EECS 242

Two-Port Gain and Stability

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Input/Output Admittance

- The input and output impedance of a two-port will play an important role in our discussions. The stability and power gain of the two-port is determined by these quantities.
- In terms of y-parameters

$$Y_{in} = \frac{I_1}{V_1} = \frac{Y_{11}V_1 + Y_{12}V_2}{V_1} = Y_{11} + Y_{12}\frac{V_2}{V_1}$$

- The voltage gain of the two-port is given by solving the following equations

$$-I_2 = V_2 Y_L = -(Y_{21}V_1 + V_2 Y_{22})$$

$$\frac{V_2}{V_1} = \frac{-Y_{21}}{Y_L + Y_{22}}$$

- Note that for a simple transistor $Y_{21} = g_m$ and so the above reduces to the familiar $g_m R_o || R_L$.

Input/Output Admittance (cont)

- We can now solve for the input and output admittance

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}$$

$$Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}}$$

- Note that if $Y_{12} = 0$, then the input and output impedance are de-coupled

$$Y_{in} = Y_{11}$$

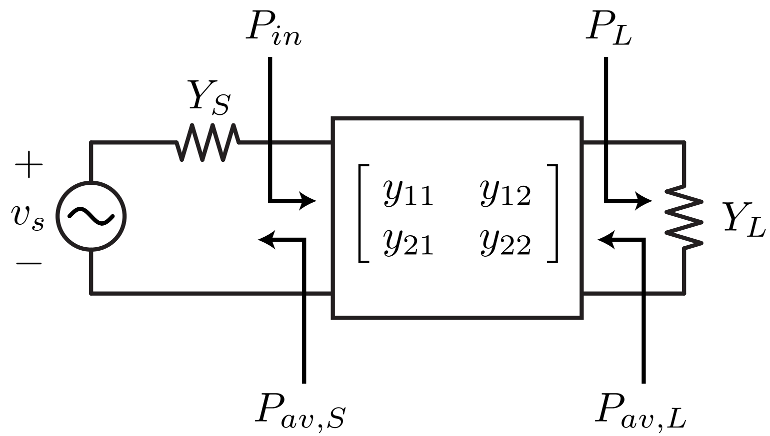
$$Y_{out} = Y_{22}$$

- But in general they are coupled and changing the load will change the input admittance.
- It's interesting to note the same formula derived above also works for the input/output impedance

$$Z_{in} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_L + Z_{22}}$$

- The same is true for the hybrid and inverse hybrid matrices.

Power Gain



- We can define power gain in many different ways. The *power gain* G_p is defined as follows

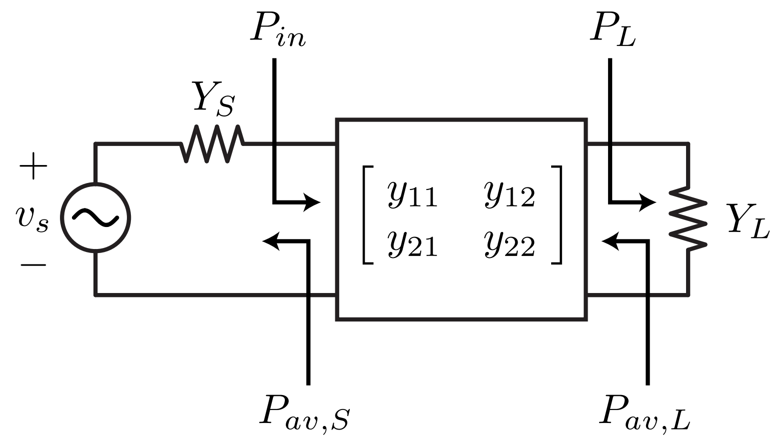
$$G_p = \frac{P_L}{P_{in}} = f(Y_L, Y_{ij}) \neq f(Y_S)$$

- We note that this power gain is a function of the load admittance Y_L and the two-port parameters Y_{ij}.
- The *available power gain* is defined as follows

$$G_a = \frac{P_{av,L}}{P_{av,S}} = f(Y_S, Y_{ij}) \neq f(Y_L)$$

- The available power from the two-port is denoted $P_{av,L}$ whereas the power available from the source is $P_{av,S}$.

