) \cdot \alpha(\omega_0 t)$$

- We can absorb this into the ISF and all the previous results follow unaltered as long as we use Γ_{eff}.

$$\Gamma_{eff}(x) = \Gamma(x) \cdot \alpha(x)$$

$1/f$ Noise Upconversion

- It's interesting to note that the $1/f$ noise up-conversion process occurs through c_0. This is related to the "DC" value of the noise sensitivity function.
- The $1/f^3$ corner is given by

$$\Delta_{1/f^3} = \omega_{1/f} \cdot \frac{c_0^2}{4\Gamma_{rms}^2} = \omega_{1/f} \cdot \left(\frac{\Gamma_{dc}}{\Gamma_{rms}} \right)^2$$

- So in theory a "balanced" oscillator can achieve much better phase noise performance.

Amplitude Noise

- In a like manner, the response of an oscillator to noise to the output amplitude (rather than phase) can be described by a LTV model

$$h_A(t,\tau) = \frac{\Lambda(\omega_0 t)}{q_{max}} d(t-\tau)$$

- where $d(t-\tau)$ is a function that defines how the excess amplitude decays. Since there is an amplitude restoration mechanism in place, this perturbation decays to zero. Assume this process is described by a damped exponential decay

$$d(t-\tau) = e^{-\omega_0(t-\tau)/Q} \cdot u(t-\tau)$$

Amplitude Response

- The excess amplitude response to an arbitrary input current $i(t)$ is simply the convolution integral

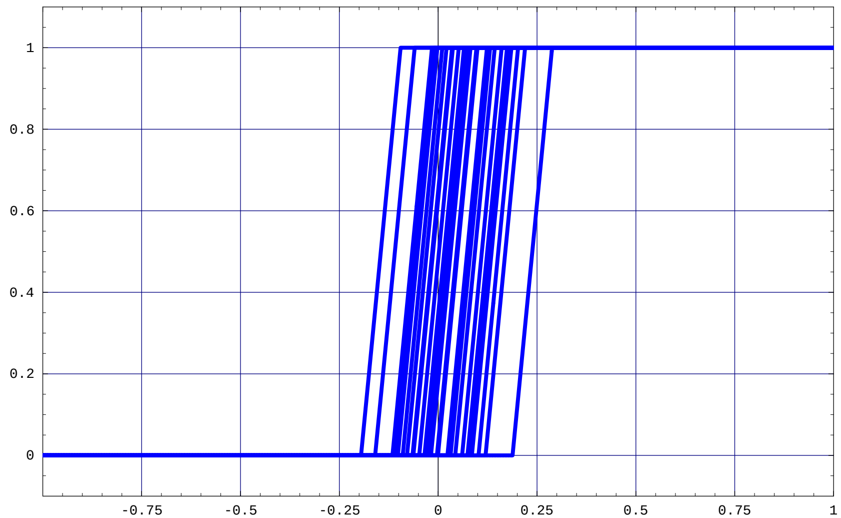
$$A(t) = \int_{-\infty}^{t} \frac{i(\tau)}{q_{max}} \Lambda(\omega_0 \tau) e^{-\omega_0 (t-\tau)/Q} d\tau$$

- If the current $i(t)$ is a white noise source, then the output spectrum is given by

$$L_{\text{amplitude}}(\Delta\omega) = \frac{\Lambda_{rms}^2}{q_{max}^2} \cdot \frac{\overline{i_n^2}/\Delta f}{2 \cdot \left(\frac{\omega_0^2}{Q^2} + \Delta\omega^2 \right)}$$

- The total noise is a sum of phase and amplitude noise. In general for close in noise, the phase term dominates.

Jitter



Jitter is the undesired fluctionations in the timing of a signal. Jitter arises due to phase noise. In other words, it is a time-domain perspective of the same general phenomena.

