EECS 217

Lecture 4: Distributed Resonant Circuits

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Open Line I/V

- The open transmission line has infinite VSWR and $\rho_L = 1$. At any given point along the transmission line

$$v(z) = V^+(e^{-j\beta z} + e^{j\beta z}) = 2V^+ \cos(\beta z)$$

whereas the current is given by

$$i(z) = \frac{V^+}{Z_0}(e^{-j\beta z} - e^{j\beta z})$$

or

$$i(z) = \frac{-2jV^+}{Z_0} \sin(\beta z)$$

Open Line Impedance (I)

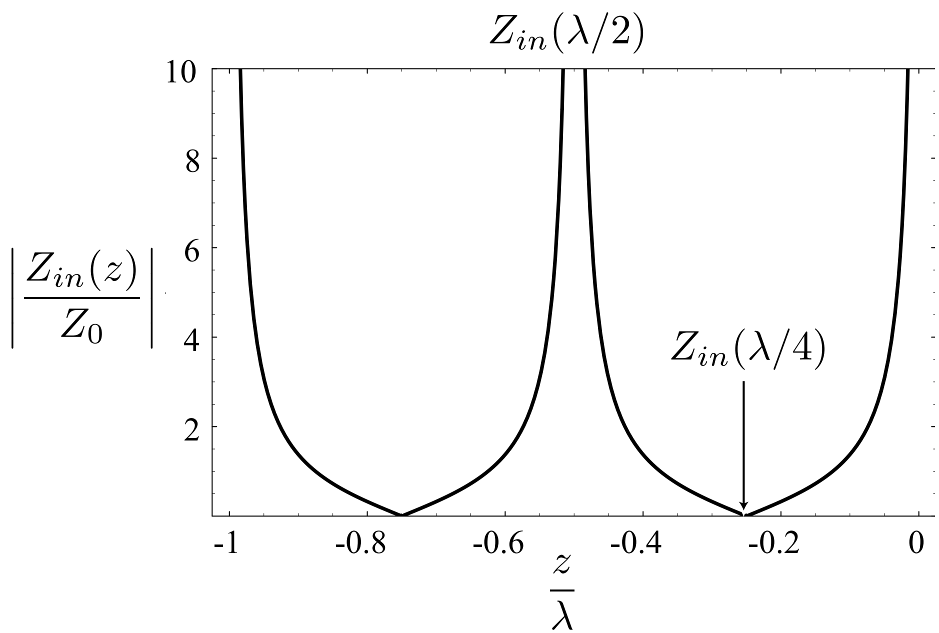
- The impedance at any point along the line takes on a simple form

$$Z_{in}(-\ell) = \frac{v(-\ell)}{i(-\ell)} = -jZ_0 \cot(\beta\ell)$$

- This is a special case of the more general transmission line equation with $Z_L = \infty$.
- Note that the impedance is purely imaginary since an open lossless transmission line cannot dissipate any power.
- We have learned, though, that the line stores reactive energy in a distributed fashion.

Open Line Impedance (II)

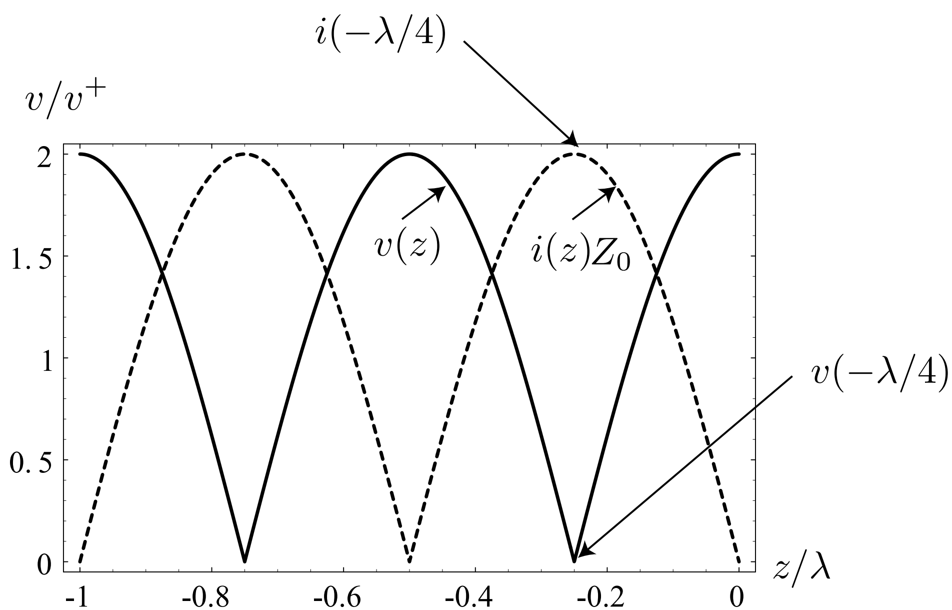
- A plot of the input impedance as a function of z is shown below



- The cotangent function takes on zero values when $\beta\ell$ approaches $\pi/2$ modulo 2π