Power Gain (cont)



- Finally, the *transducer gain* is defined by

$$G_T = \frac{P_L}{P_{av,S}} = f(Y_L, Y_S, Y_{ij})$$

- This is a measure of the efficacy of the two-port as it compares the power at the load to a simple conjugate match.

Derivation of Power Gain

- The power gain is readily calculated from the input admittance and voltage gain

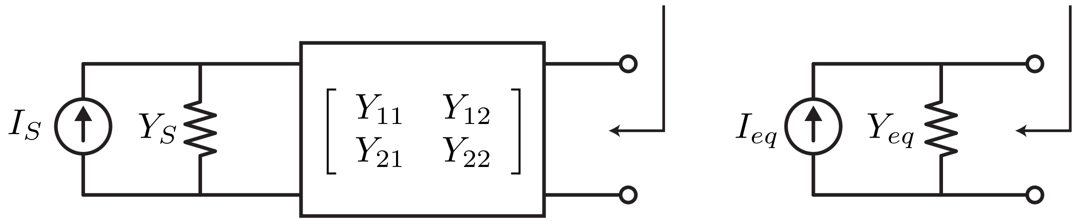
$$P_{in} = \frac{|V_1|^2}{2} \Re(Y_{in})$$

$$P_L = \frac{|V_2|^2}{2} \Re(Y_L)$$

$$G_p = \left| \frac{V_2}{V_1} \right|^2 \frac{\Re(Y_L)}{\Re(Y_{in})}$$

$$G_p = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \frac{\Re(Y_L)}{\Re(Y_{in})}$$

Derivation of Available Gain



- To derive the available power gain, consider a Norton equivalent for the two-port where

$$I_{eq} = I_2 = Y_{21}V_1 = \frac{Y_{21}}{Y_{11} + Y_S} I_S$$

- The Norton equivalent admittance is simply the output admittance of the two-port

$$Y_{eq} = Y_{22} - \frac{Y_{21}Y_{12}}{Y_{11} + Y_S}$$

- The available power at the source and load are given by

$$P_{av,S} = \frac{|I_S|^2}{8\Re(Y_S)} \qquad P_{av,L} = \frac{|I_{eq}|^2}{8\Re(Y_{eq})}$$

$$G_a = \frac{|Y_{21}|^2}{|Y_{11} + Y_S|^2} \frac{\Re(Y_S)}{\Re(Y_{eq})}$$

Transducer Gain Derivation

- The transducer gain is given by

$$G_T = \frac{P_L}{P_{av,S}} = \frac{\frac{1}{2}\Re(Y_L)|V_2|^2}{\frac{|I_S|^2}{8\Re(Y_S)}} = 4\Re(Y_L)\Re(Y_S)\left|\frac{V_2}{I_S}\right|^2$$

- We need to find the output voltage in terms of the source current. Using the voltage gain we have and input admittance we have

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right|$$

$$I_S = V(Y_S + Y_{in})$$

$$\left|\frac{V_2}{I_S}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right| \frac{1}{|Y_S + Y_{in}|}$$

$$|Y_S + Y_{in}| = \left|Y_S + Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right|$$

Transducer Gain (cont)

- We can now express the output voltage as a function of source current as

$$\left|\frac{V_2}{I_S}\right|^2 = \frac{|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

- And thus the transducer gain

$$G_T = \frac{4\Re(Y_L)\Re(Y_S)|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

- It's interesting to note that *all* of the gain expression we have derived are in the exact same form for the impedance, hybrid, and inverse hybrid matrices.

