# Review of Resonance

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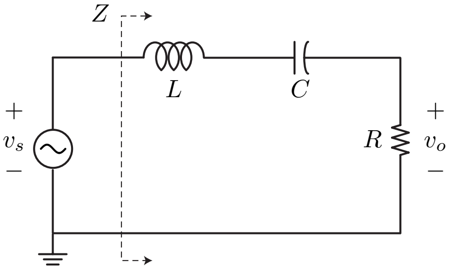
# Series RLC Circuis

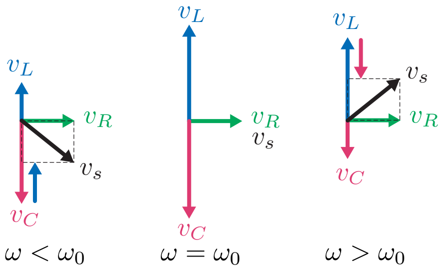



- The *RLC* circuit shown is deceptively simple. The impedance seen by the source is simply given by

$$Z = j\omega L + \frac{1}{j\omega C} + R = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right)$$

- The impedance is purely real at at the *resonant frequency* when $\Im(Z) = 0$, or $\omega = \pm\frac{1}{\sqrt{LC}}$. At resonance the impedance takes on a minimal value.

# Series Resonance




- It's worthwhile to investigate the cause of resonance, or the cancellation of the reactive components due to the inductor and capacitor. Since the inductor and capacitor voltages are always $180°$ out of phase, and one reactance is dropping while the other is increasing, there is clearly always a frequency when the magnitudes are equal.
- Resonance occurs when $\omega L = \frac{1}{\omega C}$.

- So what's the magic about this circuit? The first observation is that at resonance, the voltage across the reactances can be larger, in fact much larger, than the voltage across the resistors $R$. In other words, this circuit has voltage gain. Of course it does not have power gain, for it is a passive circuit. The voltage across the inductor is given by

$$v_L = j\omega_0 L i = j\omega_0 L \frac{v_s}{Z(j\omega_0)} = j\omega_0 L \frac{v_s}{R} = jQ \times v_s$$

- where we have defined a circuit $Q$ factor at resonance as

$$Q = \frac{\omega_0 L}{R}$$

# Voltage Multiplication

- It's easy to show that the same voltage multiplication occurs across the capacitor (the reactances are equal at resonance after all)

$$v_C = \frac{1}{j\omega_0 C} i = \frac{1}{j\omega_0 C} \frac{v_s}{Z(j\omega_0)} = \frac{1}{j\omega_0 RC} \frac{v_s}{R} = -jQ \times v_s$$

- This voltage multiplication property is the key feature of the circuit that allows it to be used as an impedance transformer.
- It's important to distinguish this $Q$ factor from the intrinsic $Q$ of the inductor and capacitor. For now, we assume the inductor and capacitor are ideal.

# More of Q

- We can re-write the $Q$ factor in several equivalent forms owing to the equality of the reactances at resonance

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C}\frac{1}{R} = \frac{\sqrt{LC}}{C}\frac{1}{R} = \sqrt{\frac{L}{C}}\frac{1}{R} = \frac{Z_0}{R}$$

- where we have defined the $Z_0 = \sqrt{\frac{L}{C}}$ as the characteristic impedance of the circuit.

# Circuit Transfer Function

- Let's now examine the transfer function of the circuit

$$H(j\omega) = \frac{v_o}{v_s} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}$$

$$H(j\omega) = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC}$$

- Obviously, the circuit cannot conduct DC current, so there is a zero in the transfer function. The denominator is a quadratic polynomial. It's worthwhile to put it into a standard form that quickly reveals important circuit parameters

$$H(j\omega) = \frac{j\omega \frac{R}{L}}{\frac{1}{LC} + (j\omega)^2 + j\omega \frac{R}{L}}$$

# Canonical Form

- Using the definition of $Q$ and $\omega_0$ for the circuit

$$H(j\omega) = \frac{j\omega\frac{\omega_0}{Q}}{\omega_0^2 + (j\omega)^2 + j\frac{\omega\omega_0}{Q}}$$

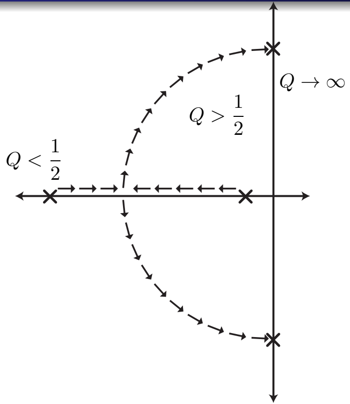
- Factoring the denominator with the assumption that $Q > \frac{1}{2}$ gives us the complex poles of the circuit

$$s^{\pm} = -\frac{\omega_0}{2Q} \pm j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

