Integrated Circuits for Communication
Berkeley

The Smith Chart

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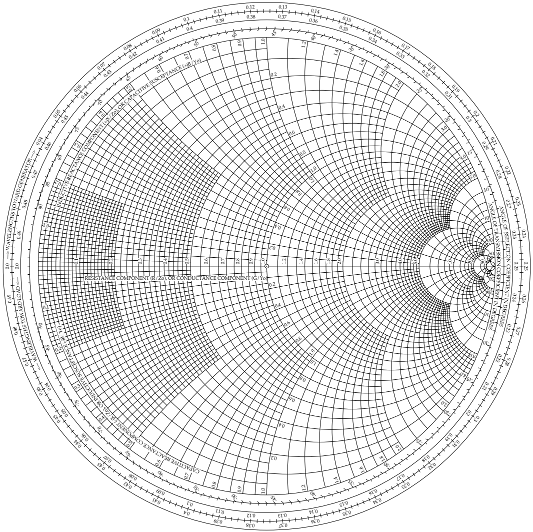
February 11, 2025

The Smith Chart

- The Smith Chart is simply a graphical calculator for computing impedance as a function of reflection coefficient $z = f(\rho)$
- More importantly, many problems can be easily visualized with the Smith Chart
- This visualization leads to a insight about the behavior of transmission lines
- All the knowledge is coherently and compactly represented by the Smith Chart
- Why else study the Smith Chart? It's beautiful!
- Aside: There are deep mathematical connections in the Smith Chart. It's the tip of the iceberg! Study complex analysis to learn more.

An Impedance Smith Chart

- Without further ado, here it is!



Generalized Reflection Coefficient

- We have found that in sinusoidal steady-state, the voltage on the line is a T-line

$$v(z) = v^+(z) + v^-(z) = V^+(e^{-\gamma z} + \rho_L e^{\gamma z})$$

- Recall that we can define the reflection coefficient anywhere by taking the ratio of the reflected wave to the forward wave

$$\rho(z) = \frac{v^-(z)}{v^+(z)} = \frac{\rho_L e^{\gamma z}}{e^{-\gamma z}} = \rho_L e^{2\gamma z}$$

- Therefore the impedance on the line ...

$$Z(z) = \frac{v^+ e^{-\gamma z}(1 + \rho_L e^{2\gamma z})}{\frac{v^+}{Z_0} e^{-\gamma z}(1 - \rho_L e^{2\gamma z})}$$

Normalized Impedance

- ...can be expressed in terms of $\rho(z)$

$$Z(z) = Z_0 \frac{1+\rho(z)}{1-\rho(z)}$$

- It is extremely fruitful to work with normalized impedance values $z = Z/Z_0$

$$z(z) = \frac{Z(z)}{Z_0} = \frac{1+\rho(z)}{1-\rho(z)}$$

- Let the normalized impedance be written as $z = r + jx$ (note small case)
- The reflection coefficient is "normalized" by default since for passive loads $|\rho| \leq 1$. Let $\rho = u + jv$

Dissection of the Transformation

- Now simply equate the real ($\Re$) and imaginary ($\Im$) components in the above equaiton

$$r + jx = \frac{(1+u)+jv}{(1-u)-jv} = \frac{(1+u+jv)(1-u+jv)}{(1-u)^2+v^2}$$

- To obtain the relationship between the (r, x) plane and the (u, v) plane

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

$$x = \frac{v(1-u)+v(1+u)}{(1-u)^2+v^2}$$

- The above equations can be simplified and put into a nice form

Completing Your Squares...