Comparison of Power Gains

- In general, $P_L \leq P_{av,L}$, with equality for a matched load. Thus we can say that

$$G_T \leq G_a$$

- The maximum transducer gain as a function of the load impedance thus occurs when the load is conjugately matched to the two-port output impedance

$$G_{T,max,L} = \frac{P_L(Y_L = Y_{out}^*)}{P_{av,S}} = G_a$$

- Likewise, since $P_{in} \leq P_{av,S}$, again with equality when the the two-port is conjugately matched to the source, we have

$$G_T \leq G_p$$

- The transducer gain is maximized with respect to the source when

$$G_{T,max,S} = G_T(Y_{in} = Y_S^*) = G_p$$

Bi-Conjugate Match

- When the input and output are simultaneously conjugately matched, or a *bi-conjugate match* has been established, we find that the transducer gain is maximized with respect to the source and load impedance

$$G_{T,max} = G_{p,max} = G_{a,max}$$

- This is thus the recipe for calculating the optimal source and load impedance in to maximize gain

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} = Y_S^*$$

$$Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}} = Y_L^*$$

- Solution of the above four equations (real/imag) results in the optimal $Y_{S,opt}$ and $Y_{L,opt}$.

Calculation of Optimal Source/Load

- Another approach is to simply equate the partial derivatives of G_T with respect to the source/load admittance to find the maximum point

$$\frac{\partial G_T}{\partial G_S} = 0 \qquad \frac{\partial G_T}{\partial G_L} = 0$$

$$\frac{\partial G_T}{\partial B_S} = 0 \qquad \frac{\partial G_T}{\partial B_L} = 0$$

- Again we have four equations. But we should be smarter about this and recall that the maximum gains are all equal. Since G_a and G_p are only a function of the source or load, we can get away with only solving two equations. For instance

$$\frac{\partial G_a}{\partial G_S} = 0 \qquad \frac{\partial G_a}{\partial B_S} = 0$$

- This yields $Y_{S,opt}$ and by setting $Y_L = Y_{out}^*$ we can find the $Y_{L,opt}$.
- Likewise we can also solve

$$\frac{\partial G_p}{\partial G_L} = 0 \qquad \frac{\partial G_p}{\partial B_L} = 0$$

- And now use $Y_{S,opt} = Y_{in}^*$.

Optimal Power Gain Derivation

- Let's outline the procedure for the optimal power gain. We'll use the power gain G_p and take partials with respect to the load. Let

$$Y_{jk} = m_{jk} + jn_{jk}$$

$$Y_L = G_L + jX_L$$

$$Y_{12}Y_{21} = P + jQ = Le^{j\phi}$$

$$G_p = \frac{|Y_{21}|^2}{D} G_L$$

$$\Re\left(Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right) = m_{11} - \frac{\Re(Y_{12}Y_{21}(Y_L + Y_{22})^*)}{|Y_L + Y_{22}|^2}$$

$$D = m_{11}|Y_L + Y_{22}|^2 - P(G_L + m_{22}) - Q(B_L + n_{22})$$

$$\frac{\partial G_p}{\partial B_L} = 0 = -\frac{|Y_{21}|^2 G_L}{D^2}\frac{\partial D}{\partial B_L}$$

Optimal Load (cont)

- Solving the above equation we arrive at the following solution

$$B_{L,opt} = \frac{Q}{2m_{11}} - n_{22}$$

- In a similar fashion, solving for the optimal load conductance

$$G_{L,opt} = \frac{1}{2m_{11}}\sqrt{(2m_{11}m_{22} - P)^2 - L^2}$$

- If we substitute these values into the equation for G_p (lot's of algebra ...), we arrive at

$$G_{p,max} = \frac{|Y_{21}|^2}{2m_{11}m_{22} - P + \sqrt{(2m_{11}m_{22} - P)^2 - L^2}}$$

Final Solution

- Notice that for the solution to exists, G_L must be a real number. In other words

$$(2m_{11}m_{22} - P)^2 > L^2$$

$$(2m_{11}m_{22} - P) > L$$

$$K = \frac{2m_{11}m_{22} - P}{L} > 1$$

- This factor K plays an important role as we shall show that it also corresponds to an unconditionally stable two-port. We can recast all of the work up to here in terms of K

$$Y_{S,opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{22})}$$

$$Y_{L,opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{11})}$$

$$G_{p,max} = G_{T,max} = G_{a,max} = \frac{Y_{21}}{Y_{12}} \frac{1}{K + \sqrt{K^2 - 1}}$$

