EECS 217

Lecture 2: Poynting's Thm/Impedance and Conductors

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Power in Fields

- From circuit intuition, we know that current times voltage is power, so we suspect that the product of $\mathbf{E}$ and $\mathbf{H}$ should be related to the power in the field. In fact, the units work out

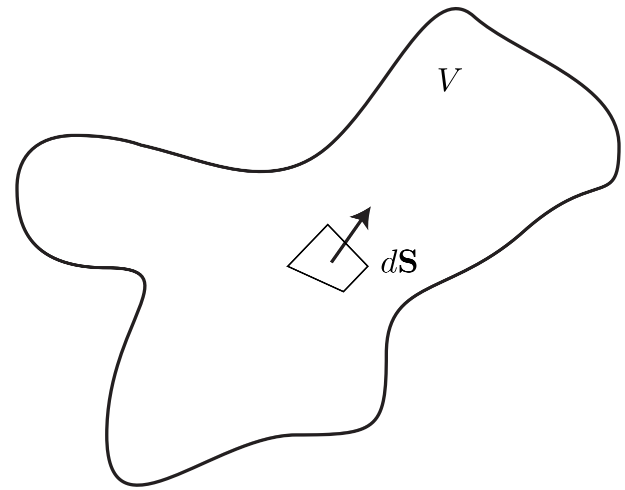
$$[E][H] = \frac{\mathrm{V}}{\mathrm{m}}\frac{\mathrm{A}}{\mathrm{m}} = \frac{\mathrm{V}\cdot\mathrm{A}}{\mathrm{m}^2} = \frac{\mathrm{W}}{\mathrm{m}^2}$$

- We expect that this may represent the energy density of the field. We need to prove this more rigorously.
- In fact, we will demonstrate that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ represents the power density of an EM field.

Poynting Vector

- As such, the surface integral of $\mathbf{S}$ should represent the power crossing a surface

$$\int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV$$



- Note that the direction of $\mathbf{S}$ represents the direction of power flow. The magnitude S is the strength of the power flow.

Poynting's Theorem

- Let's work with the divergence term

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

$$= \mathbf{H} \cdot -\frac{\partial \mathbf{B}}{\partial \mathbf{t}} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial \mathbf{t}} - \mathbf{E} \cdot \mathbf{J}$$

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \left(\frac{\partial \mu \mathbf{H}}{\partial t} \right) = \frac{1}{2} \frac{\partial \mu \mathbf{H} \cdot \mathbf{H}}{\partial t} = \frac{1}{2} \frac{\partial \mu |\mathbf{H}|^2}{\partial t}$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \left(\frac{\partial \epsilon \mathbf{E}}{\partial t} \right) = \frac{1}{2} \frac{\partial \epsilon \mathbf{E} \cdot \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial \epsilon |\mathbf{E}|^2}{\partial t}$$

Poynting's Theorem

- Collecting terms we have shown that

$$\mathbf{E} \cdot \mathbf{J} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu|\mathbf{H}|^2\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon|\mathbf{E}|^2\right) - \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

- Applying the Divergence Theorem we have

$$\int_V \mathbf{E} \cdot \mathbf{J} dV = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu|\mathbf{H}|^2 + \frac{1}{2}\epsilon|\mathbf{E}|^2\right) dV - \int_S \mathbf{E} \times \mathbf{H} dV$$

power dissipated in volume V (heat)	$=$	rate of change of energy storage in volume V	$-$	a surface integral over the volume of $\mathbf{E} \times \mathbf{H}$

Interpretation of the Poynting Vector

- We now have a physical interpretation of the last term in the above equation. By the conservation of energy, it must be equal to the energy flow into or out of the volume
- We may be so bold, then, to interpret the vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ as the energy flow density of the field
- While this seems reasonable, it's important to note that the physical meaning is only attached to the integral of $\mathbf{S}$ and not to discrete points in space