- The poles have a constant magnitude equal to the resonant frequency

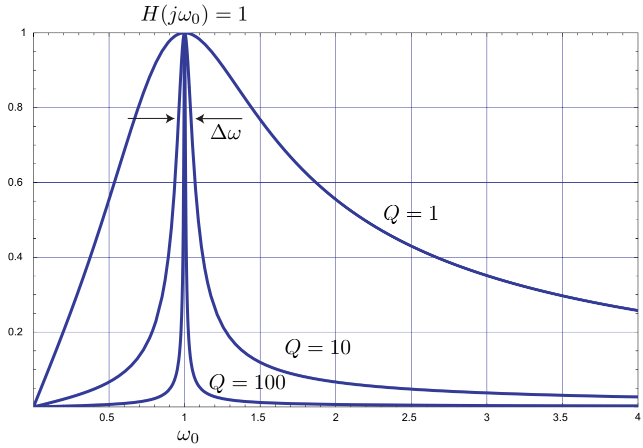
$$|s| = \sqrt{\frac{\omega_0^2}{4Q^2} + \omega_0^2\left(1 - \frac{1}{4Q^2}\right)} = \omega_0$$

# Root Locus




- A root-locus plot of the poles as a function of $Q$. As $Q \to \infty$, the poles move to the imaginary axis. In fact, the real part of the poles is inversely related to the $Q$ factor.

# Circuit Bandwidth




- As we plot the magnitude of the transfer function, we see that the selectivity of the circuit is also related inversely to the $Q$ factor.

# Selectivity

- In the limit that $Q \to \infty$, the circuit is infinitely selective and only allows signals at resonance $\omega_0$ to travel to the load.
- Note that the peak gain in the circuit is always unity, regardless of $Q$, since at resonance the $L$ and $C$ together disappear and effectively all the source voltage appears across the load.
- The selectivity of the circuit lends itself well to filter applications. To characterize the peakiness, let's compute the frequency when the magnitude squared of the transfer function drops by half

$$|H(j\omega)|^2 = \frac{\left(\omega \frac{\omega_0}{Q}\right)^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\omega \frac{\omega_0}{Q}\right)^2} = \frac{1}{2}$$

# Selectivity Bandwidth

- This happens when

$$\left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right)^2 = 1$$

- Solving the above equation yields four solutions, corresponding to two positive and two negative frequencies. The peakiness is characterized by the difference between these frequencies, or the bandwidth, given by

$$\Delta\omega = \omega_+ - \omega_- = \frac{\omega_0}{Q}$$

# Selectivity Bandwidth (cont)

- The normalized bandwidth is inversely proportional to the circuit $Q$.

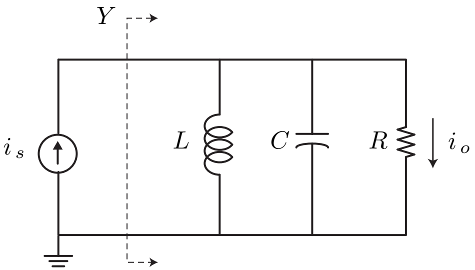
$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$$

- You can also show that the resonance frequency is the geometric mean frequency of the $3$ dB frequencies

$$\omega_0 = \sqrt{\omega_+ \omega_-}$$

# Parallel RLC Circuits

# Parallel RLC




- The parallel *RLC* circuit is the dual of the series circuit. By "dual" we mean that the role of voltage and currents are interchanged.
- Hence the circuit is most naturally probed with a current source $i_s$. In other words, the circuit has current gain as opposed to voltage gain, and the admittance minimizes at resonance as opposed to the impedance.

# Duality

- The role of capacitance and inductance are also interchanged. In principle, therefore, we don't have to repeat all the detailed calculations we just performed for the series case, but in practice it's a worthwhile exercise.
- The admittance of the circuit is given by

$$Y = j\omega C + \frac{1}{j\omega L} + G = G + j\omega C \left(1 - \frac{1}{\omega^2 LC}\right)$$

which has the same form as before. The resonant frequency also occurs when $\Im(Y) = 0$, or when $\omega = \omega_0 = \pm\frac{1}{\sqrt{LC}}$.