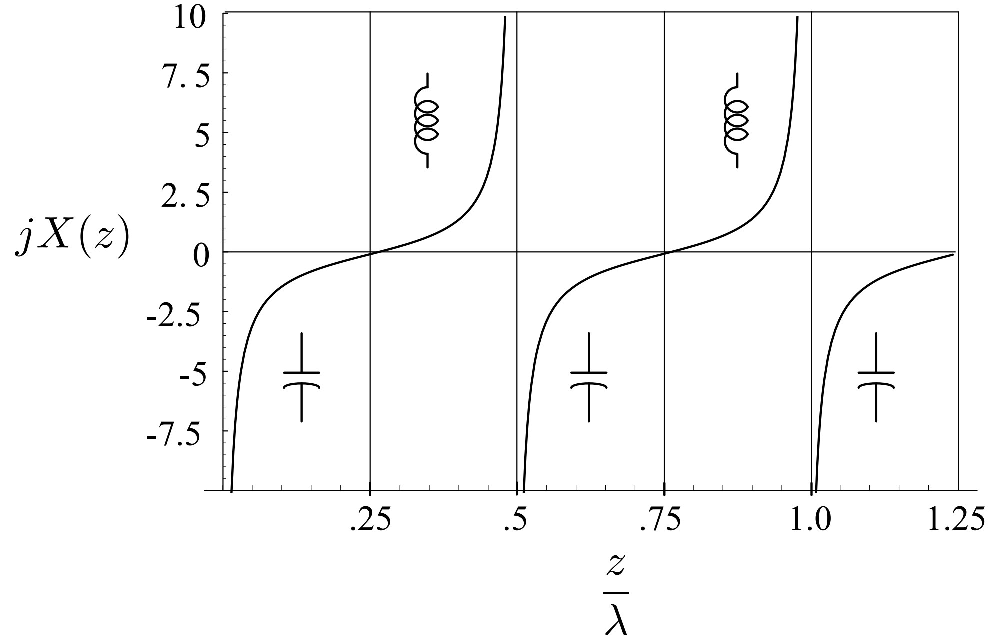
Open Line Impedance (III)

- Open transmission line can have zero input impedance!
- This is particularly surprising since the short load is in effect transformed from an open.
- A plot of the voltage/current as a function of z is shown below.



Open Line Reactance

- $\ell \ll \lambda/4 \rightarrow$ capacitor
- $\ell < \lambda/4 \rightarrow$ capacitive reactance
- $\ell = \lambda/4 \rightarrow$ short (acts like resonant series LC circuit)
- $\ell > \lambda/4$ but $\ell < \lambda/2 \rightarrow$ inductive reactance
- And the process repeats ...



$\lambda/2$ **Transmission Line**

- Plug into the general T-line equation for any multiple of $\lambda/2$

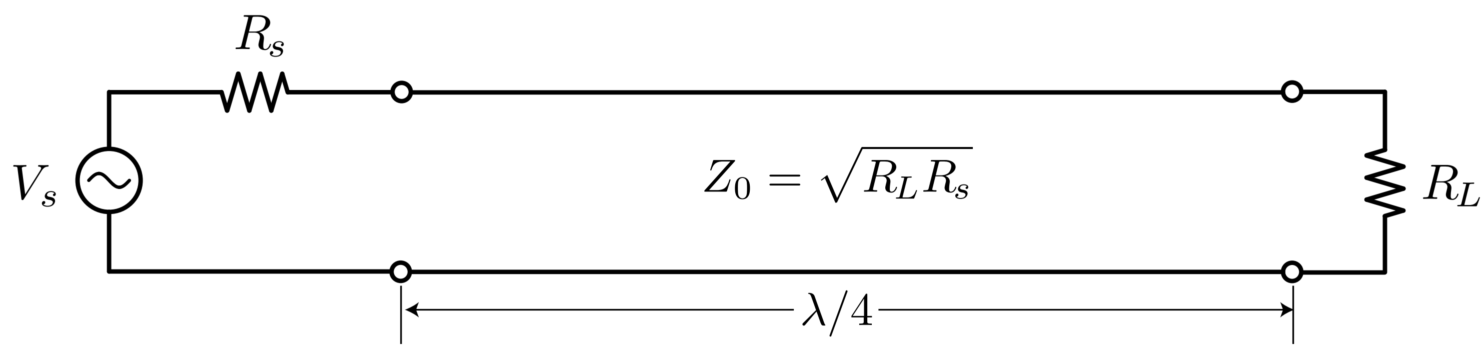
$$Z_{in}(-m\lambda/2) = Z_0 \frac{Z_L + jZ_0 \tan(-\beta\lambda/2)}{Z_0 + jZ_L \tan(-\beta\lambda/2)}$$

- $\beta\lambda m/2 = \frac{2\pi}{\lambda}\frac{\lambda m}{2} = \pi m$
- $\tan m\pi = 0$ if $m \in \mathcal{Z}$
- $Z_{in}(-\lambda m/2) = Z_0 \frac{Z_L}{Z_0} = Z_L$
- Impedance does not change ... it's periodic about $\lambda/2$ (not λ)