Complex Poynting Theorem

- We derived the Poynting Theorem for general electric/magnetic fields. We'd like to derive the Poynting Theorem for time-harmonic fields.
- We can't simply take our results and simply transform $\frac{\partial}{\partial t} \to j\omega$. This is because the Poynting vector is a non-linear function of the fields.
- Let's start from the beginning

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J} = (j\omega\epsilon + \sigma)\mathbf{E}$$

Complex Poynting Theorem (II)

- Using our knowledge of circuit theory, $P = V \times I^*$, we compute the following quantity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^*$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot (-j\omega \mathbf{B}) - \mathbf{E} \cdot (-j\omega \mathbf{D}^* + \mathbf{J}^*)$$

- Applying the Divergence Theorem

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) dV = \oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S}$$

$$\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = -\int_V \mathbf{E} \cdot \mathbf{J}^* dV + \int_V j\omega (\mathbf{E} \cdot \mathbf{D}^* - \mathbf{H}^* \cdot \mathbf{B}) dV$$

Complex Poynting Theorem (III)

- Let's define $\sigma_{\text{eff}} = \omega\epsilon'' + \sigma$, and $\epsilon = \epsilon'$. Since most materials are non-magnetic, we can ignore magnetic losses

$$\int_S (\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{S} = -\int_V \sigma\mathbf{E}\cdot\mathbf{D}^* dV - j\omega \int_V (\mu\mathbf{H}^* \cdot \mathbf{H} - \epsilon\mathbf{E}\cdot\mathbf{E}^*)\, dV$$

- Notice that the first volume integral is a real number whereas the second volume integral is imaginary

$$\Re\left(\oint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{dS}\right) = -2\int_V P_c dV$$

$$\Im\left(\oint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{dS}\right) = -4\omega\int_V (w_m - w_e) dV$$

Complex Poynting Vector

- Let's compute the average vector $\mathbf{S}$

$$\mathbf{S} = \Re\left(\mathbf{E}e^{j\omega t}\right) \times \Re\left(\mathbf{H}e^{j\omega t}\right)$$

- First observe that $\Re(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$, so that

$$\Re(\mathbf{G}) \times \Re(\mathbf{F}) = \frac{1}{2}(\mathbf{G} + \mathbf{G}^*) \times \frac{1}{2}(\mathbf{F} + \mathbf{F}^*)$$

$$= \frac{1}{4}(\mathbf{G} \times \mathbf{F} + \mathbf{G} \times \mathbf{F}^* + \mathbf{G}^* \times \mathbf{F} + \mathbf{G}^* \times \mathbf{F}^*)$$

$$= \frac{1}{4}\left[(\mathbf{G} \times \mathbf{F}^* + \mathbf{G}^* \times \mathbf{F}) + (\mathbf{G} \times \mathbf{F} + \mathbf{G}^* \times \mathbf{F}^*)\right]$$

$$= \frac{1}{2}\Re\left(\mathbf{G} \times \mathbf{F}^* + \mathbf{G} \times \mathbf{F}\right)$$

Average Complex Poynting Vector

- Finally, we have computed the complex Poynting vector with the time dependence

$$\mathbf{S} = \frac{1}{2}\Re\left(\mathbf{E}\times\mathbf{H}^* + \mathbf{E}\times\mathbf{H}e^{2j\omega t}\right)$$

- Taking the average value, the complex exponential vanishes, so that

$$\mathbf{S_{av}} = \frac{1}{2}\Re\left(\mathbf{E}\times\mathbf{H}^*\right)$$

- We have thus justified that the quantity $\mathbf{S} = \mathbf{E}\times\mathbf{H}^*$ represents the complex power stored in the field.