# Duality (cont)

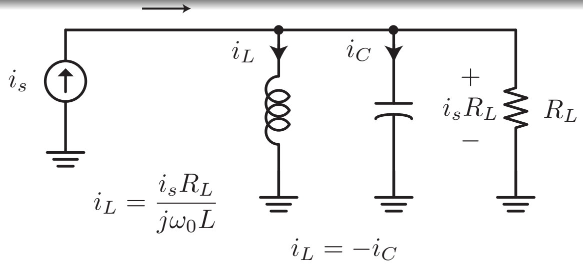
- Likewise, at resonance the admittance takes on a minimal value. Equivalently, the impedance at resonance is maximum.
- This property makes the parallel $RLC$ circuit an important element in tuned amplifier loads. It's also easy to show that at resonance the circuit has a current gain of $Q$

$$i_C = j\omega_0 C v_o = j\omega_0 C \frac{i_s}{Y(j\omega_0)} = j\omega_0 C \frac{i_s}{G} = jQ \times i_s$$

- where we have defined the circuit $Q$ factor at resonance by

$$Q = \frac{\omega_0 C}{G}$$

# Current Multiplication




- The current gain through the inductor is also easily derived

$$i_L = -jQ \times i_s$$

- The equivalent expressions for the circuit $Q$ factor are given by the inverse of the previous relations

$$Q = \frac{\omega_0 C}{G} = \frac{R}{\omega_0 L} = \frac{R}{\frac{1}{\sqrt{LC}} L} = \frac{R}{\sqrt{\frac{L}{C}}} = \frac{R}{Z_0}$$

# Phase Response

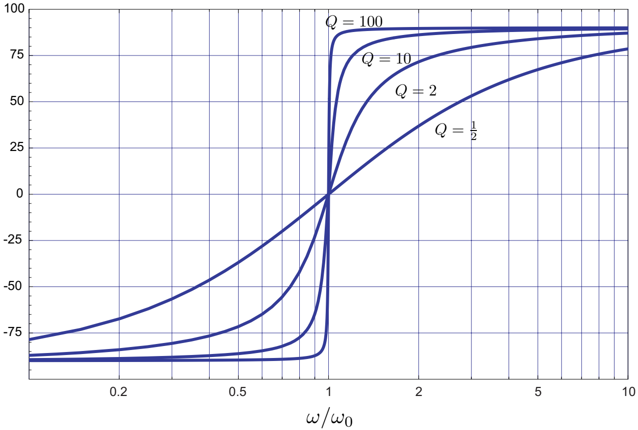
- The phase response of a resonant circuit is also related to the $Q$ factor. For the parallel $RLC$ circuit the phase of the admittance is given by

$$\angle Y(j\omega) = \tan^{-1}\left(\frac{\omega C\left(1 - \frac{1}{\omega^2 LC}\right)}{G}\right)$$

- The rate of change of phase at resonance is given by

$$\left.\frac{d\angle Y(j\omega)}{d\omega}\right|_{\omega_0} = \frac{2Q}{\omega_0}$$

# Phase Response




- A plot of the admittance phase as a function of frequency and $Q$ is shown. Higher $Q$ circuits go through a more rapid transition.

# Circuit Transfer Function

- Given the duality of the series and parallel $RLC$ circuits, it's easy to deduce the behavior of the circuit. Whereas the series $RLC$ circuit acted as a filter and was only sensitive to voltages near resonance $\omega_0$, likewise the parallel $RLC$ circuit is only sensitive to currents near resonance

$$H(j\omega) = \frac{i_o}{i_s} = \frac{v_o G}{v_o Y(j\omega)} = \frac{G}{j\omega C + \frac{1}{j\omega L} + G}$$

which can be put into the same canonical form as before

$$H(j\omega) = \frac{j\omega \frac{\omega_0}{Q}}{\omega_0^2 + (j\omega)^2 + j\frac{\omega\omega_0}{Q}}$$

- We have appropriately re-defined the circuit $Q$ to correspond the parallel $RLC$ circuit. Notice that the impedance of the circuit takes on the same form

$$Z(j\omega) = \frac{1}{Y(j\omega)} = \frac{1}{j\omega C + \frac{1}{j\omega L} + G}$$

- which can be simplified to

$$Z(j\omega) = \frac{j\frac{\omega}{\omega_0}\frac{1}{GQ}}{1 + \left(\frac{j\omega}{\omega_0}\right)^2 + j\frac{\omega}{\omega_0 Q}}$$