$\lambda/4$ Transmission Line

- Plug into the general T-line equation for any multiple of $\lambda/4$
- $\beta\lambda m/4 = \frac{2\pi}{\lambda}\frac{\lambda m}{4} = \frac{\pi}{2}m$
- $\tan m\frac{\pi}{2} = \infty$ if m is an odd integer
- $Z_{in}(-\lambda m/4) = \frac{Z_0^2}{Z_L}$
- $\lambda/4$ line transforms or “inverts” the impedance of the load

$\lambda/4$ Impedance Match



- If the source and load are real resistors, then a quarter-wave line can be used to match the source and load impedances
- Recall that the impedance looking into the quarter-wave line is the "inverse" of the load impedance

$$Z_{in}(z = -\lambda/4) = \frac{Z_0^2}{Z_L}$$

SWR on $\lambda/4$ Line

- In this case, therefore, we equate this to the desired source impedance $Z_{in} = \frac{Z_0^2}{R_L} = R_s$
- The quarter-wave line should therefore have a characteristic impedance that is the geometric mean $Z_0 = \sqrt{R_s R_L}$
- Since $Z_0 \neq R_L$, the line has a non-zero reflection coefficient

$$SWR = \frac{R_L - \sqrt{R_L R_s}}{R_L + \sqrt{R_L R_s}}$$

- It also therefore has standing waves on the T-line
- The non-unity SWR is given by $\frac{1+|\rho_L|}{1-|\rho_L|}$

Interpretation of SWR on $\lambda/4$ Line

- Consider a generic lossless transformer ($R_L > R_s$)
- Thus to make the load look smaller to match to the source, the voltage of the source should be increased in magnitude
- But since the transformer is lossless, the current will likewise decrease in magnitude by the same factor
- With the $\lambda/4$ transformer, the location of the voltage minimum to maximum is $\lambda/4$ from load (since the load is real)
- Voltage/current is thus increased/decreased by a factor of $1 + |\rho_L|$ at the load
- Hence the impedance decreased by a factor of $(1 + |\rho_L|)^2$

Lossy Transmission Line Attenuation

- The power delivered into the line at a point z is now non-constant and decaying exponentially

$$P_{av}(z) = \frac{1}{2}\Re\left(v(z)i(z)^*\right) = \frac{|v^+|^2}{2|Z_0|^2}e^{-2\alpha z}\Re\left(Z_0\right)$$

- For instance, if $\alpha = .01\mathrm{m}^{-1}$, then a transmission line of length $\ell = 10\mathrm{m}$ will attenuate the signal by $10\log(e^{2\alpha\ell})$ or 2 dB. At $\ell = 100\mathrm{m}$ will attenuate the signal by $10\log(e^{2\alpha\ell})$ or 20 dB.

Lossy Transmission Line Impedance

- Using the same methods to calculate the impedance for the low-loss line, we arrive at the following line voltage/current

$$v(z) = v^+ e^{-\gamma z}(1 + \rho_L e^{2\gamma z}) = v^+ e^{-\gamma z}(1 + \rho_L(z))$$

$$i(z) = \frac{v^+}{Z_0} e^{-\gamma z}(1 - \rho_L(z))$$

- Where $\rho_L(z)$ is the complex reflection coefficient at position z and the load reflection coefficient is unaltered from before
- The input impedance is therefore

$$Z_{in}(z) = Z_0 \frac{e^{-\gamma z} + \rho_L e^{\gamma z}}{e^{-\gamma z} - \rho_L e^{\gamma z}}$$

Lossy T-Line Impedance (cont)

- Substituting the value of ρ_L we arrive at a similar equation (now a hyperbolic tangent)

$$Z_{in}(-\ell) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}$$

- For a short line, if $\gamma\delta\ell \ll 1$, we may safely assume that

$$Z_{in}(-\delta\ell) = Z_0 \tanh(\gamma\delta\ell) \approx Z_0\gamma\delta\ell$$

- Recall that $Z_0\gamma = \sqrt{Z'/Y'}\sqrt{Z'Y'}$
- As expected, input impedance is therefore the series impedance of the line (where $R = R'\delta\ell$ and $L = L'\delta\ell$)

$$Z_{in}(-\delta\ell) = Z'\delta\ell = R + j\omega L$$

Low Loss Line

- For a low loss line, $\omega L' \gg R'$ and $\omega C' \gg G'$, so the prop. constant can be simplified

$$\gamma = \sqrt{(j\omega L' + R')(j\omega C' + G')}$$

$$\gamma = \sqrt{(j\omega)^2 L'C' \left(1 - j\left(\frac{R'}{\omega L'} + \frac{G'}{\omega C'}\right) + \frac{R'G'}{(j\omega)^2 L'C'}\right)}$$