Impedance and Power

- The circuit concept of impedance can be stated in terms of power

$$Z_{in} = \frac{V}{I} = \frac{V \cdot I^*}{|I|^2} = \frac{P}{\frac{1}{2}|I|^2} = R + jX$$

- Applying Poynting's Thm to the "black box", we can write this as

$$Z_{in} = \frac{P_0 + P_\ell + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2}$$

Resistance and Reactance

- Note that the resistive component has a radiation term, an ohmic loss term, and possibility a dielectric or permeability loss term

$$R = \frac{P_0 + P_\ell}{\frac{1}{2}|I|^2}$$

- The reactance is positive if $W_m > W_e$, and negative otherwise

$$X = \frac{2\omega(W_m - W_e)}{\frac{1}{2}|I|^2}$$

Quality Factor

- The quality factor for a "black box" (usually resonator) is defined as follows

$$Q = 2\pi \frac{\text{Peak Energy Stored}}{\text{Energy Loss Per Cycle}}$$

- The denominator can be reformulated in terms of the average power loss to give

$$Q = \omega \frac{\text{Peak Energy Stored}}{P_\ell}$$

- From Poynting's Thm, the net stored energy is given by $W_m - W_e$, so we may be tempted to write

$$Q \stackrel{?}{=} \omega \frac{W_m - W_e}{P_\ell} = \frac{1}{2} \frac{\Im(Z_{in})}{\Re(Z_{in})}$$

Q-Factor Continued

- But the peak energy is different from the *net* energy. In a resonator, the peak energy is actually *twice* the maximum energy stored in either the inductor or the capacitor (see problems), so we have

$$Q = \omega \frac{2W_m^{peak}}{P_\ell} = \omega \frac{2W_e^{peak}}{P_\ell} = \omega \frac{|W_m^{peak}| + |W_e^{peak}|}{P_\ell}$$

- For a single one-port element *not* in resonance, one often defines the Q factor as

$$Q = \frac{\Im(Z_{in})}{\Re(Z_{in})}$$

- We see that this is correct under the assumption that the one-port forms a resonant circuit!

Properties of Conductors

- It's interesting to observe that for a conductor, Ohm's law implies the absence of "free" charge

$$\nabla \times \mathbf{H} = J + j\omega\mathbf{D} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} = (\sigma + j\omega\epsilon)\mathbf{E}$$

- Since the divergence of the curl of any vector is zero

$$\nabla \cdot (\nabla \times \mathbf{H}) \equiv 0 = (\sigma + j\omega\epsilon)\nabla \cdot \mathbf{E}$$

- that implies that $\nabla \cdot \mathbf{D} = \rho = 0$, or $\rho = 0$.
- Even though current is charge in motion, in steady-state the net charge for any macroscopic region must be zero in a conductor. This condition is satisfied on a time scale of the relaxation time $t \sim \epsilon/\sigma$.

Definition of a Good Conductor

- A good conductor is defined as a material where displacement current is negligible in comparison with conduction current.
- For most good conductors, this is true at microwave frequencies.
- For example, for Al at $10\,\mathrm{GHz}$, $\sigma \approx 4 \times 10^7\,\mathrm{S/m}$, where as $\omega\epsilon < 10\,\mathrm{S/m}$.
- Lightly doped Si, with $\sigma = 10\,\mathrm{S/m}$, acts like a poor conductor at this frequency.

EM Fields Inside Good Conductor

- Helmholtz' equations for a good conductor is given by

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -j\omega\mu\nabla \times \mathbf{H}$$

- We see that by Ohm's law, the first term is zero, and for a good conductor $\nabla \times \mathbf{H} = \sigma \mathbf{E}$, so

$$\nabla^2 \mathbf{E} = j\omega\mu\sigma\mathbf{E}$$

- It's easy to show the same equation is satisfied by $\mathbf{H}$ and $\mathbf{J}$.