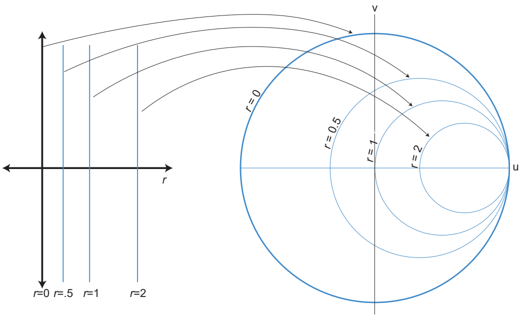
- If you remember your high school algebra, you can derive the following equivalent equations

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2}$$

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

- These are circles in the (u, v) plane! Circles are good!
- We see that vertical and horizontal lines in the (r, x) plane (complex impedance plane) are transformed to circles in the (u, v) plane (complex reflection coefficient)

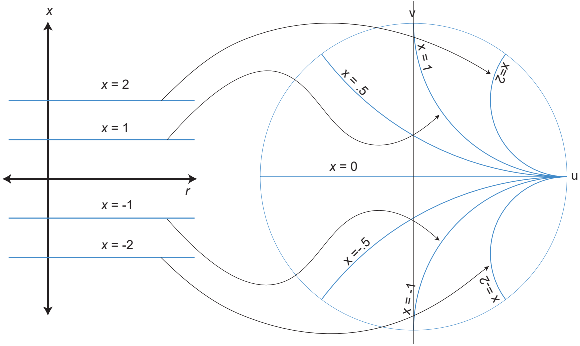
Resistance Transformations



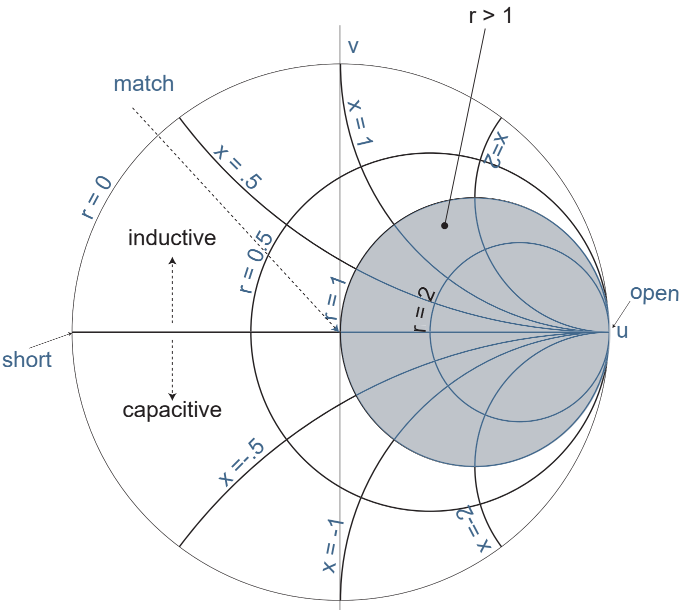
- $r = 0$ maps to $u^2 + v^2 = 1$ (unit circle)
- $r = 1$ maps to $(u - 1/2)^2 + v^2 = (1/2)^2$ (matched real part)
- $r = .5$ maps to $(u - 1/3)^2 + v^2 = (2/3)^2$ (load R less than Z_0)
- $r = 2$ maps to $(u - 2/3)^2 + v^2 = (1/3)^2$ (load R greater than Z_0)

Reactance Transformations

- $x = \pm 1$ maps to $(u-1)^2 + (v \mp 1)^2 = 1$
- $x = \pm 2$ maps to $(u-1)^2 + (v \mp 1/2)^2 = (1/2)^2$
- $x = \pm 1/2$ maps to $(u-1)^2 + (v \mp 2)^2 = 2^2$
- Inductive reactance maps to upper half of unit circle
- Capacitive reactance maps to lower half of unit circle