- Dropping the last term and using $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for small x

$$\gamma = \sqrt{(j\omega)^2 L'C'}\left(1 - j\frac{1}{2}\left(\frac{R'}{\omega L'} + \frac{G'}{\omega C'}\right)\right)$$

Low Loss Line (cont)

- Using the fact that $Z_0 \approx \sqrt{L'/C'}$

$$\gamma = \alpha + j\beta = \frac{1}{2}\left(\frac{R'}{Z_0} + G'Z_0\right) + j\omega\sqrt{L'C'}$$

- The low loss line is therefore dispersionless since α is independent of frequency and $\beta \propto \omega$.
- The imaginary part of γ is identical to a lossless line, and thus the phase relationship is the same as the lossless case (quarter wavelength on a lossy line is the same length as on a lossless line)
- For all practical purposes, then, the low loss line behaves like a lossless line except the wave attenuates by $e^{-\alpha z}$

Dispersionless Line

- To find the conditions for the transmission line to be dispersionless in terms of the R, L, C, G, expand

$$
\begin{aligned}
\gamma &= \sqrt{(j\omega L' + R')(j\omega C' + G')} \\
&= \sqrt{(j\omega)^2 LC (1 + \frac{R}{j\omega L} + \frac{G}{j\omega C} + \frac{RG}{(j\omega)^2 LC})} \\
&= \sqrt{(j\omega)^2 LC}\sqrt{\Box}
\end{aligned}
$$

- Suppose that $R/L = G/C$ and simplify the $\Box$ term

$$\Box = 1 + \frac{2R}{j\omega L} + \frac{R^2}{(j\omega)^2 L^2}$$

Dispersionless Line (II)

- For $R/L = G/C$ the propagation constant simplifies

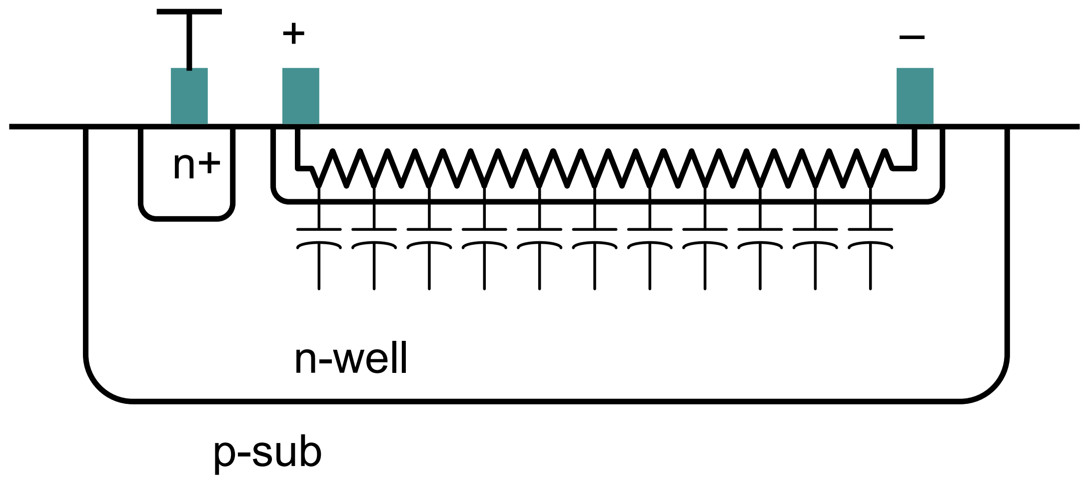
$$\square = \left(1 + \frac{R}{j\omega L}\right)^2 \qquad \gamma = -j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)$$

- Breaking γ into real and imaginary components

$$\gamma = R\sqrt{\frac{C}{L}} - j\omega\sqrt{LC} = \alpha + j\beta$$

- The attenuation constant α is independent of frequency. For low loss lines, $\alpha \approx -\frac{R}{Z_0}$ ✓
- The propagation constant β is a linear function of frequency ✓

Example: IC Resistor



- The IC resistor shown above is common. A reverse biased diffusion resistor has capacitance to substrate arising from the reverse biased junction.
- A thin film resistor has capacitance to substrate due to its close proximity.
- For simplicity, assume the substrate is a perfect ground.

Telegrapher's Equations

- The series impedance per unit length is predominantly resistive. For all frequencies of interest, $\omega L' \ll R'$

$$Z' = j\omega L' + R' \approx R'$$

- Assuming the conductance per unit is capacitive, $Y' = j\omega C'$, the propagation constant is given by

$$\gamma = \sqrt{j\omega C' R'}$$

- which has a phase of 45°. Likewise, the characteristic impedance is given by

$$Z_0 = \sqrt{\frac{Z'}{Y'}} = \sqrt{j\omega C' R'}$$

Resistor Sizing

- The optimal size of the resistor can be analyzed by noting that $R' = R_\square / W$ and $C' = W\epsilon / t_{dep}$ ($t_{dep} =$ depletion region depth). Let $C_x = \epsilon_{\mathrm{Si}} / t_{dep}$. Then

