Integrated Circuits for Communication
Berkeley

Scattering Parameters

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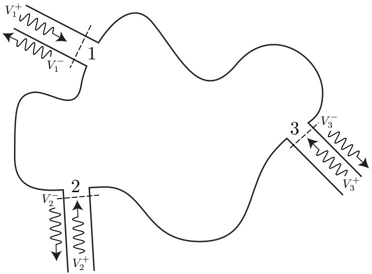
Scattering Parameters

Scattering Matrix

- Voltages and currents are difficult to measure directly at microwave freq. Z matrix requires "opens", and it's hard to create an ideal open (parasitic capacitance and radiation). Likewise, a Y matrix requires "shorts", again ideal shorts are impossible at high frequency due to the finite inductance.
- Many active devices could oscillate under the open or short termination.
- S parameters are easier to measure at high frequency. The measurement is direct and only involves measurement of relative quantities (such as the SWR or the location of the first minima relative to the load).

S-Parameters



- Scattering parameters represent the flow of power into and out of ports of an arbitrary N-port
- It's important to realize that although we associate S parameters with high frequency and wave propagation, the concept is valid for any frequency.

Power Flow in an One-Port

- We begin with the simple observation that the power flow into a one-port circuit can be written in the following form

$$P_{in} = P_{avs} - P_r$$

- where P_{avs} is the available power from the source. Unless otherwise stated, let us assume sinusoidal steady-state. If the source has a real resistance of Z_0, this is simply given by

$$P_{avs} = \frac{V_s^2}{8Z_0}$$

- Of course if the one-port is conjugately matched to the source, then it will draw the maximal available power from the source. Otherwise, the power P_{in} is always less than P_{avs}, which is reflected in our equation. In general, P_r represents the wasted or untapped power that one-port circuit is "reflecting" back to the source due to a mismatch. For passive circuits it's clear that each term in the equation is positive and $P_{in} \geq 0$.

Power Absorbed by One-Port

- The complex power absorbed by the one-port is given by

$$P_{in} = \frac{1}{2}(V_1 \cdot I_1^* + V_1^* \cdot I_1)$$

- which allows us to write

$$P_r = P_{avs} - P_{in} = \frac{V_s^2}{4Z_0} - \frac{1}{2}(V_1 I_1^* + V_1^* I_1)$$

- the factor of 4 instead of 8 is used since we are now dealing with complex power. The average power can be obtained by taking one half of the real component of the complex power. If the one-port has an input impedance of Z_{in}, then the power P_{in} is expanded to

$$P_{in} = \frac{1}{2}\left(\frac{Z_{in}}{Z_{in}+Z_0} V_s \cdot \frac{V_s^*}{(Z_{in}+Z_0)^*} + \frac{Z_{in}^*}{(Z_{in}+Z_0)^*} V_s^* \cdot \frac{V_s}{(Z_{in}+Z_0)}\right)$$

(cont.)

- The previous equation is easily simplified to (where we have assumed Z_0 is real)

$$P_{in} = \frac{|V_s|^2}{2Z_0}\left(\frac{Z_0 Z_{in} + Z_{in}^* Z_0}{|Z_{in} + Z_0|^2}\right)$$

- With the exception of a factor of 2, the premultiplier is simply the source available power, which means that our overall expression for the reflected power is given by

$$P_r = \frac{V_s^2}{4Z_0}\left(1 - 2\frac{Z_0 Z_{in} + Z_{in}^* Z_0}{|Z_{in} + Z_0|^2}\right)$$

which can be simplified

$$P_r = P_{avs}\left|\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right|^2 = P_{avs}|\Gamma|^2$$

Definition of Reflection Coefficient

$$P_r = P_{avs} \left| \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \right|^2 = P_{avs} |\Gamma|^2$$

- We have defined Γ, or the reflection coefficient, as

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

- From the definition it is clear that $|\Gamma| \leq 1$, which is just a re-statement of the conservation of energy implied by our assumption of a passive load.
- This constant Γ, also called the scattering parameter of a one-port, plays a very important role. On one hand we see that it is has a one-to-one relationship with Z_{in}.

- Given Γ we can solve for Z_{in} by inverting the above equation

$$Z_{in} = Z_0 \frac{1+\Gamma}{1-\Gamma}$$

which means that all of the information in Z_{in} is also in Γ. Moreover, since $|\Gamma| < 1$, we see that the space of the semi-infinite space of all impedance values with real positive components (the right-half plane) maps into the unit circle. This is a great compression of information which allows us to visualize the entire space of realizable impedance values by simply observing the unit circle. We shall find wide application for this concept when finding the appropriate load/source impedance for an amplifier to meet a given noise or gain specification.

Scattering Parameter as Power Flow

- More importantly, Γ expresses very direct and obviously the power flow in the circuit. If $\Gamma = 0$, then the one-port is absorbing all the possible power available from the source. If $|\Gamma| = 1$ then the one-port is not absorbing any power, but rather "reflecting" the power back to the source. Clearly an open circuit, short circuit, or a reactive load cannot absorb net power. For an open and short load, this is obvious from the definition of Γ. For a reactive load, this is pretty clear if we substitute $Z_{in} = jX$

$$|\Gamma_X| = \left|\frac{jX - Z_0}{jX + Z_0}\right| = \left|\frac{\sqrt{X^2 + Z_0^2}}{\sqrt{X^2 + Z_0^2}}\right| = 1$$

Relation between Z and Γ

- The transformation between impedance and Γ is the well known Bilinear Transform. It is a conformal mapping (meaning that it preserves angles) from vertical and horizontal lines into circles. We have already seen that the jX axis is mapped onto the unit circle.
- Since $|\Gamma|^2$ represents power flow, we may imagine that Γ should represent the flow of voltage, current, or some linear combination thereof. Consider taking the square root of the basic equation we have derived

  $$\sqrt{P_r} = \Gamma\sqrt{P_{avs}}$$

  where we have retained the positive root. We may write the above equation as

  $$b_1 = \Gamma a_1$$

  where a and b have the units of square root of power and represent signal flow in the network. How are a and b related to currents and voltage?