# Parallel Resonance

- At resonance, the real terms in the denominator cancel

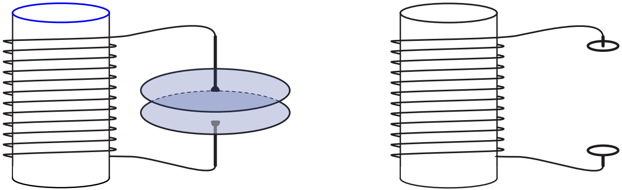
$$Z(j\omega_0) = \frac{j\frac{R}{Q}}{\underbrace{1 + \left(\frac{j\omega_0}{\omega_0}\right)^2}_{=0} + j\frac{1}{Q}} = R$$

- It's not hard to see that this circuit has the same half power bandwidth as the series $RLC$ circuit, since the denominator has the same functional form

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$$

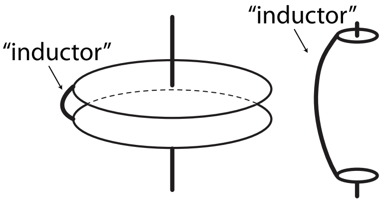
- plot of this impedance versus frequency has the same form as before multiplied by the resistance $R$.

# $LC$ Tank Frequency Limit



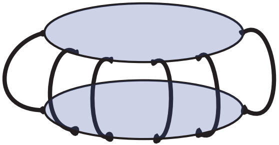
- What's the highest resonance frequency we can achieve with a "lumped" component $LC$ tank?
- Can we make $C$ and $L$ arbitrarily small?
- Clearly, to make $C$ small, we just move the plates apart and use smaller plates. But what about $L$?

# Small Capacitor → Bigger Inductor



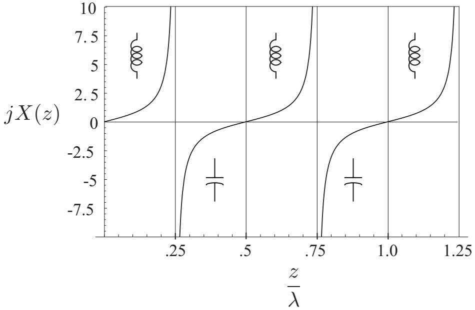

- As shown above, the smallest "inductor" has zero turns, or it's just straight wire connected to the capacitor.
- Since inductance is defined for a loop, the capacitor is actually now part of the inductor and defines the inductance of the circuit.
- To make the inductance smaller requires that we increase the capacitor (bring plates closer).

# Feynman's Can



- Given a fixed plate spacing and size, we can also be clever and keep adding inductors in parallel to reduce the inductance.
- In the limit, we end up with a "can"!
- High frequency resonators are built this way from the outset, rather than from lumped components.

# Shorted Line Impedance




- Recall the behavior of a shorted line when its length is a quarter wavelength
- Notice that the line is exactly $\lambda/4$ at a particular frequency. If we change the frequency, the line becomes either inductive or capacitive
- Also, if the line has loss, we might expect that the line cannot be a true open circuit, but a high impedance.

# Lossy Transmission Line Impedance

- Using the same methods to calculate the impedance for the low-loss line, we arrive at the following line voltage/current

$$v(z) = v^+ e^{-\gamma z}(1 + \rho_L e^{2\gamma z}) = v^+ e^{-\gamma z}(1 + \rho_L(z))$$

$$i(z) = \frac{v^+}{Z_0} e^{-\gamma z}(1 - \rho_L(z))$$

- Where $\rho_L(z)$ is the complex reflection coefficient at position $z$ and the load reflection coefficient is unaltered from before
- The input impedance is therefore

$$Z_{in}(z) = Z_0 \frac{e^{-\gamma z} + \rho_L e^{\gamma z}}{e^{-\gamma z} - \rho_L e^{\gamma z}}$$

## Lossy T-Line Impedance (cont)

- Substituting the value of $\rho_L$ we arrive at a similar equation (now a hyperbolic tangent)

$$Z_{in}(-\ell) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}$$

- For a short line, if $\gamma\delta\ell \ll 1$, we may safely assume that

$$Z_{in}(-\delta\ell) = Z_0 \tanh(\gamma\delta\ell) \approx Z_0\gamma\delta\ell$$