$$\gamma = \sqrt{jR_\square \omega C_x}$$

- which is independent of the width W. The impedance, though, drops with W

$$Z_0 = \frac{1}{W}\sqrt{\frac{R_\square}{j\omega C_x}}$$

Resistance versus W

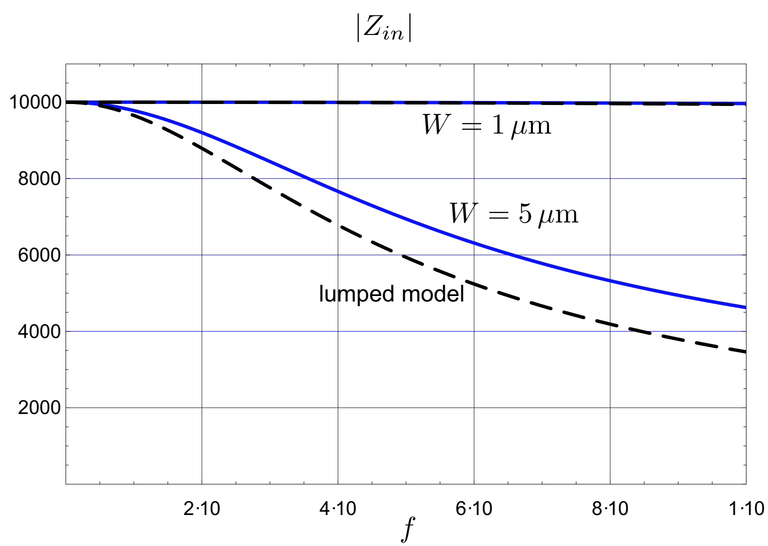
- For a shorted resistor, the input impedance is given by

$$Z_{in} = Z_0 \tanh \gamma \ell = \frac{1}{W}\sqrt{\frac{R_\square}{j\omega C_x}} \tanh\left(\sqrt{jR_\square \omega C_x \ell}\right)$$

- For a given desired resistance $\frac{\ell}{W} R_\square = R_0$, we can substitute for ℓ

$$Z_{in} = Z_0 \tanh \gamma \ell = \frac{1}{W}\sqrt{\frac{R_\square}{j\omega C_x}} \tanh\left(\sqrt{jR_\square \omega C_x}\frac{R_0}{R_\square} W\right)$$

Plot of Input Impedance

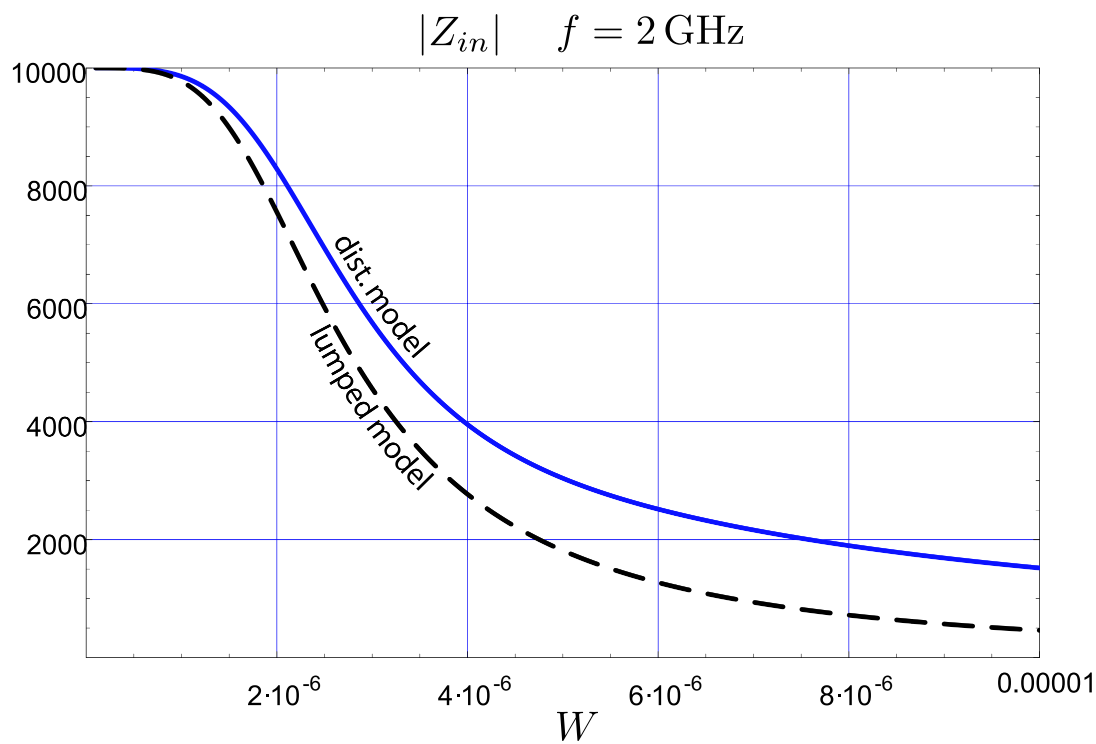


- The plot of $|Z_{in}|$ for a nominally $10\,\mathrm{k}\Omega$ resistor versus frequency is shown above.
- Say that for a thin film resistor has $R_\square = 100\,\Omega/\square$ and