Definition of a and b

- Let

$$a_1 = \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}}$$

and

$$b_1 = \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}}$$

It is now easy to show that for the one-port circuit, these relations indeed represent the available and reflected power:

$$|a_1|^2 = \frac{|V_1|^2}{4Z_0} + \frac{Z_0 |I_1|^2}{4} + \frac{V_1^* \cdot I_1 + V_1 \cdot I_1^*}{4}$$

Now substitute $V_1 = Z_{in} V_s / (Z_{in} + Z_0)$ and $I_1 = V_s / (Z_{in} + Z_0)$ we have

$$|a_1|^2 = \frac{|V_s|^2}{4Z_0} \frac{|Z_{in}|^2}{|Z_{in} + Z_0|^2} + \frac{Z_0 |V_s|^2}{4|Z_{in} + Z_0|^2} + \frac{|V_s|^2}{4Z_0} \frac{Z_{in}^* Z_0 + Z_{in} Z_0}{|Z_{in} + Z_0|^2}$$

a/b and Power Flow

- We have now shown that a_1 is associated with the power available from the source:

$$|a_1|^2 = \frac{|V_s|^2}{4Z_0}\left(\frac{|Z_{in}|^2 + Z_0^2 + Z_{in}^* Z_0 + Z_{in}Z_0}{|Z_{in}+Z_0|^2}\right)$$

$$= \frac{|V_s|^2}{4Z_0}\left(\frac{|Z_{in}+Z_0|^2}{|Z_{in}+Z_0|^2}\right) = P_{avs}$$

- In a like manner, the square of b is given by many similar terms

$$|b_1|^2 = \frac{|V_s|^2}{4Z_0}\left(\frac{|Z_{in}|^2 + Z_0^2 - Z_{in}^* Z_0 - Z_{in}Z_0}{|Z_{in}+Z_0|^2}\right) =$$

$$P_{avs}\left|\frac{Z_{in}-Z_0}{Z_{in}+Z_0}\right|^2 = P_{avs}|\Gamma|^2$$

$$= |a_1|^2|\Gamma|^2$$

as expected.

One-Port Equation

- We can now see that the expression $b = \Gamma \cdot a$ is analogous to the expression $V = Z \cdot I$ or $I = Y \cdot V$ and so it can be generalized to an N-port circuit. In fact, since a and b are linear combinations of v and i, there is a one-to-one relationship between the two. Taking the sum and difference of a and b we arrive at

$$a_1 + b_1 = \frac{2V_1}{2\sqrt{Z_0}} = \frac{V_1}{\sqrt{Z_0}}$$

which is related to the port voltage and

$$a_1 - b_1 = \frac{2Z_0 I_1}{2\sqrt{Z_0}} = \sqrt{Z_0} I_1$$

which is related to the port current.

Incident and Scattered Waves

Incident and Scattered Waves

- Let's define the vector v^+ as the incident "forward" waves on each transmission line connected to the N port. Define the reference plane as the point where the transmission line terminates onto the N port.
- The vector v^- is then the reflected or "scattered" waveform at the location of the port.

$$v^+ = \begin{pmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \\ \vdots \end{pmatrix} \qquad v^- = \begin{pmatrix} V_1^- \\ V_2^- \\ V_3^- \\ \vdots \end{pmatrix}$$

Scattering Waves (cont)

- Because the N port is linear, we expect that scattered field to be a linear function of the incident field

$$v^- = S v^+$$

- S is the scattering matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & \cdots \\ S_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

Relation to Voltages

- The fact that the S matrix exists can be easily proved if we recall that the voltage and current on each transmission line termination can be written as

$$V_i = V_i^+ + V_i^- \qquad I_i = Y_0(I_i^+ - I_i^-)$$

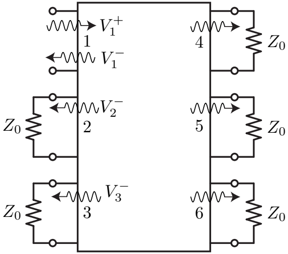
- Inverting these equations

$$V_i + Z_0 I_i = V_i^+ + V_i^- + V_i^+ - V_i^- = 2V_i^+$$

$$V_i - Z_0 I_i = V_i^+ + V_i^- - V_i^+ + V_i^- = 2V_i^-$$

- Thus v^+,v^- are simply linear combinations of the port voltages and currents. By the uniqueness theorem, then, $v^- = Sv^+$.

Measure S_{ij}



- The term S_{ij} can be computed directly by the following formula

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+ = 0 \, \forall \, k \neq j}$$

- In other words, to measure S_{ij}, drive port j with a wave amplitude of V_j^+ and terminate all other ports with the characteristic impedance of the lines (so that $V_k^+ = 0$ for $k \neq j$). Then observe the wave amplitude coming out of the port i

Termination

- It's important to realize that our definition of scattering parameters is independent of transmission lines and can be defined completely in terms of voltage and currents.
- So then why do we terminate the line? Because to make $V^+ = 0$ we solve:

$$V^+ = \frac{1}{2}(V_i + Z_0 I_i) = 0$$

- Solving this equation we find the conditions

$$V_i = -Z_0 I_i$$

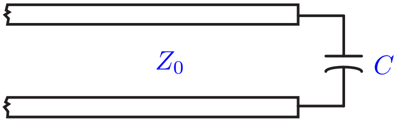
or

$$Z_L = \frac{V_i}{-I_i} = Z_0$$

- We see that there are no transmission lines to terminate, this rather follows from the definition of scattering parameters.

S Matrix for a 1-Port Capacitor

- Note you can solve this problem with T-lines or by definition of the S matrix:



- Let's calculate the S parameter for a capacitor

$$S_{11} = \frac{V_1^-}{V_1^+}$$

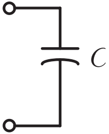
- This is of course just the reflection coefficient for a capacitor

$$S_{11} = \rho_L = \frac{Z_C - Z_0}{Z_C + Z_0} = \frac{\frac{1}{j\omega C} - Z_0}{\frac{1}{j\omega C} + Z_0}$$

$$= \frac{1 - j\omega C Z_0}{1 + j\omega C Z_0}$$

S Matrix for a 1-Port Cap (cont)

- Let's calculate the S parameter for a capacitor directly from the definition of S parameters



$$S_{11} = \frac{V_1^-}{V_1^+}$$

- Substituting for the current in a capacitor

$$V_1^- = V - IZ_0 = V - j\omega CV = V(1 - j\omega CZ_0)$$

$$V_1^+ = V + IZ_0 = V + j\omega CV = V(1 + j\omega CZ_0)$$

- We arrive at the same answer as expected

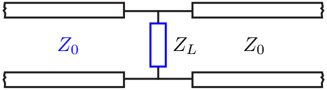
$$= \frac{1 - j\omega CZ_0}{1 + j\omega CZ_0}$$

S Matrix for a 2-Port Shunt Element

- Consider a shunt impedance connected at the junction of two transmission lines. The voltage at the junction is of course continuous. The currents, though, differ

$$V_1 = V_2$$

$$I_1 + I_2 = Y_L V_2$$



- To compute S_{11}, enforce $V_2^+ = 0$ by terminating the line. Thus we can be re-write the above equations

$$V_1^+ + V_1^- = V_2^-$$

$$Y_0(V_1^+ - V_1^-) = Y_0 V_2^- + Y_L V_2^- = (Y_L + Y_0) V_2^-$$

Shunt Element (cont)

- We can now solve the above eq. for the reflected and transmitted wave

$$V_1^- = V_2^- - V_1^+ = \frac{Y_0}{Y_L + Y_0}(V_1^+ - V_1^-) - V_1^+$$

$$V_1^-(Y_L + Y_0 + Y_0) = (Y_0 - (Y_0 + Y_L))V_1^+$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \frac{Y_0 - (Y_0 + Y_L)}{Y_0 + (Y_L + Y_0)} = \frac{Z_0||Z_L - Z_0}{Z_0||Z_L + Z_0}$$