- Recall that $Z_0\gamma = \sqrt{Z'/Y'}\sqrt{Z'Y'}$
- As expected, input impedance is therefore the series impedance of the line (where $R = R'\delta\ell$ and $L = L'\delta\ell$)

$$Z_{in}(-\delta\ell) = Z'\delta\ell = R + j\omega L$$

# Review of Resonance (I)

- We'd like to find the impedance of a series resonator near resonance $Z(\omega) = j\omega L + \frac{1}{j\omega C} + R$
- Recall the definition of the circuit $Q$

$$Q = \omega_0 \frac{\text{time average energy stored}}{\text{energy lost per cycle}}$$

- For a series resonator, $Q = \omega_0 L/R$. For a small frequency shift from resonance $\delta\omega \ll \omega_0$

$$Z(\omega_0 + \delta\omega) = j\omega_0 L + j\delta\omega L + \frac{1}{j\omega_0 C}\left(\frac{1}{1 + \frac{\delta\omega}{\omega_0}}\right) + R$$

# Review of Resonance (II)

- Which can be simplified using the fact that $\omega_0 L = \frac{1}{\omega_0 C}$

$$Z(\omega_0 + \delta\omega) = j2\delta\omega L + R$$

- Using the definition of $Q$

$$Z(\omega_0 + \delta\omega) = R\left(1 + j2Q\frac{\delta\omega}{\omega_0}\right)$$

- For a parallel line, the same formula applies to the admittance

$$Y(\omega_0 + \delta\omega) = G\left(1 + j2Q\frac{\delta\omega}{\omega_0}\right)$$

- Where $Q = \omega_0 C/G$

# $\lambda/2$ T-Line Resonators (Series)

- A shorted transmission line of length $\ell$ has input impedance of $Z_{in} = Z_0 \tanh(\gamma\ell)$
- For a low-loss line, $Z_0$ is almost real
- Expanding the tanh term into real and imaginary parts

$$\tanh(\alpha\ell + j\beta\ell) = \frac{\sinh(2\alpha\ell)}{\cos(2\beta\ell) + \cosh(2\alpha\ell)} + \frac{j\sin(2\beta\ell)}{\cos(2\beta\ell) + \cosh(2\alpha\ell)}$$

- Since $\lambda_0 f_0 = c$ and $\ell = \lambda_0/2$ (near the resonant frequency), we have $\beta\ell = 2\pi\ell/\lambda = 2\pi\ell f/c = \pi + 2\pi\delta f\ell/c = \pi + \pi\delta\omega/\omega_0$
- If the lines are low loss, then $\alpha\ell \ll 1$

- Simplifying the above relation we come to

$$Z_i n = Z_0 \left( \alpha\ell + j \frac{\pi \delta\omega}{\omega_0} \right)$$

- The above form for the input impedance of the series resonant T-line has the same form as that of the series LRC circuit
- We can define equivalent elements

$$R_{eq} = Z_0 \alpha \ell = Z_0 \alpha \lambda / 2$$

$$L_{eq} = \frac{\pi Z_0}{2\omega_0} \qquad C_{eq} = \frac{2}{Z_0 \pi \omega_0}$$

# $\lambda/2$ Series Resonance Q

- The equivalent $Q$ factor is given by

$$Q = \frac{1}{\omega_0 R_{eq} C_{eq}} = \frac{\pi}{\alpha\lambda_0} = \frac{\beta_0}{2\alpha}$$

- For a low-loss line, this $Q$ factor can be made very large. A good T-line might have a $Q$ of 1000 or 10,000 or more
- It's difficult to build a lumped circuit resonator with such a high $Q$ factor

# $\lambda/4$ T-Line Resonators (Parallel)

- For a short-circuited $\lambda/4$ line

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell = Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell}$$

- Multiply numerator and denominator by $-j \cot \beta\ell$

$$Z_{in} = Z_0 \frac{1 - j \tanh \alpha\ell \cot \beta\ell}{\tanh \alpha\ell - j \cot \beta\ell}$$

- For $\ell = \lambda/4$ at $\omega = \omega_0$ and $\omega = \omega_0 + \delta\omega$

$$\beta\ell = \frac{\omega_0 \ell}{v} + \frac{\delta\omega \ell}{v} = \frac{\pi}{2} + \frac{\pi \delta\omega}{2\omega_0}$$

# $\lambda/4$ T-Line Resonators (Parallel)

- So $\cot\beta\ell = -\tan\frac{\pi\delta\omega}{2\omega_0} \approx \frac{-\pi\delta\omega}{2\omega_0}$ and $\tanh\alpha\ell \approx \alpha\ell$

$$Z_{in} = Z_0 \frac{1 + j\alpha\ell\pi\delta\omega/2\omega_0}{\alpha\ell + j\pi\delta\omega/2\omega_0} \approx \frac{Z_0}{\alpha\ell + j\pi\delta\omega/2\omega_0}$$