$$C_x = \epsilon_{\mathrm{SiO_2}}/t_0 = 3.45\times10^{-5}\,\mathrm{F/m^2}$$

- The $W = 1\,\mu\mathrm{m}$ resistor has a relatively flat frequency response up to $1\,\mathrm{GHz}$, whereas the $W = 5\,\mu\mathrm{m}$ resistor rolls off quickly and is about half of its nominal size at $1\,\mathrm{GHz}$.

Impedance versus W



- The variation of the impedance magnitude versus W is shown above. Larger W resistors have better precision and matching, but clearly the extra capacitance hurts at high frequency.

Review of Resonance (I)

- We'd like to find the impedance of a series resonator near resonance $Z(\omega) = j\omega L + \frac{1}{j\omega C} + R$
- Recall the definition of the circuit Q

$$Q = \omega_0 \frac{\text{time average energy stored}}{\text{energy lost per cycle}}$$

- For a series resonator, $Q = \omega_0 L/R$. For a small frequency shift from resonance $\delta\omega \ll \omega_0$

$$Z(\omega_0 + \delta\omega) = j\omega_0 L + j\delta\omega L + \frac{1}{j\omega_0 C}\left(\frac{1}{1+\frac{\delta\omega}{\omega_0}}\right) + R$$

Review of Resonance (II)

- Which can be simplified using the fact that $\omega_0 L = \frac{1}{\omega_0 C}$

$$Z(\omega_0 + \delta\omega) = j2\delta\omega L + R$$

- Using the definition of Q

$$Z(\omega_0 + \delta\omega) = R\left(1 + j2Q\frac{\delta\omega}{\omega_0}\right)$$

- For a parallel line, the same formula applies to the admittance

$$Y(\omega_0 + \delta\omega) = G\left(1 + j2Q\frac{\delta\omega}{\omega_0}\right)$$

- Where $Q = \omega_0 C/G$

$\lambda/2$ T-Line Resonators (Series)

- A shorted transmission line of length ℓ has input impedance of $Z_{in} = Z_0 \tanh(\gamma\ell)$
- For a low-loss line, Z_0 is almost real
- Expanding the $\tanh$ term into real and imaginary parts

$$\tanh(\alpha\ell + j\beta\ell) = \frac{\sinh(2\alpha\ell)}{\cos(2\beta\ell) + \cosh(2\alpha\ell)} + \frac{j\sin(2\beta\ell)}{\cos(2\beta\ell) + \cosh(2\alpha\ell)}$$

- Since $\lambda_0 f_0 = c$ and $\ell = \lambda_0/2$ (near the resonant frequency), we have $\beta\ell = 2\pi\ell/\lambda = 2\pi\ell f/c = \pi + 2\pi\delta f\ell/c = \pi + \pi\delta\omega/\omega_0$
- If the lines are low loss, then $\alpha\ell \ll 1$

$\lambda/2$ Series Resonance

- Simplifying the above relation we come to

$$Z_{in} = Z_0 \left(\alpha\ell + j\frac{\pi\delta\omega}{\omega_0} \right)$$

- The above form for the input impedance of the series resonant T-line has the same form as that of the series LRC circuit
- We can define equivalent elements

$$R_{eq} = Z_0\alpha\ell = Z_0\alpha\lambda/2$$

$$L_{eq} = \frac{\pi Z_0}{2\omega_0} \qquad C_{eq} = \frac{2}{Z_0\pi\omega_0}$$

$\lambda/2$ Series Resonance Q

- The equivalent Q factor is given by

$$Q = \frac{1}{\omega_0 R_{eq} C_{eq}} = \frac{\pi}{\alpha \lambda_0} = \frac{\beta_0}{2\alpha}$$

- For a low-loss line, this Q factor can be made very large. A good T-line might have a Q of 1000 or 10,000 or more
- It's difficult to build a lumped circuit resonator with such a high Q factor

$\lambda/4$ T-Line Resonators (Parallel)

- For a short-circuited $\lambda/4$ line

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell = Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell}$$

- Multiply numerator and denominator by $-j \cot \beta\ell$

$$Z_{in} = Z_0 \frac{1 - j \tanh \alpha\ell \cot \beta\ell}{\tanh \alpha\ell - j \cot \beta\ell}$$

- For $\ell = \lambda/4$ at $\omega = \omega_0$ and $\omega = \omega_0 + \delta\omega$

$$\beta\ell = \frac{\omega_0 \ell}{v} + \frac{\delta\omega\ell}{v} = \frac{\pi}{2} + \frac{\pi\delta\omega}{2\omega_0}$$

$\lambda/4$ T-Line Resonators (Parallel)