- The above eq. can be written by inspection since $Z_0||Z_L$ is the effective load seen at the junction of port 1.
- Thus for port 2 we can write

$$S_{22} = \frac{Z_0||Z_L - Z_0}{Z_0||Z_L + Z_0}$$

Shunt Element (cont)

- Likewise, we can solve for the transmitted wave, or the wave scattered into port 2

$$S_{21} = \frac{V_2^-}{V_1^+}$$

- Since $V_2^- = V_1^+ + V_1^-$, we have

$$S_{21} = 1 + S_{11} = \frac{2Z_0||Z_L}{Z_0||Z_L + Z_0}$$

- By symmetry, we can deduce S_{12} as

$$S_{12} = \frac{2Z_0||Z_L}{Z_0||Z_L + Z_0}$$

Conversion Formula

- Since V^+ and V^- are related to V and I, it's easy to find a formula to convert for Z or Y to S

$$V_i = V_i^+ + V_i^- \;\rightarrow\; v = v^+ + v^-$$

$$Z_{i0} I_i = V_i^+ - V_i^- \;\rightarrow\; Z_0 i = v^+ - v^-$$

- Now starting with $v = Zi$, we have

$$v^+ + v^- = ZZ_0^{-1}(v^+ - v^-)$$

- Note that Z_0 is the scalar port impedance

$$v^-(I + ZZ_0^{-1}) = (ZZ_0^{-1} - I)v^+$$

$$v^- = (I + ZZ_0^{-1})^{-1}(ZZ_0^{-1} - I)v^+ = Sv^+$$

Conversion (cont)

- We now have a formula relating the Z matrix to the S matrix

$$S = (ZZ_0^{-1} + I)^{-1}(ZZ_0^{-1} - I) = (Z + Z_0 I)^{-1}(Z - Z_0 I)$$

- Recall that the reflection coefficient for a load is given by the same equation!

$$\overline{\rho} = \frac{Z/Z_0 - 1}{Z/Z_0 + 1}$$

- To solve for Z in terms of S, simply invert the relation

$$Z_0^{-1} ZS + IS = Z_0^{-1} Z - I$$

$$Z_0^{-1} Z(I - S) = S + I \qquad\qquad Z = Z_0(I + S)(I - S)^{-1}$$

- As expected, these equations degenerate into the correct form for a 1×1 system $Z_{11} = Z_0 \frac{1+S_{11}}{1-S_{11}}$

Properties of S-Parameters

Shift in Reference Planes

- Note that if we move the reference planes, we can easily recalculate the S parameters.
- We'll derive a new matrix S' related to S. Let's call the waves at the new reference ν

$$v^- = Sv^+$$

$$\nu^- = S'\nu^+$$

- Since the waves on the lossless transmission lines only experience a phase shift, we have a phase shift of $\theta_i = \beta_i \ell_i$

$$\nu_i^- = v^- e^{-j\theta_i}$$

$$\nu_i^+ = v^+ e^{j\theta_i}$$

Reference Plane (cont)

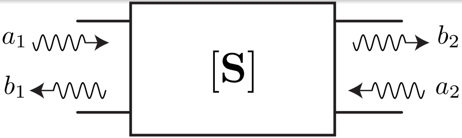
- Or we have

$$\begin{bmatrix} e^{j\theta_1} & 0 & \cdots & \\ 0 & e^{j\theta_2} & \cdots & \\ 0 & 0 & e^{j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix} \nu^- = S \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & \\ 0 & e^{-j\theta_2} & \cdots & \\ 0 & 0 & e^{-j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix} \nu^+$$

- So we see that the new S matrix is simply

$$S' = \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & \\ 0 & e^{-j\theta_2} & \cdots & \\ 0 & 0 & e^{-j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix} S \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & \\ 0 & e^{-j\theta_2} & \cdots & \\ 0 & 0 & e^{-j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix}$$

Normalized S-Parameters



- Let's introduce normalized voltage waves

$$a(x) = \frac{v^+(x)}{\sqrt{Z_0}} \qquad b(x) = \frac{v^-(x)}{\sqrt{Z_0}}$$

- So now $|a|^2$ and $|b|^2$ represent the power of the forward and reverse wave. Define the scattering matrix as before

$$b = Sa$$

- For a 2×2 system, this is simply

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Generalized Scattering Parameters

- We can use different impedances $Z_{0,n}$ at each port and so we have the generalized incident and reflected waves

$$a_n = \frac{v_n^+}{\sqrt{Z_{0,n}}} \qquad b_n = \frac{v_n^-}{\sqrt{Z_{0,n}}}$$

- The scattering parameters are now given by

$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_{k \neq j}=0} \qquad S_{ij} = \left.\frac{V_i^-}{V_j^+}\frac{\sqrt{Z_{0,j}}}{\sqrt{Z_{0,i}}}\right|_{V_{k \neq j}^+=0}$$

- Consider the current and voltage in terms of a and b

$$V_n = v_n^+ + v_n^- = \sqrt{Z_{0,n}}(a_n + b_n)$$

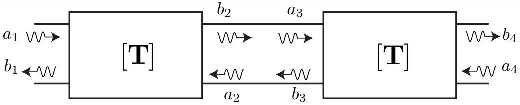
$$I_n = \frac{1}{Z_{0,n}}\left(v_n^- - v_n^-\right) = \frac{1}{\sqrt{Z_{0,n}}}(a_n - b_n)$$

(cont)

- The power flowing into this port is given by

$$\frac{1}{2}\Re\left(V_n I_n^*\right) = \frac{1}{2}\Re\left(|a_n|^2 - |b_n|^2 + (b_n a_n^* - b_n^* a_n)\right) = \frac{1}{2}\left(|a_n|^2 - |b_n|^2\right)$$

Scattering Transfer Parameters



- Up to now we found it convenient to represent the scattered waves in terms of the incident waves. But what if we wish to cascade two ports as shown?
- Since b_2 flows into a_1', and likewise b_1' flows into a_2, would it not be convenient if we defined the a relationship between a_1,b_1 and b_2,a_2?
- In other words we have

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

- Notice carefully the order of waves (a, b) in reference to the figure above. This allows us to cascade matrices

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = T_1 \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = T_1 \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = T_1 T_2 \begin{bmatrix} b_4 \\ a_4 \end{bmatrix}$$

Reciprocal Networks

Reciprocal Networks

- Suppose the Z/Y matrix are symmetric. Now let's see what we can infer about the S matrix.

$$v^+ = \frac{1}{2}(v + Z_0 i)$$

$$v^- = \frac{1}{2}(v - Z_0 i)$$

- Substitute $v = Zi$ in the above equations

$$v^+ = \frac{1}{2}(Zi + Z_0 i) = \frac{1}{2}(Z + Z_0)i$$

$$v^- = \frac{1}{2}(Zi - Z_0 i) = \frac{1}{2}(Z - Z_0)i$$

- Since $i = i$, the above eq. must result in consistent values of i. Or

$$2(Z + Z_0)^{-1}v^+ = 2(Z - Z_0)^{-1}v^-$$

Reciprocal Networks (cont)