Semi-Infinite Conductor

- For a semi-infinite conductor, assume a uniform field $\mathbf{E_0}$ impinges on the surface of the conductor. By symmetry, the wave equation is one-dimensional

$$\frac{\partial^2 E_z}{\partial x^2} = j\omega\mu\sigma E_z = \tau^2 E_z$$

$$\tau = \sqrt{j\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} = \frac{1+j}{\delta}$$

- The general solution for the field is simply

$$E_z = C_1 e^{-\tau x} + C_2 e^{\tau x}$$

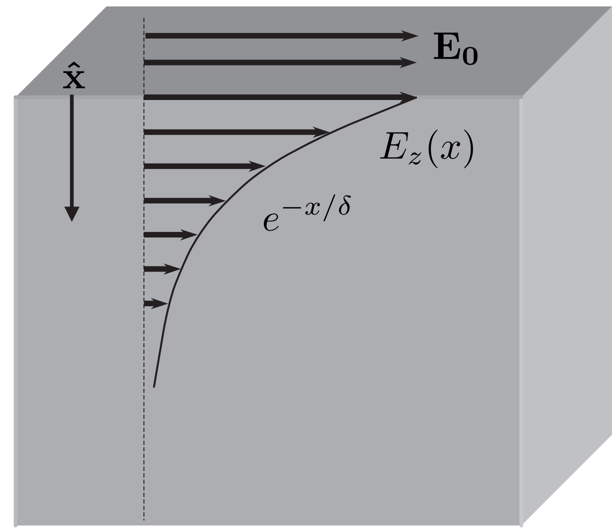
- Clearly, $C_2 \equiv 0$ and $C_1 = E_0$ to satisfy the boundary conditions.

$$E_z = E_0 e^{-x/\delta} e^{jx/\delta}$$

Semi-Infinite Conductor (cont)

- The parameter δ, also called the skin depth, determines the penetration depth of the field

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}}$$



- For Al at $10\,\mathrm{GHz}$, $\delta \approx 0.4\,\mu\mathrm{m}$. Thus the fields decays rapidly as we enter the conductor. Since $J \propto E$, the current density likewise drops and essentially flows on the "skin" of the conductor.

Internal Impedance of Plane (cont)

- The total current flowing past a unit width of conductor is given by

$$J_{sz} = \int_0^\infty J_z dx = \int_0^\infty J_0 e^{-\tau x} dx = \frac{J_0 \delta}{1+j}$$

- At the surface, $E_{z0} = J_0/\sigma$. The internal impedance for a unit length and width is defined as

$$Z_s = \frac{E_{Z0}}{J_{sz}} = \frac{1+j}{\sigma\delta} = R_s + j\omega L_i$$

Surface Impedance

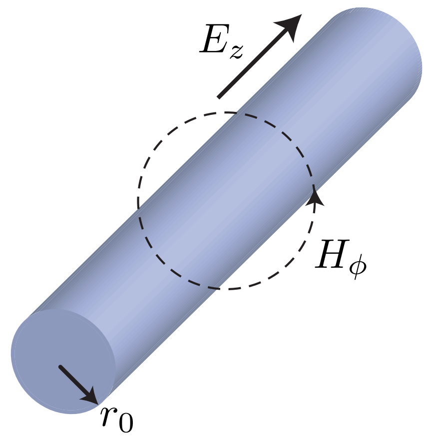
- The surface resistance R_s can be interpreted as the loss due to uniform current flow over a thickness δ of the top of the conductor

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

- Also, the conductor appears inductive with $\omega L_i = R_s$.
- It can be shown through Poynting's Thm that the power loss into the conductor is given by

$$P_\ell = \tfrac{1}{2}\Re(Z_s J_s J_s^*) = \tfrac{1}{2} R_s |J_s|^2$$

Round Wires



- For a long round wire, the current J_z is invariant with z and the angle θ as shown. Therefore, the Helmholtz eq. simplifies

$$\nabla^{\mathbf{2}}\mathbf{J} = j\omega\mu\sigma\mathbf{J} = \tau^2\mathbf{J}$$

$$\frac{\partial^2 J_z}{\partial r^2} + \frac{1}{r}\frac{\partial J_z}{\partial r} + \tau^2 J_z = 0$$