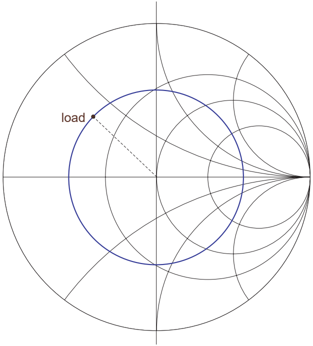


Complete Smith Chart



Reading the Smith Chart

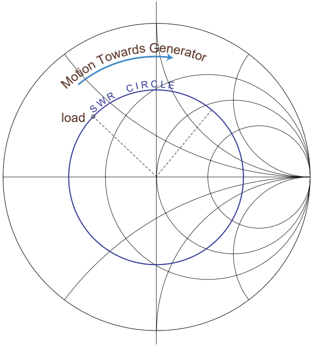
Reading the Smith Chart



- First map z_L on the Smith Chart as ρ_L
- To read off the impedance on the T-line at any point on a lossless line, simply move on a circle of constant radius since

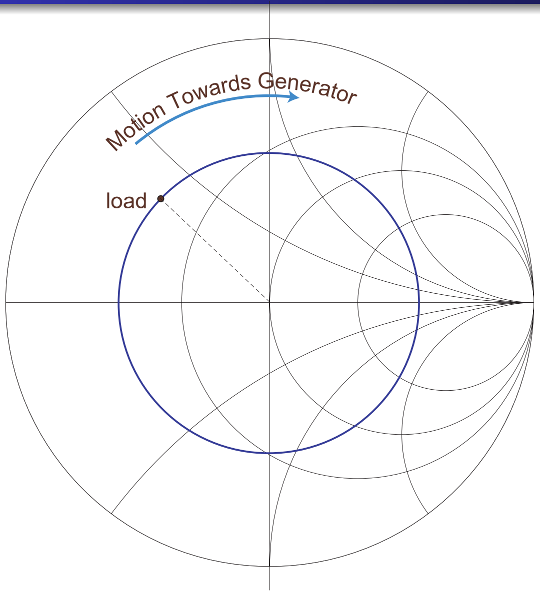
$$\rho(z) = \rho_L e^{2j\beta z}$$

Motion Towards Generator



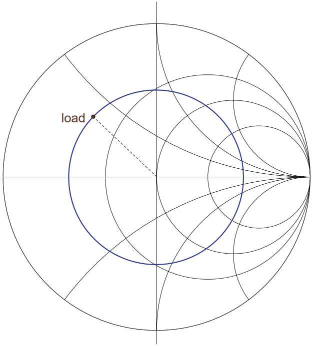
- Moving towards generator means $\rho(-\ell) = \rho_L e^{-2j\beta\ell}$, or clockwise motion
- We're back to where we started when $2\beta\ell = 2\pi$, or $\ell = \lambda/2$
- Thus the impedance is periodic (as we know)
- Aside: For a lossy line, this corresponds to a spiral motion and so the Smith Chart is more useful for low-loss lines

SWR Circle



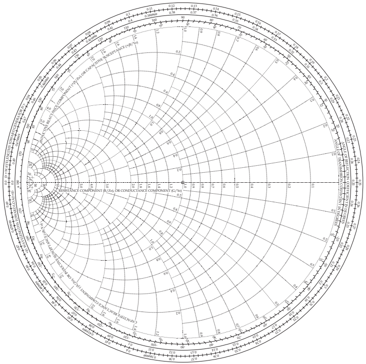
- Since SWR is a function of $|\rho|$, a circle at origin in (u, v) plane is called an SWR circle
- Recall the voltage max occurs when the reflected wave is in phase with the forward wave, so $\rho(z_{min}) = |\rho_L|$
- This corresponds to the intersection of the SWR circle with the positive real axis (read off SWR by just reading the value of r)
- Likewise, the intersection with the negative real axis is the location of the voltage min

Example of Smith Chart Visualization



- Prove that if Z_L has an inductance reactance, then the position of the first voltage maximum occurs before the voltage minimum as we move towards the generator
- Proof: On the Smith Chart start at any point in the upper half of the unit circle. Moving towards the generator corresponds to clockwise motion on a circle. Therefore we will always cross the positive real axis first and then the negative real axis.