- From the above, we have

$$S = (Z - Z_0)(Z + Z_0)^{-1}$$

- Consider the transpose of the S matrix

$$S^t = \left((Z + Z_0)^{-1}\right)^t (Z - Z_0)^t$$

- Recall that Z_0 is a diagonal matrix

$$S^t = (Z^t + Z_0)^{-1}(Z^t - Z_0)$$

- If $Z^t = Z$ (reciprocal network), then we have

$$S^t = (Z + Z_0)^{-1}(Z - Z_0)$$

(cont)

- Previously we found that

$$S = (Z + Z_0)^{-1}(Z - Z_0)$$

- So that we see that the S matrix is also symmetric (under reciprocity) $S^t = S$
- Note that in effect we have shown that

$$(Z + I)^{-1}(Z - I) = (Z - I)(Z + I)^{-1}$$

- This is easy to demonstrate if we note that

$$Z^2 - I = Z^2 - I^2 = (Z + I)(Z - I) = (Z - I)(Z + I)$$

- In general matrix multiplication does not commute, but here it does

$$(Z - I) = (Z + I)(Z - I)(Z + I)^{-1} \qquad (Z + I)^{-1}(Z - I) = (Z - I)(Z + I)^{-1}$$

S-Parameters of a Lossless Network

- Consider the total power dissipated by a network (must sum to zero)

$$P_{av} = \frac{1}{2}\Re\left(v^t i^*\right) = 0$$

- Expanding in terms of the wave amplitudes

$$= \frac{1}{2}\Re\left((v^+ + v^-)^t Z_0^{-1}(v^+ - v^-)^*\right)$$

- Where we assume that Z_0 are real numbers and equal. The notation is about to get ugly

$$= \frac{1}{2Z_0}\Re\left({v^+}^t {v^+}^* - {v^+}^t {v^-}^* + {v^-}^t {v^+}^* - {v^-}^t {v^-}^*\right)$$

Lossless (cont)

- Notice that the middle terms sum to a purely imaginary number. Let $x = v^+$ and $y = v^-$

$$y^t x^* - x^t y^* = y_1 x_1^* + y_2 x_2^* + \cdots - x_1 y_1^* + x_2 y_2^* + \cdots = a - a^*$$

- We have shown that

$$P_{av} = \frac{1}{2Z_0}\left(\underbrace{v^{+t} v^{+}}_{\text{total incident power}} - \underbrace{v^{-t} v^{-*}}_{\text{total reflected power}}\right) = 0$$

(cont)

- This is a rather obvious result. It simply says that the incident power is equal to the reflected power (because the N port is lossless). Since $v^- = Sv^+$

$$v^{+t} v^{+*} = (Sv^+)^t (Sv^+)^* = v^{+t} S^t S^* v^{+*}$$

- This can only be true if S is a unitary matrix

$$S^t S^* = I$$

$$S^* = (S^t)^{-1}$$

Orthogonal Properties of S

- Expanding out the matrix product

$$\delta_{ij} = \sum_k (S^t)_{ik} S^*_{kj} = \sum_k S_{ki} S^*_{kj}$$

- For $i = j$ we have

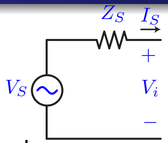
$$\sum_k S_{ki} S^*_{ki} = 1$$

- For $i \neq j$ we have

$$\sum_k S_{ki} S^*_{kj} = 0$$

- The dot product of any column of S with the conjugate of that column is unity while the dot product of any column with the conjugate of a different column is zero. If the network is reciprocal, then $S^t = S$ and the same applies to the rows of S.
- Note also that $|S_{ij}| \leq 1$.

S-Parameter Representation of a Source



$$V_i = V_s - I_s Z_s$$

- The voltage source can be represented directly for s-parameter analysis as follows. First note that

$$V_i^+ + V_i^- = V_s + \left(\frac{V_i^+}{Z_0} - \frac{V_i^-}{Z_0} \right) Z_s$$

- Solve these equations for V_i^-, the power flowing *away* from the source

$$V_i^- = V_i^+ \frac{Z_s - Z_0}{Z_s + Z_0} + \frac{Z_0}{Z_0 + Z_s} V_s$$

- Dividing each term by $\sqrt{Z_0}$, we have

$$\frac{V_i^-}{\sqrt{Z_0}} = \frac{V_i^+}{\sqrt{Z_0}} \Gamma_s + \frac{\sqrt{Z_0}}{Z_0 + Z_s} V_s \quad b_i = a_i \Gamma_s + b_s \quad b_s = V_s \sqrt{Z_0}/(Z_0 + Z_s)$$

Available Power from Source

- A useful quantity is the available power from a source under conjugate matched conditions. Since

$$P_{avs} = |b_i|^2 - |a_i|^2$$

- If we let $\Gamma_L = \Gamma_S^*$, then using $a_i = \Gamma_L b_i$, we have

$$b_i = b_s + a_i \Gamma_S = b_s + \Gamma_S^* b_i \Gamma_S$$

- Solving for b_i we have

$$b_i = \frac{b_s}{1 - |\Gamma_S|^2}$$

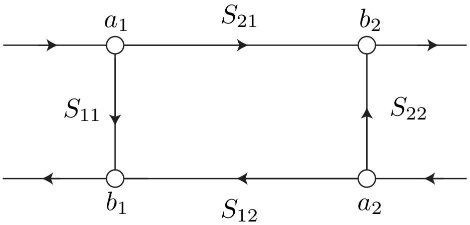
- So the P_{avs} is given by

$$P_{avs} = |b_i|^2 - |a_i|^2 = |b_s|^2 \left(\frac{1 - |\Gamma_S|^2}{(1 - |\Gamma_S|^2)^2} \right)$$

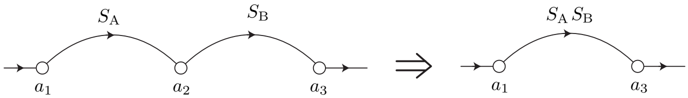
$$= \frac{|b_s|^2}{1 - |\Gamma_S|^2}$$

Signal Flow Analysis

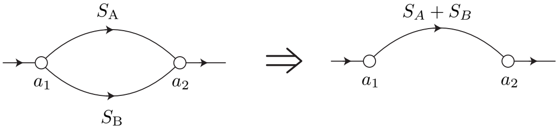
Signal-Flow Analysis



- Each signal *a* and *b* in the system is represented by a node. Branches connect nodes with “strength” given by the scattering parameter. For example, a general two-port is represented above.
- Using three simple rules, we can simplify signal flow graphs to the point that detailed calculations are done by inspection. Of course we can always “do the math” using algebra, so pick the technique that you like best.

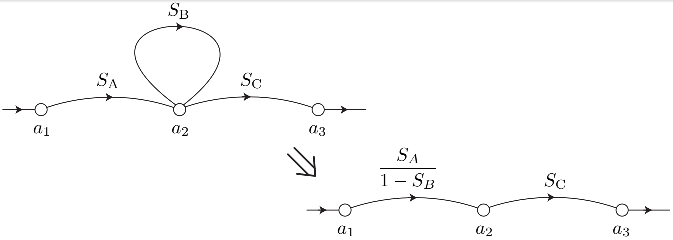


- Rule 1: (series rule) By inspection, we have the cascade.