Maximum Gain

- The maximum gain is usually written in the following insightful form

$$G_{max} = \frac{Y_{21}}{Y_{12}}(K - \sqrt{K^2 - 1})$$

- For a reciprocal network, such as a passive element, $Y_{12} = Y_{21}$ and thus the maximum gain is given by the second factor

$$G_{r,max} = K - \sqrt{K^2 - 1}$$

- Since $K > 1$, $|G_{r,max}| < 1$. The reciprocal gain factor is known as the efficiency of the reciprocal network.
- The first factor, on the other hand, is a measure of the non-reciprocity.

Unilateral Maximum Gain

- For a unilateral network, the design for maximum gain is trivial. For a bi-conjugate match

$$Y_S = Y_{11}^*$$

$$Y_L = Y_{22}^*$$

$$G_{T,max} = \frac{|Y_{21}|^2}{4m_{11}m_{22}}$$

Stability of a Two-Port

- A two-port is unstable if the admittance of either port has a negative conductance for a passive termination on the second port. Under such a condtion, the two-port can oscillate.
- Consider the input admittance

$$Y_{in} = G_{in} + jB_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_L}$$

- Using the following definitions

$$Y_{11} = g_{11} + jb_{11} \qquad Y_{12}Y_{21} = P + jQ = L\angle\phi$$

$$Y_{22} = g_{22} + jb_{22} \qquad Y_L = G_L + jB_L$$

- Now substitute real/imag parts of the above quantities into Y_{in}

$$Y_{in} = g_{11} + jb_{11} - \frac{P + jQ}{g_{22} + jb_{22} + G_L + jB_L}$$

$$= g_{11} + jb_{11} - \frac{(P + jQ)(g_{22} + G_L - j(b_{22} + B_L))}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2}$$

Input Conductance

- Taking the real part, we have the input conductance

$$\Re(Y_{in}) = G_{in} = g_{11} - \frac{P(g_{22}+G_L)+Q(b_{22}+B_L)}{(g_{22}+G_L)^2+(b_{22}+B_L)^2}$$

$$= \frac{(g_{22}+G_L)^2+(b_{22}+B_L)^2 - \frac{P}{g_{11}}(g_{22}+G_L) - \frac{Q}{g_{11}}(b_{22}+B_L)}{D}$$

- Since $D > 0$ if $g_{11} > 0$, we can focus on the numerator. Note that $g_{11} > 0$ is a requirement since otherwise oscillations would occur for a short circuit at port 2.
- The numerator can be factored into several positive terms

$$N = (g_{22}+G_L)^2+(b_{22}+B_L)^2 - \frac{P}{g_{11}}(g_{22}+G_L) - \frac{Q}{g_{11}}(b_{22}+B_L)$$

$$= \left(G_L + \left(g_{22} - \frac{P}{2g_{11}}\right)\right)^2 + \left(B_L + \left(b_{22} - \frac{Q}{2g_{11}}\right)\right)^2 - \frac{P^2+Q^2}{4g_{11}^2}$$

Input Conductance (cont)

- Now note that the numerator can go negative only if the first two terms are smaller than the last term. To minimize the first two terms, choose $G_L = 0$ and $B_L = -\left(b_{22} - \frac{Q}{2g_{11}}\right)$ (reactive load)

$$N_{min} = \left(g_{22} - \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2}$$

- And thus the above must remain positive, $N_{min} > 0$, so

$$\left(g_{22} + \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} > 0$$

$$g_{11}g_{22} > \frac{P + L}{2} = \frac{L}{2}(1 + \cos\phi)$$

Linvill/Llewellyn Stability Factors

- Using the above equation, we define the Linvill stability factor

$$L < 2g_{11}g_{22} - P$$

$$C = \frac{L}{2g_{11}g_{22} - P} < 1$$

- The two-port is stable if $0 < C < 1$.
- It's more common to use the inverse of C as the stability measure

$$\frac{2g_{11}g_{22} - P}{L} > 1$$

- The above definition of stability is perhaps the most common

$$K = \frac{2\Re(Y_{11})\Re(Y_{22}) - \Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} > 1$$

- The above expression is identical if we interchnage ports 1/2. Thus it's the general condition for stability.
- Note that $K > 1$ is the same condition for the maximum stable gain derived last lecture. The connection is now more obvious. If $K < 1$, then the maximum gain is infinity!