Admittance Chart



- Since $y = 1/z = \frac{1-\rho}{1+\rho}$, you can imagine that an Admittance Smith Chart looks very similar
- In fact everything is switched around a bit and you can construct a combined admittance/impedance Smith Chart. You can also use an impedance chart for admittance if you simply map $x \to b$ and $r \to g$
- Be careful ... the caps are now on the top of the chart and the inductors on the bottom
- The short and open likewise swap positions

Admittance on Smith Chart

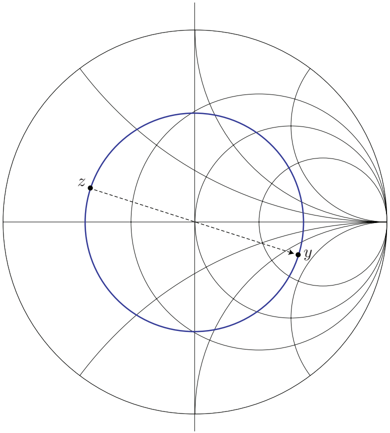
- Sometimes you may need to work with both impedances and admittances.
- This is not easy on the Smith Chart and requires proficient use of the impedance inversion property of a $\lambda/4$ line (it actually can get pretty confusing)

$$Z' = \frac{Z_0^2}{Z}$$

- If we normalize Z' we get y

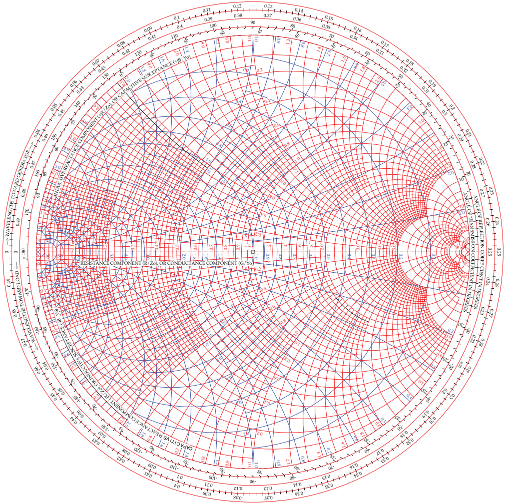
$$\frac{Z'}{Z_0} = \frac{Z_0}{Z} = \frac{1}{z} = y$$

Admittance Conversion



- Thus if we simply rotate π degrees on the Smith Chart and read off the impedance, we're actually reading off the admittance!
- Rotating π degrees is easy. Simply draw a line through origin and z_L and read off the second point of intersection on the SWR circle

Combined Chart

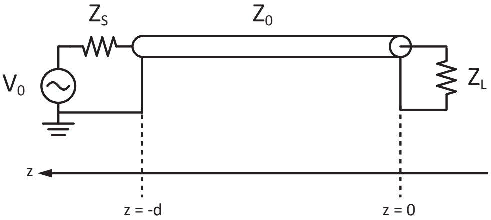


- On a combined chart, you can read off both y and z. You can also easily visualize and draw motions along constant r, g, x, and b circles all on the same chart.