- So $\cot\beta\ell = -\tan\frac{\pi\delta\omega}{2\omega_0} \approx \frac{-\pi\delta\omega}{2\omega_0}$ and $\tanh\alpha\ell \approx \alpha\ell$

$$Z_{in} = Z_0 \frac{1 + j\alpha\ell\pi\delta\omega/2\omega_0}{\alpha\ell + j\pi\delta\omega/2\omega_0} \approx \frac{Z_0}{\alpha\ell + j\pi\delta\omega/2\omega_0}$$

- This has the same form for a parallel resonant RLC circuit

$$Z_{in} = \frac{1}{1/R + 2j\delta\omega C}$$

- The equivalent circuit elements are

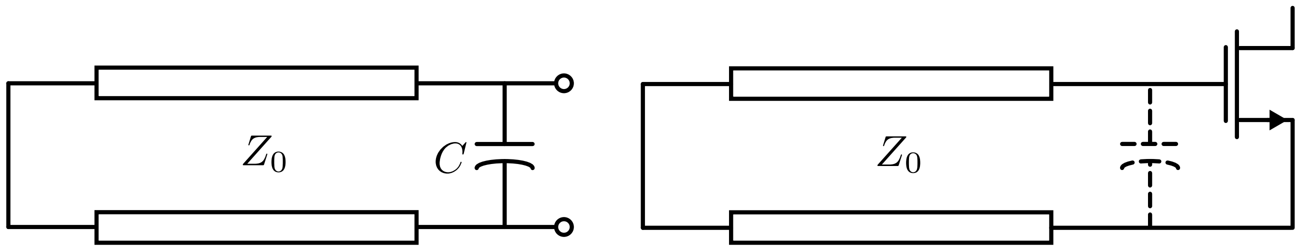
$$R_{eq} = \frac{Z_0}{\alpha\ell} \qquad C_{eq} = \frac{\pi}{4\omega_0 Z_0} \qquad L_{eq} = \frac{1}{\omega_0^2 C_{eq}}$$

$\lambda/4$ T-Line Resonators Q Factor

- The quality factor is thus

$$Q = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha}$$

T-Line-C Resonator Q Factor



- Often transmission lines are used as resonant elements along with lumped elements.
- A good example, shown above, is a short section of transmission line resonating with the input capacitance of a transistor. For simplicity assume that the lumped input capacitance is lossless. What's the the Q factor of the resulting resonant circuit?

Magnetic Energy Storage

- It's important to note that $Q \neq \frac{1}{2}\beta/\alpha$ since this only applies to the transmission line in resonance, when the magnetic and electric energy are equal on the transmission line.
- In our case, we would like to use the transmission line as an inductor, so we will be concerned with the net magnetic energy on the line. The Q factor is therefore given by

$$Q = 2\omega_0 \frac{\text{net energy stored}}{\text{avg. power loss}} = \frac{2\omega_0(W_m - W_e)}{P_R + P_G}$$

- where W_m and W_e are the average magnetic and electric energy stored, and P_R represent the "series" resistive losses and P_G the "shunt" conductive losses.

Inductive/Capacitive Q

- Defining the series inductive and shunt capacitive Q we have

$$Q_L = 2\omega_0 \frac{W_m}{P_R} \qquad Q_C = 2\omega_0 \frac{W_e}{P_G}$$

we can express the overall Q as

$$\frac{1}{Q} = \frac{1}{\eta_L Q_L} + \frac{1}{\eta_C Q_C}$$

$$\eta_L = 1 - \frac{W_e}{W_m} \qquad \eta_C = \frac{W_m}{W_e} - 1$$

Magnetic/Electric Energy

- For a shorted transmission line, under the assumption of low loss, one can show that

$$W_m \approx \frac{1}{2}\frac{LV^{+2}\ell}{Z_0^2}\left(1+\mathrm{sinc}\left(\frac{4\pi\ell}{\lambda}\right)\right)$$

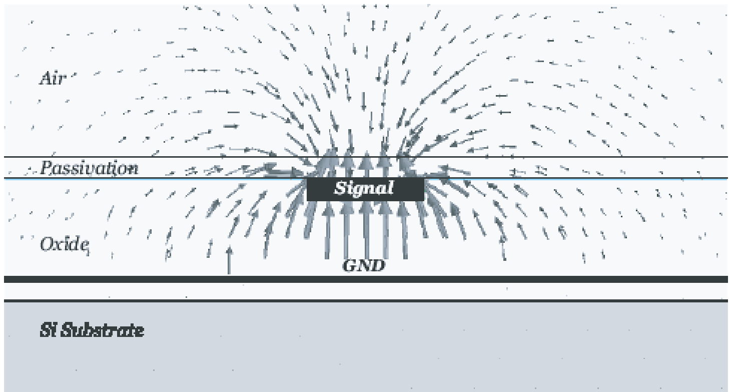
$$W_e \approx \frac{1}{2}CV^{+2}\ell\left(1-\mathrm{sinc}\left(\frac{4\pi\ell}{\lambda}\right)\right)$$