Stability From Another Perspective

- We can also derive stability in terms of the input reflection coefficient. For a general two-port with load Γ_L we have

$$v_2^- = \Gamma_L^{-1} v_2^+ = S_{21} v_1^+ + S_{22} v_2^+$$

$$v_2^+ = \frac{S_{21}}{\Gamma_L^{-1} - S_{22}} v_1^-$$

$$v_1^- = \left(S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}} \right) v_1^+$$

$$\Gamma = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}}$$

- If $|\Gamma| < 1$ for all Γ_L, then the two-port is stable

$$\Gamma = \frac{S_{11}(1 - S_{22}\Gamma_L) + S_{12} S_{21} \Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} + \Gamma_L (S_{21} S_{12} - S_{11} S_{22})}{1 - S_{22}\Gamma_L}$$

$$= \frac{S_{11} - \Delta \Gamma_L}{1 - S_{22}\Gamma_L}$$

Stability Circle

- To find the boundary between stability/instability, let's set $|\Gamma| = 1$

$$\left|\frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}\right| = 1$$

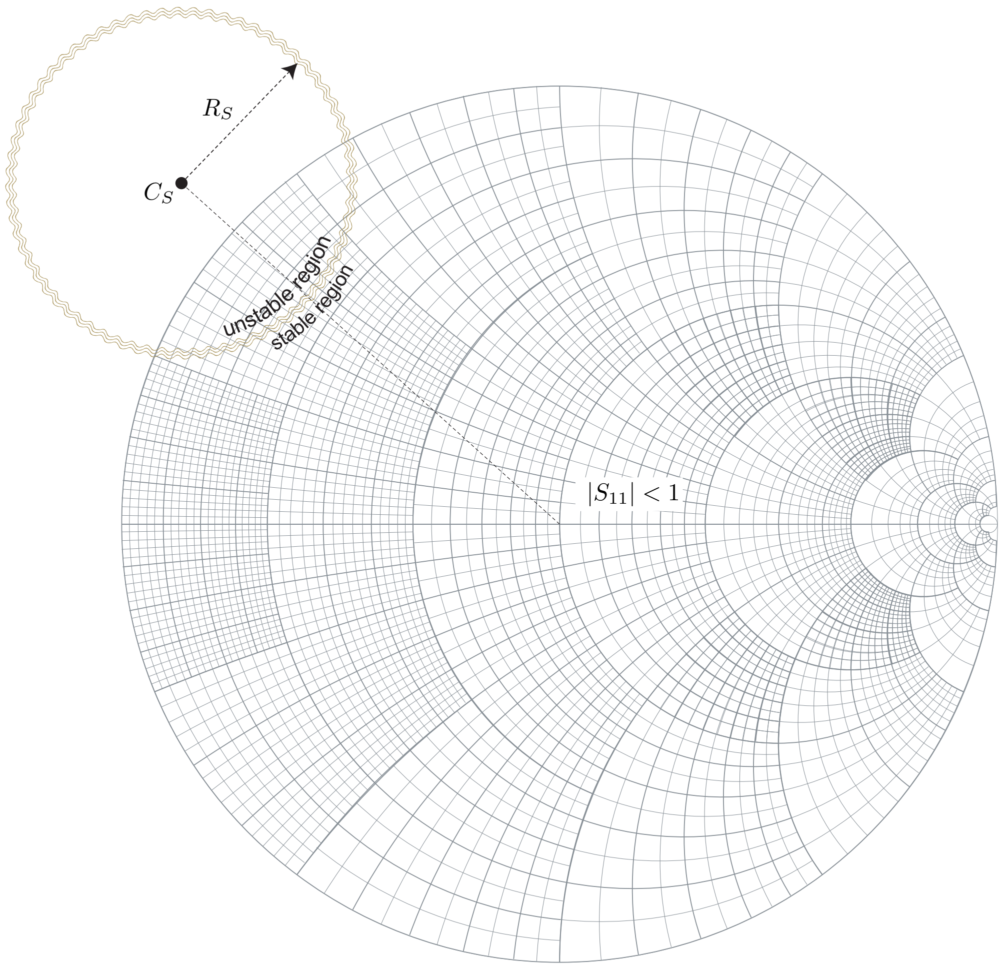
$$|S_{11} - \Delta\Gamma_L| = |1 - S_{22}\Gamma_L|$$