Example Calculation

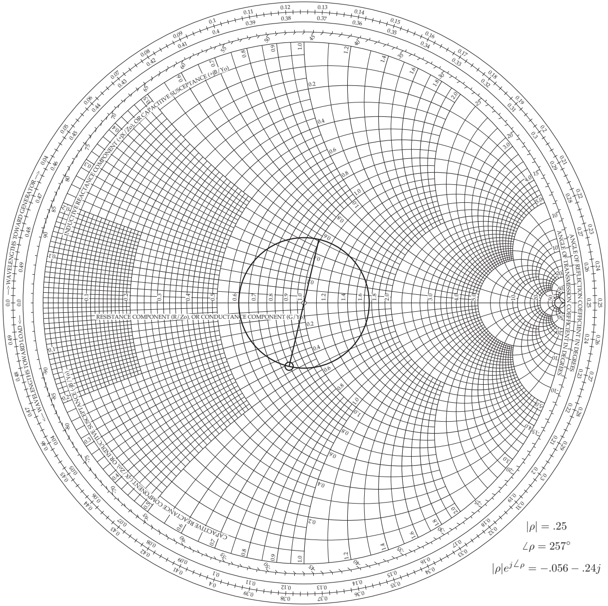
Example Calculation

Example Problem



1. Consider the transmission line circuit shown below. A voltage source generating 10V amplitude of sinewave at 10 GHz is driving a transmission line terminated with load Z_L = 80 - j40 ohm. The transmission line has a characteristic impedance of Z_0 (= 100Ω), effective dielectric constant of 4, and length d =22.5 mm.
    1. Find the reflection coefficient at the load ($z = 0$) and at the source ($z = -d$). [Note this is 1.5λ]
    2. Find the input impedance at the source (z = -d) and at z = 18.75 mm. [Note this is 1.25λ]
    3. Plot the magnitude of the voltage along the line. Find voltage maximum, voltage minimum, and standing wave ratio.