- This has the same form for a parallel resonant $RLC$ circuit

$$Z_{in} = \frac{1}{1/R + 2j\delta\omega C}$$

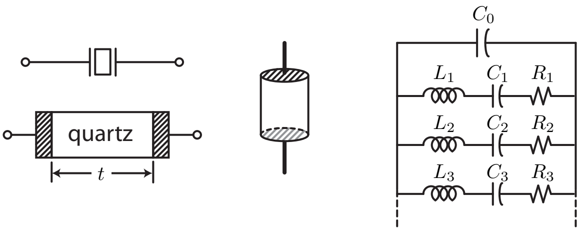
- The equivalent circuit elements are

$$R_{eq} = \frac{Z_0}{\alpha\ell} \qquad C_{eq} = \frac{\pi}{4\omega_0 Z_0} \qquad L_{eq} = \frac{1}{\omega_0^2 C_{eq}}$$

# $\lambda/4$ T-Line Resonators Q Factor

- The quality factor is thus

$$Q = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha}$$




- Quartz crystal is a piezoelectric material. An electric field causes a mechanical displacement and vice versa. Thus it is a electromechanical transducer.
- The equivalent circuit contains series *LCR* circuits that represent resonant modes of the XTAL. The capacitor $C_0$ is a physical capacitor that results from the parallel plate capacitance due to the leads.

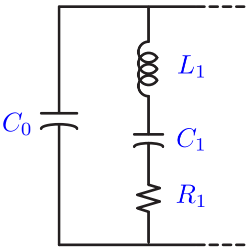
# Fundamental Resonant Mode

- Acoustic waves through the crystal have phase velocity $v = 3 \times 10^3 \mathrm{m/s}$. For a thickness $t = 1\mathrm{mm}$, the delay time through the XTAL is given by $\tau = t/v = (10^{-3}\mathrm{m})/(3 \times 10^3 \mathrm{m/s}) = 1/3\mu\mathrm{s}$.
- This corresponds to a fundamental resonant frequency $f_0 = 1/\tau = v/t = 3\mathrm{MHz} = \frac{1}{2\pi\sqrt{L_1 C_1}}$.
- The quality factor is extremely high, with $Q \sim 3 \times 10^6$ (in vacuum) and about $Q = 1 \times 10^6$ (air). This is much higher than can be acheived with electrical circuit elements (inductors, capacitors, transmission lines, etc). This high $Q$ factor leads to good frequency stability (low phase noise).

# MEMS Resonators

- The highest frequency, though, is limited by the thickness of the material. For $t \approx 15\mu\mathrm{m}$, the frequency is about $200\mathrm{MHz}$. MEMS resonators have been demonstrated up to $\sim \mathrm{GHz}$ frequencies. MEMS resonators are an active research area.
- Integrated MEMS resonators are fabricated from polysilicon beams (forks), disks, and other mechanical structures. These resonators are electrostatically induced structures.

# Example XTAL

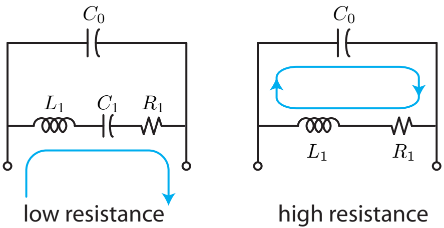



- Some typical numbers for a fundamental mode resonator are $C_0 = 3\text{pF}$, $L_1 = 0.25\text{H}$, $C_1 = 40\text{fF}$, $R_1 = 50\Omega$, and $f_0 = 1.6\text{MHz}$. Note that the values of $L_1$ and $C_1$ are modeling parameters and not physical inductance/capacitance. The value of $L$ is large in order to reflect the high quality factor.

- The quality factor is given by

$$Q = \frac{\omega L_1}{R_1} = 50 \times 10^3 = \frac{1}{\omega R_1 C_1}$$

# Series and Parallel Mode




- Due to the external physical capacitor, there are two resonant modes between a series branch and the capacitor. In the series mode $\omega_s$, the $LCR$ is a low impedance ("short"). But beyond this frequency, the $LCR$ is an equivalent inductor that resonates with the external capacitance to produce a parallel resonant circuit at frequency $\omega_p > \omega_s$.