- After some algebraic manipulations, we arrive at the following equation

$$\left|\Gamma - \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2}\right| = \frac{|S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2}$$

- This is of course an equation of a circle, $|\Gamma - C| = R$, in the complex plane with center at C and radius R
- Thus a circle on the Smith Chart divides the region of instability from stability.

Example: Stability Circle



- In this example, the origin of the circle lies outside the stability circle but a portion of the circle falls inside the unit circle. Is the region of stability inside the circle or outside?
- This is easily determined if we note that if $\Gamma_L = 0$, then $\Gamma = S_{11}$. So if $S_{11} < 1$, the origin should be in the stable region. Otherwise, if $S_{11} > 1$, the origin should be in the unstable region.

Stability: Unilateral Case

- Consider the stability circle for a unilateral two-port

$$C_S = \frac{S_{11}^* - (S_{11}^* S_{22}^*) S_{22}}{|S_{11}|^2 - |S_{11} S_{22}|^2} = \frac{S_{11}^*}{|S_{11}|^2}$$

$$R_S = 0$$

$$|C_S| = \frac{1}{|S_{11}|}$$

- The cetner of the circle lies outside of the unit circle if $|S_{11}|$ < 1. The same is true of the load stability circle. Since the radius is zero, stability is only determined by the location of the center.
- If $S_{12} = 0$, then the two-port is unconditionally stable if $S_{11} < 1$ and $S_{22} < 1$.
- This result is trivial since

$$\Gamma_S \Big|_{S_{12}=0} = S_{11}$$

- The stability of the source depends only on the device and not on the load.

Mu Stability Test

- If we want to determine if a two-port is unconditionally stable, then we should use the μ test

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} > 1$$

- The μ test not only is a test for unconditional stability, but the magnitude of μ is a measure of the stability. In other words, if one two port has a larger μ, it is more stable.
- The advantage of the μ test is that only a single parameter needs to be evaluated. There are no auxiliary conditions like the K test derivation earlier.
- The derivaiton of the μ test can proceed as follows. First let $\Gamma_S = |\rho_s|e^{j\phi}$ and evaluate Γ_{out}

$$\Gamma_{out} = \frac{S_{22} - \Delta|\rho_s|e^{j\phi}}{1 - S_{11}|\rho_s|e^{j\phi}}$$

- Next we can manipulate this equation into the following eq. for a circle $|\Gamma_{out} - C| = R$

$$\left|\Gamma_{out} + \frac{|\rho_s|S_{11}^*\Delta - S_{22}}{1 - |\rho_s||S_{11}|^2}\right| = \frac{\sqrt{|\rho_s|}|S_{12}S_{21}|}{(1 - |\rho_s||S_{11}|^2)}$$

Mu Test (cont)

- For a two-port to be unconditionally stable, we'd like Γ_{out} to fall within the unit circle

$$||C| + R| < 1$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| < 1 - |\rho_s||S_{11}|^2$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| + |\rho_s||S_{11}|^2 < 1$$

- The worse case stability occurs when $|\rho_s| = 1$ since it maximizes the left-hand side of the equation. Therefore we have

$$\mu = \frac{1 - |S_{11}|^2}{|S_{11}^*\Delta - S_{22}| + |S_{12}S_{21}|} > 1$$

K-Δ *Test*

- The K stability test has already been derived using Y parameters. We can also do a derivation based on S parameters. This form of the equation has been attributed to Rollett and Kurokawa.
- The idea is very simple and similar to the μ test. We simply require that all points in the instability region fall outside of the unit circle.
- The stability circle will intersect with the unit circle if

$$|C_L| - R_L > 1$$

or

$$\frac{|S_{22}^* - \Delta^* S_{11}| - |S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2} > 1$$