Homework Problem with Aid from Mr. Smith



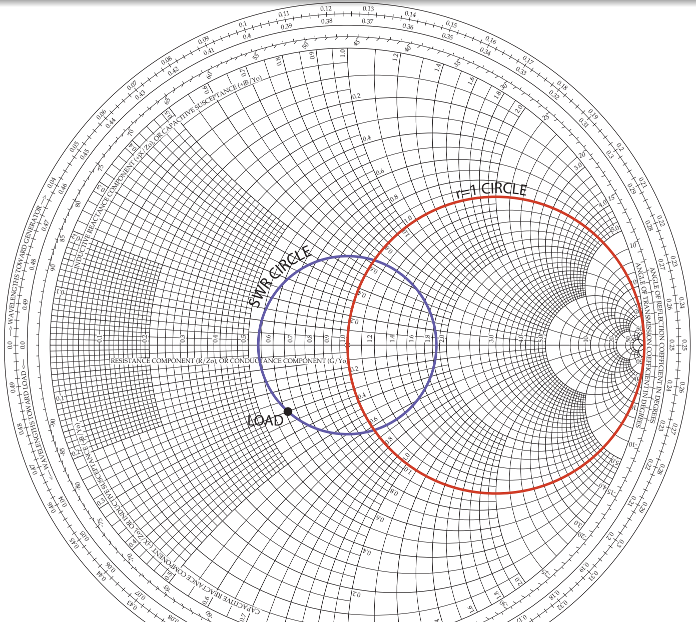
$|\rho| = .25$

$\angle\rho = 257°$

$|\rho|e^{j\angle\rho} = -.056 - .24j$

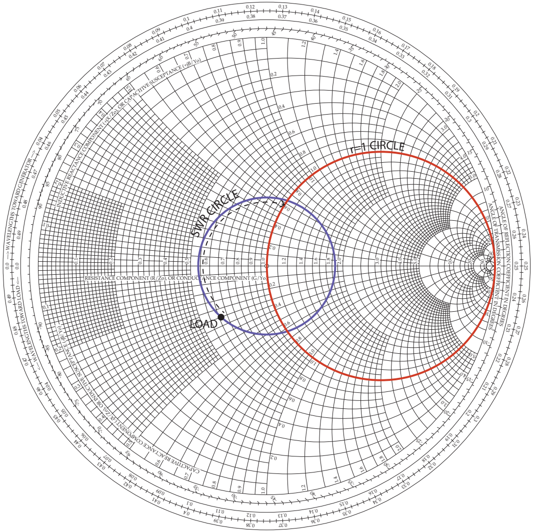
Impedance Matching with Smith Chart

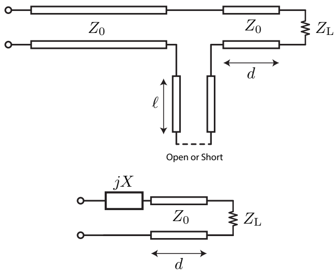
Impedance Matching Example



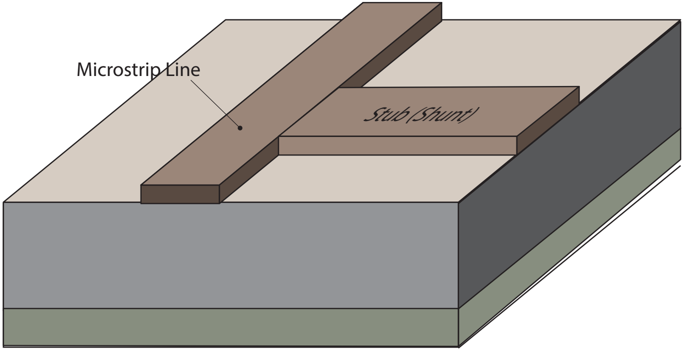
- Single stub impedance matching is easy to do with the Smith Chart
- Simply find the intersection of the SWR circle with the $r = 1$ circle
- The match is at the center of the circle. Grab a reactance in series or shunt to move you there!

Series Stub Match



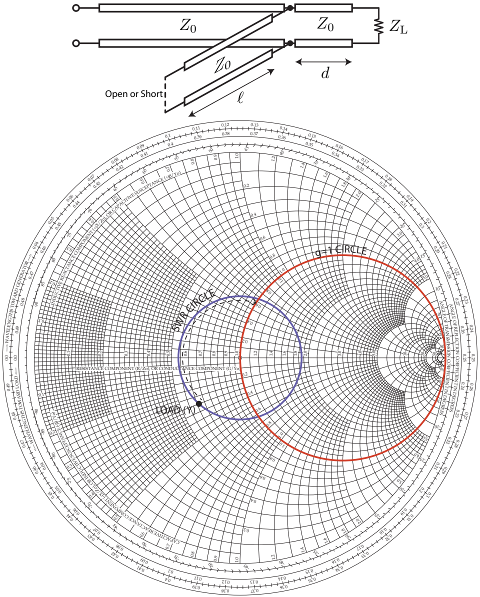


Series Sub Using Microstrip ?



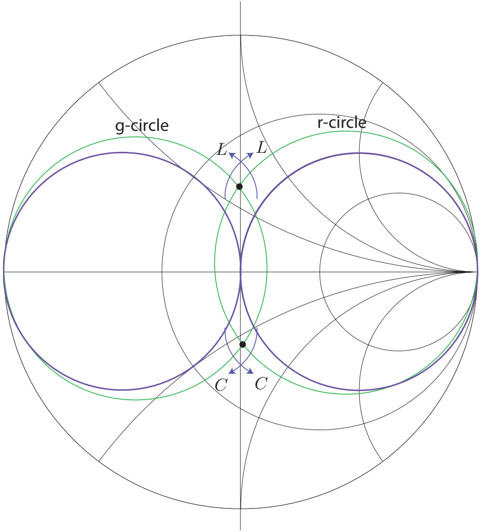
- Shunt stubs are easy but series stubs are not easy and require cuts in the ground plane.