- Rule 2: (parallel rule) Clear by inspection.

Self-Loop Rule



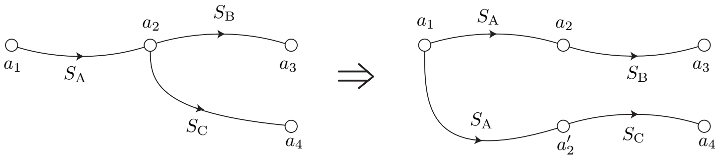
- Rule 3: (self-loop rule) We can remove a "self-loop" by multiplying branches feeding the node by $1/(1 - S_B)$ since

$$a_2 = S_A a_1 + S_B a_2$$

$$a_2(1 - S_B) = S_A a_1$$

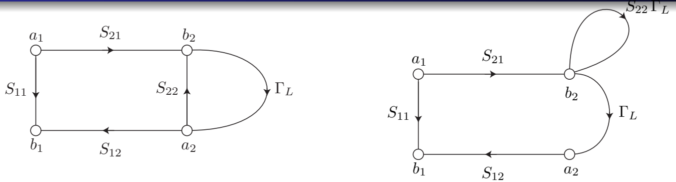
$$a_2 = \frac{S_A}{1 - S_B} a_1$$

Splitting Rule

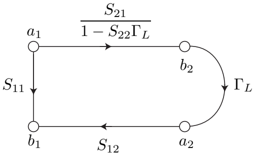


- We can duplicate node a_2 by splitting the signals at an earlier phase

Example: Signal Flow Analysis



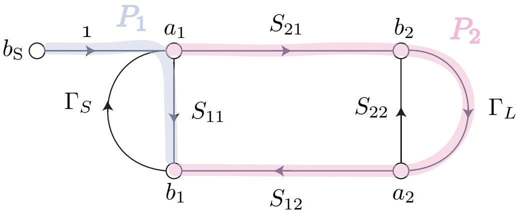
- Using the above rules, we can calculate the input reflection coefficient of a two-port terminated by $\Gamma_L = b_1/a_1$ using a couple of steps.
- First we notice that there is a self-loop around b_2.



- Next we remove the self loop and from here it's clear that the

$$\Gamma_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{21}S_{12}\Gamma_L}{1 - S_{22}\Gamma_L}$$

Mason's Rule

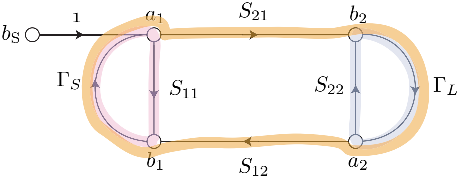


- Using Mason's Rule, you can calculate the transfer function for a signal flow graph by "inspection"

$$T = \frac{P_1\left(1 - \sum \mathcal{L}(1)^{(1)} + \sum \mathcal{L}(2)^{(1)} - \cdots\right) + P_2\left(1 - \sum \mathcal{L}(1)^{(2)} + \cdots\right) + \cdots}{1 - \sum \mathcal{L}(1) + \sum \mathcal{L}(2) - \sum \mathcal{L}(3) + \cdots}$$

- Each P_i defines a *path*, a directed route from the input to the output not containing each node more than once. The value of P_i is the product of the branch coefficients along the path.
- For instance the path from b_s to b_1 ($T = b_1/b_s$) has two paths, $P_1 = S_{11}$ and $P_2 = S_{21}\Gamma_L S_{12}$

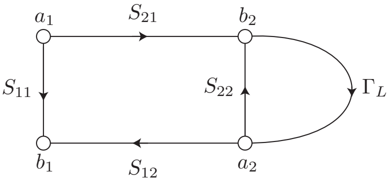
Loop of Order Summation Notation



The notation $\sum \mathcal{L}(1)$ is the sum over all first order loops.

- A "first order loop" is defined as product of the branch values in a loop in the graph. For the given example we have $\Gamma_s S_{11}$, $S_{22}\Gamma_L$, and $\Gamma_s S_{21} \Gamma_L S_{12}$.
- A "second order loop" $\mathcal{L}(2)$ is the product of two non-touching first-order loops. For instance, since loops $S_{11}\Gamma_s$ and $S_{22}\Gamma_L$ do not touch, their product is a second order loop.
- A "third order loop" $\mathcal{L}(3)$ is likewise the product of three non-touching first order loops.
- The notation $\sum \mathcal{L}(1)^{(p)}$ is the sum of all first-order loops that do not touch the path p. For path P_1, we have $\sum \mathcal{L}(1)^{(1)} = \Gamma_L S_{22}$ but for path P_2 we have $\sum \mathcal{L}(1)^{(2)} = 0$.

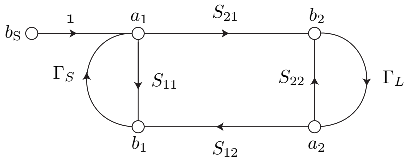
Example: Input Reflection of Two-Port



- Using Mason's rule, you can quickly identify the relevant paths for a $\Gamma_{in} = b_1/a_1$.
- There are two paths $P_1 = S_{11}$ and $P_2 = S_{21}\Gamma_L S_{12}$
- There is only one first-order loop: $\sum \mathcal{L}(1) = S_{22}\Gamma_L$ and so naturally there are no higher order loops.
- Note that the loop does not touch path P_1, so $\sum \mathcal{L}(1)^{(1)} = S_{22}\Gamma_L$.
- Now let's apply Mason's general formula

$$\Gamma_{in} = \frac{S_{11}(1 - S_{22}\Gamma_L) + S_{21}\Gamma_L S_{12}}{1 - S_{22}\Gamma_L} = S_{11} + \frac{S_{21}\Gamma_L S_{12}}{1 - S_{22}\Gamma_L}$$

Example: Transducer Power Gain



- By definition, the transducer power gain is given by

$$G_T = \frac{P_L}{P_{AVS}} = \frac{|b_2|^2(1-|\Gamma_L|^2)}{\frac{|b_s|^2}{1-|\Gamma_S|^2}} = \left|\frac{b_2}{b_S}\right|^2 (1-|\Gamma_L|^2)(1-|\Gamma_S|^2)$$

- By Mason's Rule, there is only one path $P_1 = S_{21}$ from b_S to b_2 so we have

$$\sum \mathcal{L}(1) = \Gamma_S S_{11} + S_{22}\Gamma_L + \Gamma_S S_{21} \Gamma_L S_{12}$$

$$\sum \mathcal{L}(2) = \Gamma_S S_{11} \Gamma_L S_{22} \qquad \sum \mathcal{L}(1)^{(1)} = 0$$

Transducer Gain (cont)

- The gain expression is thus given by

$$\frac{b_2}{b_S} = \frac{S_{21}(1-0)}{1 - \Gamma_S S_{11} - S_{22}\Gamma_L - \Gamma_S S_{21}\Gamma_L S_{12} + \Gamma_S S_{11}\Gamma_L S_{22}}$$