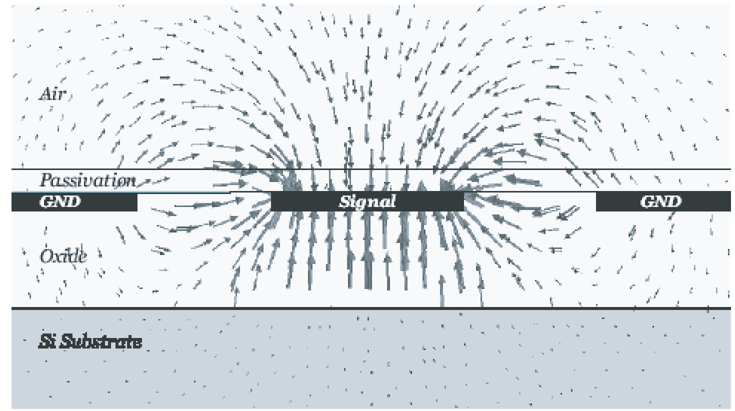
Thus we have

$$\frac{1}{\eta_L} = \frac{1}{2\,\mathrm{sinc}(\frac{4\pi\ell}{\lambda})} + \frac{1}{2}$$

$$\frac{1}{\eta_C} = \frac{1}{2\,\mathrm{sinc}(\frac{4\pi\ell}{\lambda})} - \frac{1}{2}$$

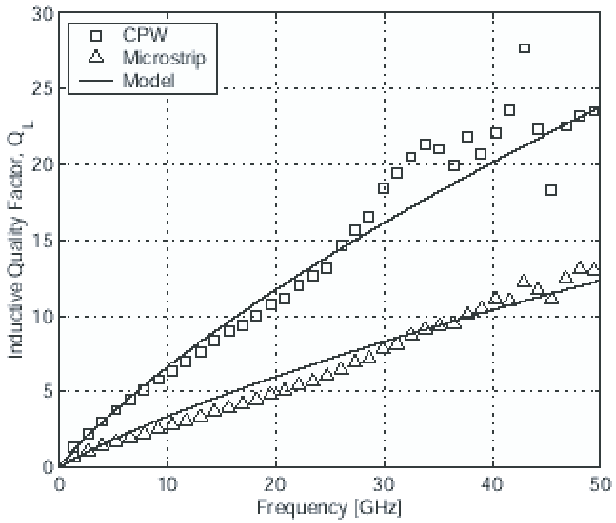
"Shorted" T-Line

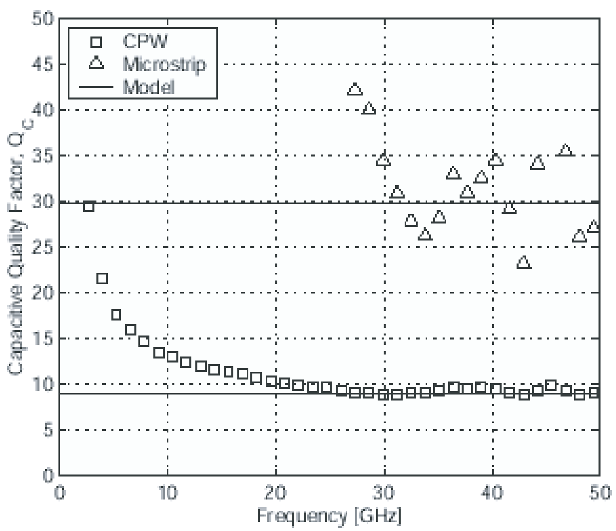




- For a shorted line, say $\ell \ll \lambda$, then $\eta_C \gg \eta_L$. For instance, if $\ell < 0.1\lambda$, then $\eta_C > 7\eta_L$. The net Q of such a resonant circuit is therefore $Q \approx \eta_L Q_L$.
- This explains why a Si coplanar line is preferred over a microstrip line in such an application.
- Due to the Si substrate losses, the resonant Q of the microstrip is higher. But the inductive Q_L of the coplanar line is higher since more magnetic energy can be stored per unit length.

Co-Planar/Microstrip Tradeoff





- Notice that the capacitive Q_C factor is larger for the microstrip, since most of the fields terminate on the M1 shield ground plane.
- The coplanar line, though, has electric fields that penetrate the substrate and cause loss due to the finite conductivity. This can be modeled as an effective frequency dependent dielectric loss.

Co-planar/Microstrip Tradeoff (cont)

- The inductive Q_L is larger, though, since the width of the coplanar line can be made wider. The spacing predominately controls the impedance of the line.
- On the other hand, for a microstrip line, the spacing between the signal and ground is fixed, and thus the impedance can only be increased by reducing the conductor width.

For more details:

“Millimeter-Wave CMOS Design”

Doan, C.H.; Emami, S.; Niknejad, A.M.; Brodersen, R.W.,

IEEE Journal of Solid-State Circuits, Volume: 40 , Issue: 1 , Jan. 2005, Pages:144 - 155