Shunt Stub Match



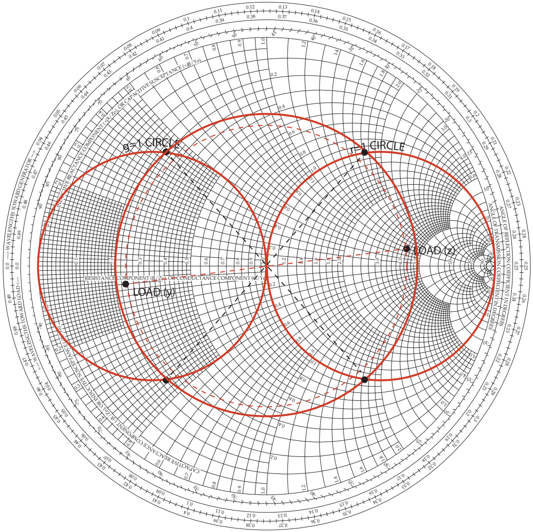
- Let's now solve the same matching problem with a shunt stub.
- To find the shunt stub value, simply convert the value of $z = 1 + jx$ to $y = 1 + jb$ and use the Smith chart as an Admittance chart.
- Find the distance to move on an SWR circle to reach the $1 + jb$ circle (same as $1 + jx$ circle since we're using it as Y-chart) and read off the distance in wavelengths. To find the shunt stub length, start at a short (open) and move until you reach the desired susceptance.

Lumped Components On Smith Chart



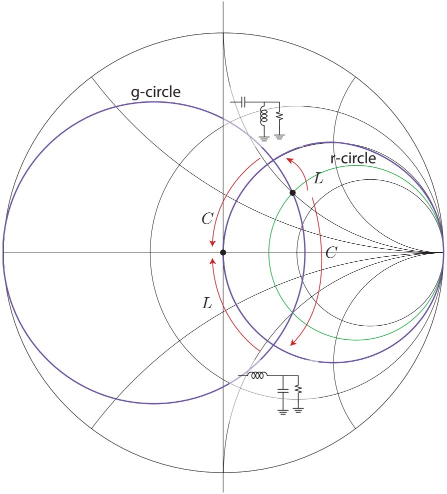
- Adding an inductor in series moves us on the constant *r* circle clock-wise (CW). Adding a capacitor in series moves counter clock-wise (CCW).
- On the Y-plane, to go CW, add a shunt C. To move CCW, add a shunt L.

Matching with Lumped Components (Inside)



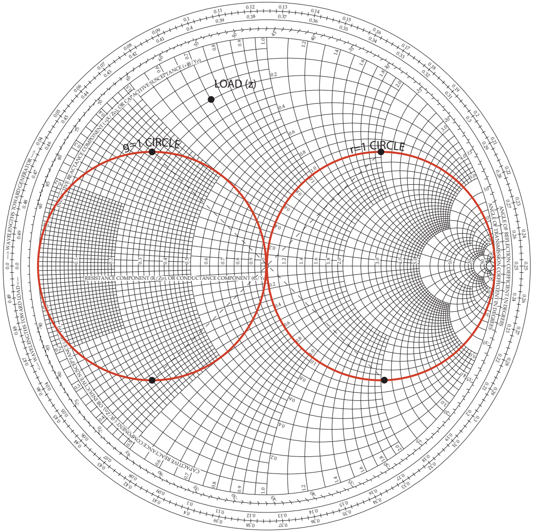
Suppose the load is inside the $1 + jx$ circle.

Escaping from “Insdie”



- Notice that there are two paths to get to the center. The difference is one is AC coupled versus DC coupled, so often the application will determine the choice.

Matching with Lumped Components (Outside)



Suppose the load is outside the $1 + jx$ circle.

Q Circles

- For designing matching networks, it's very convenient to know the required Q to achieve a match (related to bandwidth and loss). Let's find the contours of constant Q on the Smith Chart. Recall that $z = r + jx$ can be related to the reflection coefficient $\rho = u + jv$ by the following equations

$$r = \frac{1 - u^2 - v^2}{(1-u)^2 + v^2}$$

$$x = \frac{v(1-u) + v(1+u)}{(1-u)^2 + v^2}$$

So that the Q is given by:

$$Q = \frac{x}{r} = \frac{2v}{1 - u^2 - v^2}$$

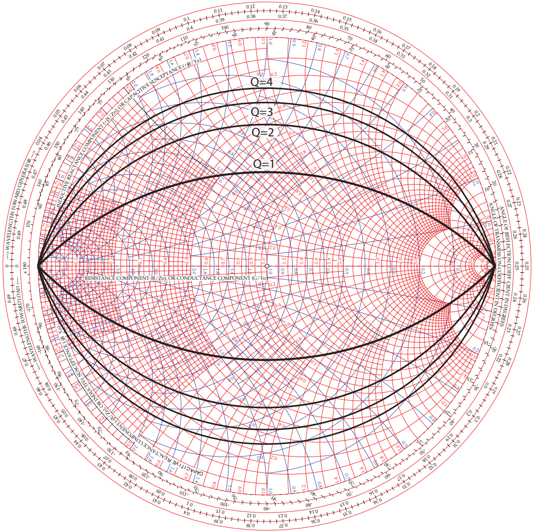
Q Circles (cont)

- We can write this as

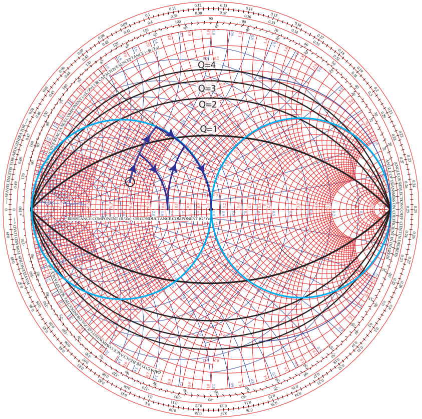
$$Q(1-u^2-v^2)=2v$$
$$(1-u^2)=v^2+2v/Q=v^2+2v/Q+1/Q^2-1/Q^2$$
$$1-1/Q^2=u^2+(v+1/Q)^2$$

- This is an equation for another circle, centered at $(u,v)=(0,-1/Q)$ with a radius of $R=\sqrt{1+\frac{1}{Q^2}}$.
- For a capacitive element, the same derivation holds except the circle is centered at $(u,v)=(0,+1/Q)$

Q Circles on Smith Chart

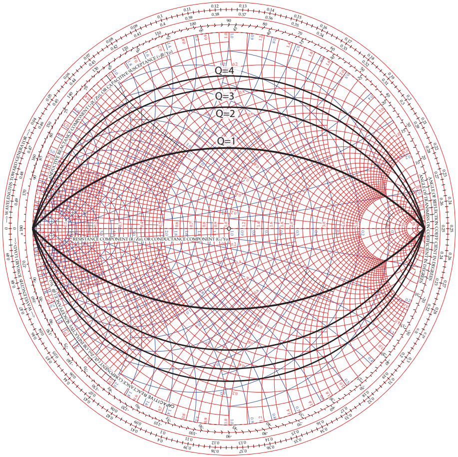


Single Stage vs Two-Stage Matching Network



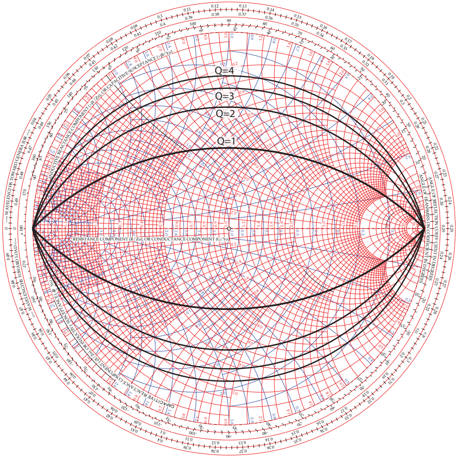
- Notice that with a one stage matching network, the Q is fixed (we already knew this but now we see it graphically).
- On the other hand, with a two-stage matching network we can actually take a different path and lower the Q.

Multi-Stage Matching Network



- For broadband matching, “hug” the x-axis. At high frequency, we can also include transmission line leads into the network.

Constant Q Matching Network



- Pick a desired Q (set by bandwidth), and then never allow the matching ratio to go above this value. The number of steps required will be set by the matching ratio.