- The denominator is in the form of $1 - x - y + xy$ which allows us to write

$$G_T = \frac{|S_{21}|^2(1 - |\Gamma_S|^2)(1 - |\Gamma_L|^2)}{|(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_L) - S_{21}S_{12}\Gamma_L\Gamma_S|^2}$$

- Recall that $\Gamma_{in} = S_{11} + S_{21}S_{12}\Gamma_L/(1 - S_{22}\Gamma_L)$. Factoring out $1 - S_{22}\Gamma_L$ from the denominator we have

$$\text{den} = \left(1 - S_{11}\Gamma_S - \frac{S_{21}S_{12}\Gamma_L}{1 - S_{22}\Gamma_L}\Gamma_S\right)(1 - S_{22}\Gamma_L)$$

$$\text{den} = \left(1 - \Gamma_S\left(S_{11} + \frac{S_{21}S_{12}\Gamma_L}{1 - S_{22}\Gamma_L}\right)\right)(1 - S_{22}\Gamma_L)$$

$$= (1 - \Gamma_S\Gamma_{in})(1 - S_{22}\Gamma_L)$$

Transducer Gain Expression

- This simplifications allows us to write the transducer gain in the following convenient form

$$G_T = \frac{1-|\Gamma_S|^2}{|1-\Gamma_{in}\Gamma_S|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}$$

- Which can be viewed as a product of the action of the input match "gain", the intrinsic two-port gain $|S_{21}|^2$, and the output match "gain". Since the general two-port is not unilateral, the input match is a function of the load.
- Likewise, by symmetry we can also factor the expression to obtain

$$G_T = \frac{1-|\Gamma_S|^2}{|1-S_{11}\Gamma_S|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-\Gamma_{out}\Gamma_L|^2}$$

Stability From Another Perspective

- We can also derive stability in terms of the input reflection coefficient. For a general two-port with load Γ_L we have

$$v_2^- = \Gamma_L^{-1} v_2^+ = S_{21} v_1^+ + S_{22} v_2^+$$

$$v_2^+ = \frac{S_{21}}{\Gamma_L^{-1} - S_{22}} v_1^-$$

$$v_1^- = \left(S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}} \right) v_1^+$$

$$\Gamma = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}}$$

- If $|\Gamma| < 1$ for all Γ_L, then the two-port is stable

$$\Gamma = \frac{S_{11}(1 - S_{22}\Gamma_L) + S_{12} S_{21} \Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} + \Gamma_L (S_{21} S_{12} - S_{11} S_{22})}{1 - S_{22}\Gamma_L}$$

$$= \frac{S_{11} - \Delta \Gamma_L}{}$$

Stability Circle

- To find the boundary between stability/instability, let's set $|\Gamma| = 1$

$$\left|\frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}\right| = 1$$

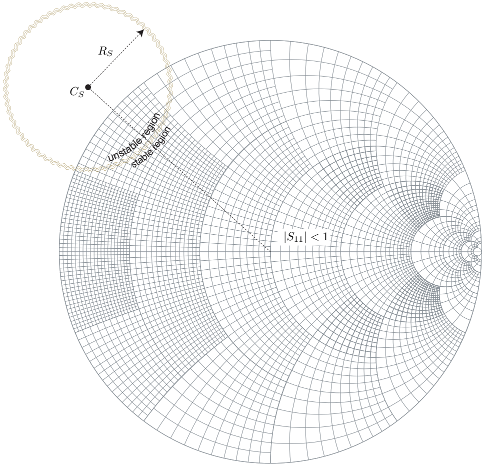
$$|S_{11} - \Delta\Gamma_L| = |1 - S_{22}\Gamma_L|$$

- After some algebraic manipulations, we arrive at the following equation

$$\left|\Gamma_L - \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2}\right| = \frac{|S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2}$$

- This is of course an equation of a circle, $|\Gamma_L - C| = R$, in the complex plane with center at C and radius R
- Thus a circle on the Smith Chart divides the region of instability from stability.

Example: Stability Circle



- In this example, the origin of the circle lies outside the stability circle but a portion of the circle falls inside the unit circle. Is the region of stability inside the circle or outside?
- This is easily determined if we note that if $\Gamma_L = 0$, then $\Gamma = S_{11}$. So if $S_{11} < 1$, the origin should be in the stable region. Otherwise, if $S_{11} > 1$, the origin should be in the unstable region.

Stability: Unilateral Case

- Consider the stability circle for a unilateral two-port

$$C_S = \frac{S_{11}^* - (S_{11}^* S_{22}^*) S_{22}}{|S_{11}|^2 - |S_{11} S_{22}|^2} = \frac{S_{11}^*}{|S_{11}|^2}$$

$$R_S = 0 \qquad\qquad |C_S| = \frac{1}{|S_{11}|}$$

- The cetner of the circle lies outside of the unit circle if $|S_{11}| < 1$. The same is true of the load stability circle. Since the radius is zero, stability is only determined by the location of the center.
- If $S_{12} = 0$, then the two-port is unconditionally stable if $S_{11} < 1$ and $S_{22} < 1$.
- This result is trivial since

$$\Gamma_S \big|_{S_{12}=0} = S_{11}$$

- The stability of the source depends only on the device and not on the load.

Mu Stability Test

- If we want to determine if a two-port is unconditionally stable, then we should use the μ test

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} > 1$$

- The μ test not only is a test for unconditional stability, but the magnitude of μ is a measure of the stability. In other words, if one two port has a larger μ, it is more stable.
- The advantage of the μ test is that only a single parameter needs to be evaluated. There are no auxiliary conditions like the K test derivation earlier.
- The derivation of the μ test proceeds as follows. First let $\Gamma_S = |\rho_s|e^{j\phi}$ and evaluate Γ_{out}

$$\Gamma_{out} = \frac{S_{22} - \Delta|\rho_s|e^{j\phi}}{1 - S_{11}|\rho_s|e^{j\phi}}$$

Mu Test (cont)

- Next we can manipulate this equation into the following circle $|\Gamma_{out} - C| = R$

$$\left|\Gamma_{out} + \frac{|\rho_s|S_{11}^*\Delta - S_{22}}{1 - |\rho_s||S_{11}|^2}\right| = \frac{\sqrt{|\rho_s|}|S_{12}S_{21}|}{(1 - |\rho_s||S_{11}|^2)}$$

- For a two-port to be unconditionally stable, we'd like Γ_{out} to fall within the unit circle

$$||C| + R| < 1$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| < 1 - |\rho_s||S_{11}|^2$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| + |\rho_s||S_{11}|^2 < 1$$

- The worse case stability occurs when $|\rho_s| = 1$ since it maximizes the left-hand side of the equation. Therefore we have

$$\mu = \frac{1 - |S_{11}|^2}{|S_{11}^*\Delta - S_{22}| + |S_{12}S_{21}|} > 1$$

K-Δ Test

- The K stability test has already been derived using Y parameters. We can also do a derivation based on S parameters. This form of the equation has been attributed to Rollett and Kurokawa.
- The idea is very simple and similar to the μ test. We simply require that all points in the instability region fall outside of the unit circle.
- The stability circle will intersect with the unit circle if

$$|C_L| - R_L > 1$$

or

$$\frac{|S_{22}^* - \Delta^* S_{11}| - |S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2} > 1$$