Wire Solution

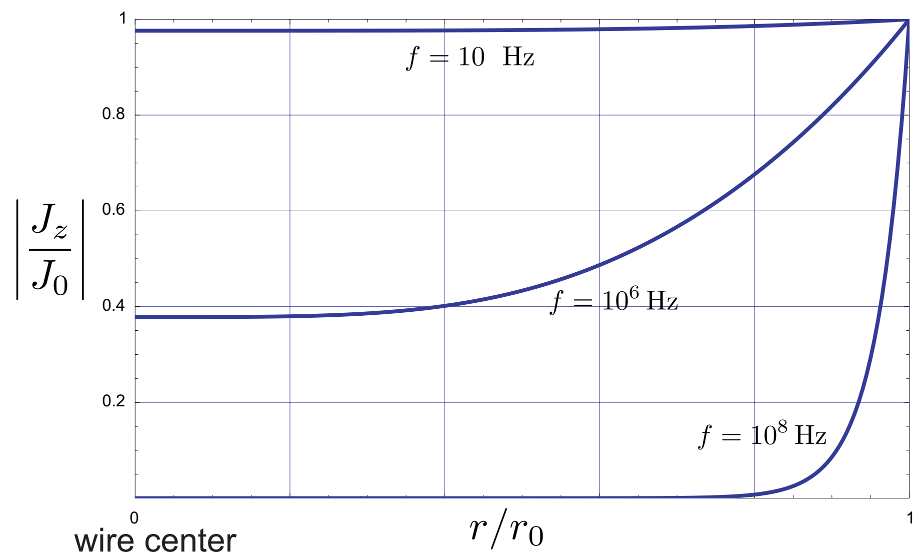
- Two linearity independent solutions are the Bessel function and the Hankel function of the first kind

$$J_z = A J_0(\tau r) + B H_0^{(1)}(\tau r)$$

- Since the Hankel function has a singularity at $r = 0$, it cannot be a solution. Normalizing to the current at the surface of the wire

$$J_z = \frac{\sigma E_0}{J_0(\tau r_0)} J_0(\tau r)$$

Current Density



- A plot of the current density in the wire is shown above. In the plot, a Cu wire with $1\,\mathrm{mm}$ diameter is used.
- Note that at low frequencies the current is essentially uniform. At high frequency, though, the current decays exponentially as we penetrate the conductor.

Large Radius Limit/Impedance of Wire

- In the limit that the radius is large, or equivalently $r_0/\delta \gg 1$, then the wire should behave like our plane conductor. In fact,

$$|\frac{J_z}{\sigma E_0}| \approx e^{-(r_0-r)/\delta}$$

- The impedance of a round wire can be computed by noting that only E_z and H_ϕ are present. Furthermore, we have

$$\oint \mathbf{H} \cdot \mathbf{d}\ell = I = 2\pi r_0 H_\phi$$

- By $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$, it's easy to show that

$$H_\phi = \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial r}$$

Impedance of Round Wire

- Using $E_z = J_z/\sigma = E_0 J_0(\tau r)/J_0(\tau r_0)$, the magnetic field is given by

$$H_\phi = \frac{E_0 \tau}{j\omega\mu} \frac{J_0'(\tau r)}{J_0(\tau r_0)}$$

- Recall that $J_0'(x) = -J_1(x)$. Solving for the current

$$I = \frac{2\pi r_0 \sigma E_0}{\tau} \frac{J_1(\tau r_0)}{J_0(\tau r_0)}$$

- Finally we can write the internal impedance of the wire

$$Z_i = \frac{E_z(r_0)}{I} = \frac{\tau J_0(\tau r_0)}{2\pi r_0 \sigma J_1(\tau r_0)}$$

Low/High Frequency Limit

- In the low frequency limit, the internal impedance of the wire reduces to

$$Z_i \approx \frac{1}{\pi r_0^2 \sigma}\left[1 + \frac{1}{48}\left(\frac{r_0}{\delta}\right)^2\right] + j\omega\frac{\mu}{8\pi}$$

- The real part corresponds to a correction to the DC resistance of the wire (per unit length). The imaginary term corresponds exactly to the static internal inductance of the wire.
- As expected, the high frequency limit matches the analysis for the semi-infinite plane

$$Z_i = \frac{(1+j)R_s}{2\pi r_0}$$