- Another way to see jitter is to observe the zero-crossing time of a periodic signal. Due to jitter, the zero-crossing time will vary slightly from the ideal location since the signal is not strictly periodic due to noise.
- Jitter is often specified by its peak-to-peak value or the RMS value. Note that for a Gaussian distribution of zero-crossings, the peak-to-peak value actually is unbounded so the RMS value is a more useful measure.

Jitter Computation

- Let $v_j(t) = v(t + j(t))$ be a waveform in the time which has jitter. Note that $j = \phi/(2\pi f_0)$, which shows that the jitter arises from phase noise.
- For example, a noisy voltage can create jitter as follows

$$v_n(t) = v(t) + n(t) = v(t + j(t)) = v(t) + \frac{dv(t)}{dt} j(t) + \cdots$$

which means that

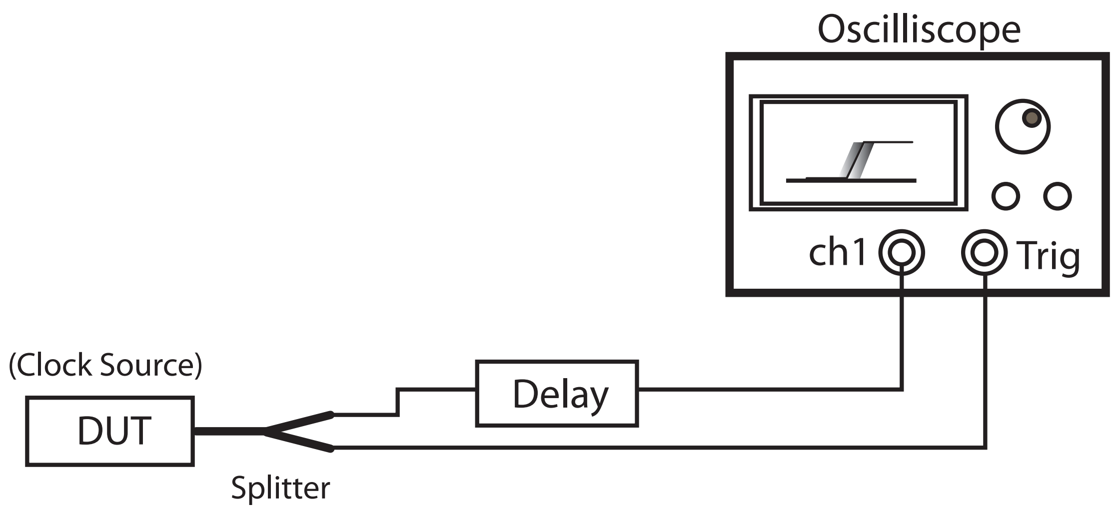
$$n(t) \approx \frac{dv(t)}{dt} j(t)$$

- This allows us to relate the variance of the jitter in terms of the noise.

$$\mathrm{var}(j(t_c)) \approx \frac{\mathrm{var}(n(t_c))}{\left(\frac{dv(t_c)}{dt}\right)^2}$$

where t_c is the expected time of the threshold crossing.

Jitter Measurement



- The above setup shows how to measure jitter on an oscilliscope.
- Most modern scopes have built-in functions for generating an eye-diagram, which can be used to estimate the statistics of the jitter.

Jitter Relation to Phase Noise

- Since jitter and phase noise are really two different ways of seeing the same thing, it's not surprising that you can calculate the jitter from the phase noise (but not the other way around).
- The variance of jitter can be computed from

$$\sigma_j = < \phi(t)^2 > = \int_{-\infty}^{+\infty} S_\phi(f) df$$

- Often the RMS jitter is quoted, which is just the square root of the above quantity. Furthermore, the jitter is normalized to the carrier freqeuncy.

$$J_{PER} = \frac{\phi(t)}{2\pi f_0}$$

so that

$$J_{PER,RMS} = \frac{\sqrt{< \phi(t)^2 >}}{2\pi f_0}$$

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- K. Kouznetsov and R. Meyer, “Phase noise in LC oscillators,” *IEEE J. Solid-State Circuits*, vol. 35, no. 8, pp. 1244-1248, Aug. 2000.