- This can be recast into the following form (assuming $|\Delta| < 1$)

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}||S_{21}|} > 1$$

Two-Port Power and Scattering Parameters

- The power flowing into a two-port can be represented by

$$P_{in} = \frac{|V_1^+|^2}{2Z_0}(1 - |\Gamma_{in}|^2)$$

- The power flowing to the load is likewise given by

$$P_L = \frac{|V_2^-|^2}{2Z_0}(1 - |\Gamma_L|^2)$$

- We can solve for V_1^+ using circuit theory

$$V_1^+ + V_1^- = V_1^+(1 + \Gamma_{in}) = \frac{Z_{in}}{Z_{in} + Z_S} V_S$$

- In terms of the input and source reflection coefficient

$$Z_{in} = \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} Z_0 \qquad Z_S = \frac{1 + \Gamma_S}{1 - \Gamma_S} Z_0$$

Two-Port Incident Wave

- Solve for V_1^+

$$V_1^+(1+\Gamma_{in}) = \frac{V_S(1+\Gamma_{in})(1-\Gamma_S)}{(1+\Gamma_{in})(1-\Gamma_S)+(1+\Gamma_S)(1-\Gamma_{in})}$$

$$V_1^+ = \frac{V_S}{2}\frac{1-\Gamma_S}{1-\Gamma_{in}\Gamma_S}$$

- The voltage incident on the load is given by

$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ = S_{21}V_1^+ + S_{22}\Gamma_L V_2^-$$

$$V_2^- = \frac{S_{21}V_1^+}{1-S_{22}\Gamma_L}$$

$$P_L = \frac{|S_{21}|^2\left|V_1^+\right|^2}{|1-S_{22}\Gamma_L|^2}\frac{1-|\Gamma_L|^2}{2Z_0}$$

Operating Gain and Available Power

- The operating power gain can be written in terms of the two-port s-parameters and the load reflection coefficient

$$G_p = \frac{P_L}{P_{in}} = \frac{|S_{21}|^2 (1 - |\Gamma_L|^2)}{|1 - S_{22}\Gamma_L|^2 (1 - |\Gamma_{in}|^2)}$$

- The available power can be similarly derived from V_1^+

$$P_{avs} = P_{in}|_{\Gamma_{in}=\Gamma_S^*} = \frac{|V_{1a}^+|^2}{2Z_0}(1 - |\Gamma_S^*|^2)$$

$$V_{1a}^+ = V_1^+|_{\Gamma_{in}=\Gamma_S^*} = \frac{V_S}{2}\frac{1 - \Gamma_S^*}{1 - |\Gamma_S|^2}$$

$$P_{avs} = \frac{|V_S|^2}{8Z_0}\frac{|1 - \Gamma_S|^2}{1 - |\Gamma_S|^2}$$

Transducer Gain

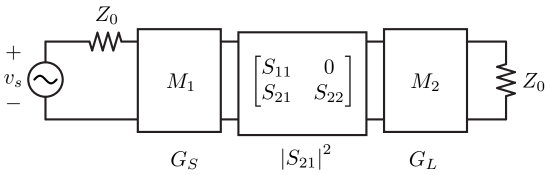
- The transducer gain can be easily derived

$$G_T = \frac{P_L}{P_{avs}} = \frac{|S_{21}|^2 (1 - |\Gamma_L|^2)(1 - |\Gamma_S|^2)}{|1 - \Gamma_{in}\Gamma_S|^2 |1 - S_{22}\Gamma_L|^2}$$

- Note that as expected, G_T is a function of the two-port s-parameters and the load and source impedance.
- If the two port is connected to a source and load with impedance Z_0, then we have $\Gamma_L = \Gamma_S = 0$ and

$$G_T = |S_{21}|^2$$

Unilateral Gain



- If $S_{12} \approx 0$, we can simplify the expression by just assuming $S_{12} = 0$. This is the *unilateral* assumption

$$G_{TU} = \frac{1 - |\Gamma_S|^2}{|1 - S_{11}\Gamma_S|^2} |S_{21}|^2 \frac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2} = G_S |S_{21}|^2 G_L$$

- The gain partitions into three terms, which can be interpreted as the gain from the source matching network, the gain of the two port, and the gain of the load.

Maximum Unilateral Gain

- We know that the maximum gain occurs for the biconjugate match

$$\Gamma_S = S_{11}^*$$

$$\Gamma_L = S_{22}^*$$

$$G_{S,max} = \frac{1}{1 - |S_{11}|^2}$$

$$G_{L,max} = \frac{1}{1 - |S_{22}|^2}$$

$$G_{TU,max} = \frac{|S_{21}|^2}{(1 - |S_{11}|^2)(1 - |S_{22}|^2)}$$

- Note that if $|S_{11}| = 1$ of $|S_{22}| = 1$, the maximum gain is infinity. This is the unstable case since $|S_{ii}| > 1$ is potentially unstable.

Design for Gain

- So far we have only discussed power gain using bi-conjugate matching. This is possible when the device is unconditionally stable. In many case, though, we'd like to design with a potentially unstable device.
- Moreover, we would like to introduce more flexibility in the design. We can trade off gain for
  - bandwidth
  - noise
  - gain flatness
  - linearity
  - etc.
- We can make this tradeoff by identifying a range of source/load impedances that can realize a given value of power gain. While maximum gain is acheived for a single point on the Smith Chart, we will find that a lot more flexibility if we back-off from the peak gain.

Unilateral Design

- No real transistor is unilateral. But most are predominantly unilateral, or else we use cascades of devices (such as the cascode) to realize such a device.
- The *unilateral figure of merit* can be used to test the validity of the unilateral assumption

$$U_m = \frac{|S_{12}|^2 \, |S_{21}|^2 \, |S_{11}|^2 \, |S_{22}|^2}{(1 - |S_{11}|^2)(1 - |S_{22}|^2)}$$

- It can be shown that the transducer gain satisfies the following inequality

$$\frac{1}{(1+U)^2} < \frac{G_T}{G_{TU}} < \frac{1}{(1-U)^2}$$

- Where the actual power gain G_T is compared to the power gain under the unilateral assumption G_{TU}. If the inequality is tight, say on the order of $0.1\,\mathrm{dB}$, then the amplifier can be assumed to be unilateral with negligible error.