- This can be recast into the following form (assuming $|\Delta| < 1$)

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}||S_{21}|} > 1$$

N-Port Passivity

- We would like to find if an N-port is active or passive. By definition, an N-port is passive if it can only absorb net power. The total net complex power flowing into or out of a N port is given by

$$P = (V_1^* I_1 + V_2^* I_2 + \cdots) = (I_1^* V_1 + I_2^* V_2 + \cdots)$$

- If we sum the above two terms we have

$$P = \frac{1}{2}(v^*)^T i + \frac{1}{2}(i^*)^T v$$

- For vectors of current and voltage i and v. Using the admittanc ematrix $i = Yv$, this can be recast as

$$P = \frac{1}{2}(v^*)^T Y v + \frac{1}{2}(Y^* v^*)^T v = \frac{1}{2}(v^*)^T Y v + \frac{1}{2}(v^*)^T (Y^*)^T v$$

$$P = (v^*)^T \frac{1}{2}(Y + (Y^*)^T) v = (v^*)^T Y_H v$$

- Thus for a network to be passive, the Hermitian part of the matrix Y_H should be positive semi-definite.

Two-Port Passivity

- For a two-port, the condition for passivity can be simplified as follows. Let the general hybrid admittance matrix for the two-port be given by

$$H(s) = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} + j \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$$

$$H_H(s) = \frac{1}{2}(H(s) + H^*(s))$$

$$= \begin{pmatrix} m_{11} & \frac{1}{2}((m_{12} + m_{21}) + j(n_{12} - n_{21})) \\ ((m_{12} + m_{21}) + j(n_{21} - n_{12})) & m_{22} \end{pmatrix}$$

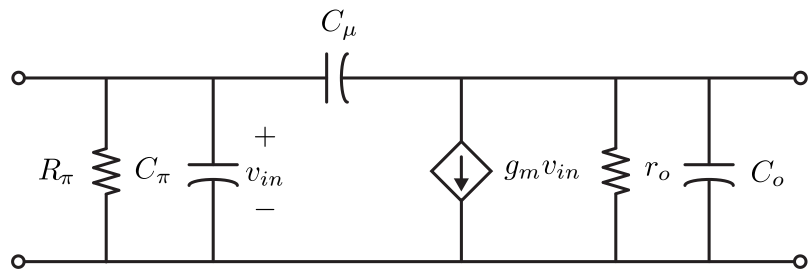
- This matrix is positive semi-definite if

$$m_{11} > 0 \qquad m_{22} > 0 \qquad det H_n(s) \geq 0$$ or

$$4m_{11}m_{22} - |k_{12}|^2 - |k_{21}|^2 - 2\Re(k_{12}k_{21}) \geq 0$$

$$4m_{11}m_{22} \geq |k_{12} + k_{21}^*|^2$$

Hybrid-Pi Example



- The hybrid-pi model for a transistor is shown above. Under what conditions is this two-port active? The hybrid matrix is given by

$$H(s) = \frac{1}{G_\pi + s(C_\pi + C_\mu)} \begin{pmatrix} 1 & sC_\mu \\ g_m - sC_\mu & q(s) \end{pmatrix}$$

$$q(s) = (G_\pi + sC_\pi)(G_0 + sC_\mu) + sC_\mu(G_\pi + g_m)$$

- Applying the condition for passivity we arrive at

$$4G_\pi G_0 \geq g_m^2$$

- The above equation is either satisfied for the two-port or not, regardless of frequency. Thus our analysis shows that the hybrid-pi model is not physical. We know from experience that real two-ports are active up to some frequency f_{max}.

References

- *High-Frequency Amplifiers*, R. Carson, Wiley, New York, NY, 1982.
- *Active Network Analysis*, Wai-Kai Chen, World Scientific Publishing Co., 1991.