Gain Circles

- We now can plot gain circles for the source and load. Let

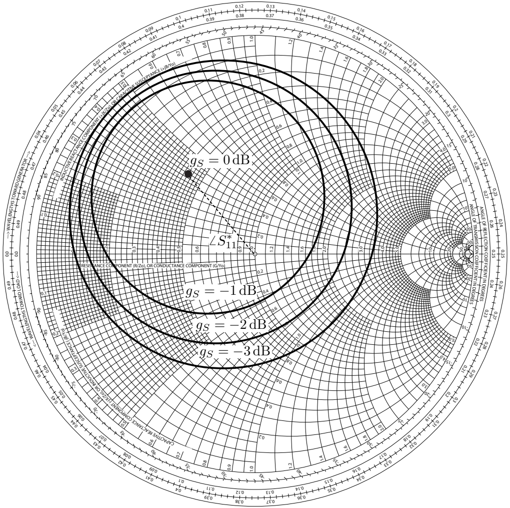
$$g_S = \frac{G_S}{G_{S,max}}$$

$$g_L = \frac{G_L}{G_{L,max}}$$

- By definition, $0 \leq g_S \leq 1$ and $0 \leq g_L \leq 1$. One can show that a fixed value of g_S represents a circle on the Γ_S plane

$$\left| \Gamma_S - \frac{S_{11}^* g_S}{|S_{11}|^2 (g_S - 1) + 1} \right| = \left| \frac{\sqrt{1 - g_S}(1 - |S_{11}|^2)}{|S_{11}|^2 (g_S - 1) + 1} \right|$$

- More simply, $|\Gamma_S - C_S| = R_S$. A similar equation can be derived for the load. Note that for $g_S = 1$, $R_S = 0$, and $C_S = S_{11}^*$ corresponding to the maximum gain.



- All gain circles lie on the line given by the angle of S_{ii}^*. We can select any desired value of source/load reflection coefficient to acheive the desired gain. To minimize

Extended Smith Chart

- For $|\Gamma| > 1$, we can still employ the Smith Chart if we make the following mapping. The reflection coefficient for a negative resistance is given by

$$\Gamma(-R + jX) = \frac{-R + jX - Z_0}{-R + jX + Z_0} = \frac{(R + Z_0) - jX}{(R - Z_0) - jX}$$

$$\frac{1}{\Gamma^*} = \frac{(R - Z_0) + jX}{(R + Z_0) + jX}$$

- We see that Γ can be mapped to the unit circle by taking $1/\Gamma^*$ and reading the resistance value (and noting that it's actually negative).

Potentially Unstable Unilateral Amplifier

- For a unilateral two-port with $|S_{11}| > 1$, we note that the input impedance has a negative real part. Thus we can still design a stable amplifier as long as the source resistance is larger than $\Re(Z_{in})$

$$\Re(Z_S) > |\Re(Z_{in})|$$

- The same is true of the load impedance if $|S_{22}| > 1$. Thus the design procedure is identical to before as long as we avoid source or load reflection coefficients with real part less than the critical value.

Pot. Unstable Unilateral Amp Example

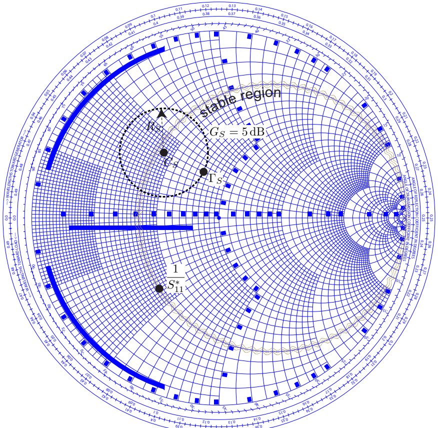
- Consider a transistor with the following S-Parameters

$$S_{11} = 2.02\angle - 130.4^\circ$$

$$S_{12} = 0$$

$$S_{22} = 0.50\angle - 70^\circ$$

$$S_{21} = 5.00\angle 60^\circ$$



- Since $|S_{11}| > 1$, the amplifier is potentially unstable. We begin by plotting $1/S_{11}^*$ to find the negative real input resistance.
- Now any source inside this circle is stable, since $\Re(Z_S) > \Re(Z_{in})$.
- We also draw the source gain circle for $G_S = 5\,\mathrm{dB}$.

Amp Example (cont)

- The input impedance is read off the Smith Chart from $1/S_{11}^*$. Note the real part is interpreted as negative

$$Z_{in} = 50(-0.4 - 0.4j)$$

- The $G_S = 5\,\mathrm{dB}$ gain circle is calculated as follows

$$g_S = 3.15(1 - |S_{11}|^2)$$

$$R_S = \frac{\sqrt{1-g_S}(1-|S_{11}|^2)}{1-|S_{11}|^2(1-g_S)} = 0.236$$

$$C_S = \frac{g_S S_{11}^*}{1-|S_{11}|^2(1-g_S)} = -.3 + 0.35j$$

- We can select any point on this circle and obtain a stable gain of $5\,\mathrm{dB}$. In particular, we can pick a point near the origin (to maximize the BW) but with as large of a real impedance as possible:

$$Z_S = 50(0.75 + 0.4j)$$

Bilateral Amp Design

- In the bilateral case, we will work with the power gain G_p. The transducer gain is not used since the source impedance is a function of the load impedaance. G_p, on the other hand, is only a function of the load.

$$G_p = \frac{|S_{21}|^2 (1 - |\Gamma_L|^2)}{\left(1 - \left|\frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}\right|^2\right) |1 - S_{22}\Gamma_L|^2} = |S_{21}|^2 g_p$$

- It can be shown that g_p is a circle on the Γ_L plane. The radius and center are given by

$$R_L = \frac{\sqrt{1 - 2K|S_{12}S_{21}|g_p + |S_{12}S_{21}|^2 g_p^2}}{\left|-1 - |S_{22}|^2 g_p + |\Delta|^2 g_p\right|^2}$$

$$C_L = \frac{g_p(S_{22}^* - \Delta^* S_{11})}{1 + g_p(|S_{22}|^2 - |\Delta|^2)}$$

Bilateral Amp (cont)

- We can also use this formula to find the maximum gain. We know that this occurs when $R_L = 0$, or

$$1 - 2K|S_{12}S_{21}|g_{p,max} + |S_{12}S_{21}|^2 g_{p,max}^2 = 0$$

$$g_{p,max} = \frac{1}{|S_{12}S_{21}|}\left(K - \sqrt{K^2 - 1}\right)$$

$$G_{p,max} = \left|\frac{S_{21}}{S_{12}}\right|\left(K - \sqrt{K^2 - 1}\right)$$

- The design procedure is as follows
  1. Specify g_p
  2. Draw operating gain circle.
  3. Draw load stability circle. Select Γ_L that is in the stable region and not too close to the stability circle.
  4. Draw source stability circle.
  5. To maximize gain, calculate Γ_{in} and check to see if $\Gamma_S = \Gamma_{in}^*$ is in the stable region. If not, iterate on Γ_L or compromise.

Refs

- “S Parameter Design,” *Hewlett-Packard Application Note 154*, April 1972.
- *Microwave Transistor Amplifiers, Analysis and Design*, Guillermo Gonzalez, Prentice Hall 1984.
- *Microwave Engineering*, David Pozer, Third Edition, Wiley 2005.
- *Microwave Circuit Design Using Linear and Nonlinear Techniques*, by George Vendelin, Anthony M. Pavio, & Ulrich L. Rohde, Wiley 1995.