Introduction to Transmission Lines

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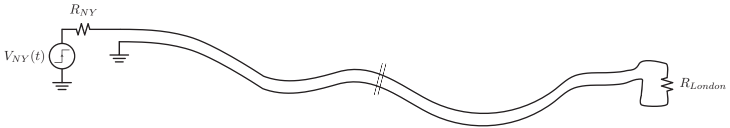
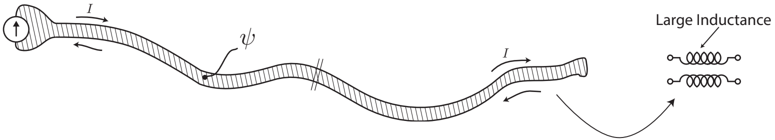
First Trans-Atlantic Cable

- Problem: A long cable – the trans-atlantic telephone cable – is laid out connecting NY to London. We would like analyze the electrical properties of this cable.
- For simplicity, assume the cable has a uniform cross-sectional configuration (shown as two wires here)

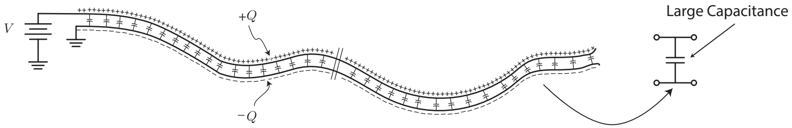


- Can we do it with circuit theory?
- Fundamental problem with circuit theory is that it assumes that the speed of light is infinite. So all signals are in phase: $V(z) = V(z + \ell)$
- Consequently, all variations in space are ignored: $\frac{\partial}{\partial z} \to 0$
- This allows the *lumped* circuit approximation.

- Shorted Line: The long loop has *inductance* since the magnetic flux ψ is not negligible (long cable) ($\psi = LI$)

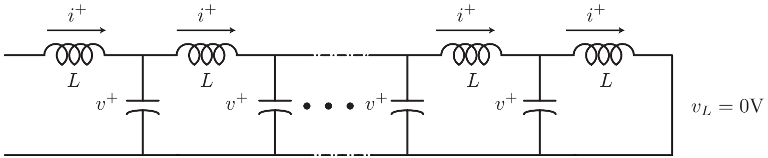


- Open Line: The cable also has substantial capacitance ($Q = CV$)



Sectional Model (I)

- So do we model the cable as an inductor or as a capacitor? Or both? How?
- Try a *distributed* model: Inductance and capacitance occur together. They are intermingled.

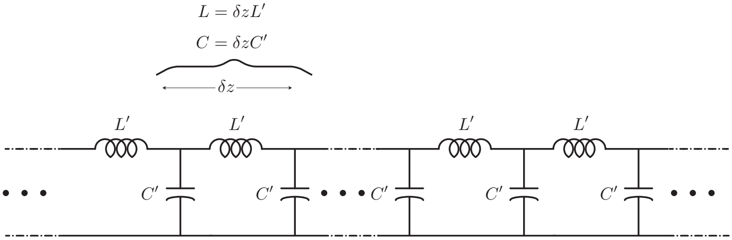


- Can add loss (series and shunt resistors) but let's keep it simple for now.
- Add more sections and solution should converge

Sectional Model (II)

- More sections $\rightarrow$ The equiv LC circuit represents a smaller and smaller section and therefore lumped circuit approximation is more valid
- This is an easy problem to solve with SPICE.
- But people in 1866 didn't have computers ... how did they analyze a problem with hundreds of inductors and capacitors?

Distributed Model



- Go to a fully distributed model by letting the number of sections go to infinity
- Define inductance and capacitance per unit length $L' = L/\ell$, $C' = C/\ell$
- For an infinitesimal section of the line, circuit theory applies since signals travel instantly over an infinitesimally small length

KCL and KVL for a small section

- KCL: $i(z) = \delta z C' \frac{\partial v(z)}{\partial t} + i(z + \delta z)$
- KVL: $v(z) = \delta z L' \frac{\partial i(z+\delta z)}{\partial t} + v(z + \delta z)$
- Take limit as $\delta z \to 0$

We arrive at "Telegrapher's Equatins"

Derivation of Wave Equations

- We have two coupled equations and two unkowns (i and v) ... can reduce it to two de-coupled equations:
- note order of partials can be changed (at least in EE)
- Same equation can be derived for current:

The Wave Equation

The Wave Equation

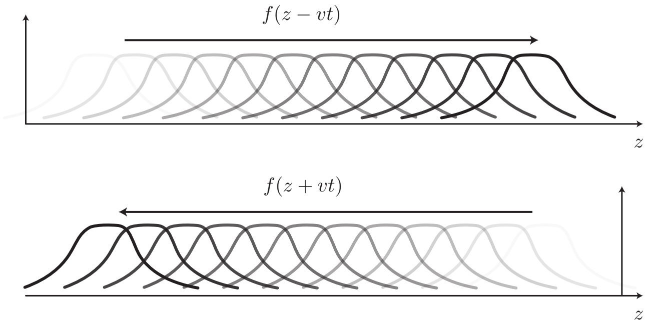
We see that the currents and voltages on the transmission line satisfy the one-dimensional wave equation. This is a partial differential equation. The solution depends on boundary conditions and the initial condition.

Wave Equation Solution

Consider the function $f(z, t) = f(z \pm vt) = f(u)$:

It satisfies the wave equation!

Wave Motion



- General voltage solution: $v(z,t) = f^{+}(z - vt) + f^{-}(z + vt)$
- Where $v = \sqrt{\frac{1}{L'C'}}$

Wave Speed

- Speed of motion can be deduced if we observe the speed of a point on the aveform
- To follow this point as time elapses, we must move the z coordinate in step. This point moves with velocity
- This is the speed at which we move with speed $\frac{dz}{dt} = \pm v$
- v is the velocity of wave propagation

“Ohm’s Law” for T-Lines

Current / Voltage Relationship (I)

- Since the current also satisfies the wave equation
- Recall that on a transmission line, current and voltage are related by
- For the general function this gives

Current / Voltage Relationship (II)

- Since the forward waves are independent of the reverse waves
- Within a constant we have
- Where $Z_0 = \sqrt{\frac{L'}{C'}}$ is the "Characteristic Impedance" of the line

A Side Note on Current

- Notice that the currents in the forward wave has the same sign

- But the reverse wave has a negative sign

- This is related to the definition of current. If positive charges are moving left, then the corresponding current is negative.
- Clearly understand the definition of currents on a transmission line with respect to the two conductors. This is a "odd" mode current, since the top and bottom conductors carry equal and opposite currents. There's also an "even" mode current that we are neglecting for now.

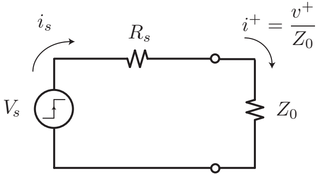
Example: Step Into Infinite Line

- Excite a step function onto a transmission line
- The line is assumed uncharged: $Q(z, 0) = 0$, $\psi(z, 0) = 0$ or equivalently $v(z, 0) = 0$ and $i(z, 0) = 0$
- By physical intuition, we would only expect a forward traveling wave since the line is infinite in extent
- The general form of current and voltage on the line is given by

- The T-line looks like a resistor of Z_0 ohms!

Example 1 (cont)

- We may therefore model the line with the following simple equivalent circuit

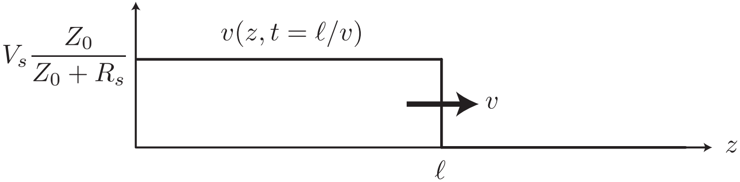


- Since $i_s = i^+$, the excited voltage wave has an amplitude of

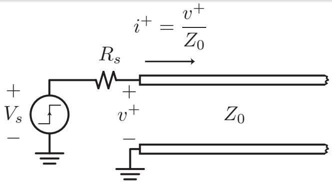
- It's surprising that the voltage on the line is not equal to the source voltage

Example 1 (cont)

- The voltage on the line is a delayed version of the source voltage



Energy to "Charge" Transmission Line

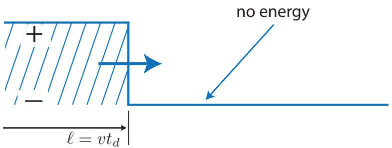


- The power flow into the line is given by

- Or in terms of the source voltage

Energy Stored in Inds and Caps (I)

- But where is the power going? The line is lossless!
- Energy stored by a cap/ind is $\frac{1}{2}CV^2/\frac{1}{2}LI^2$
- At time t_d, a length of $\ell = vt_d$ has been "charged":



- The total energy is thus

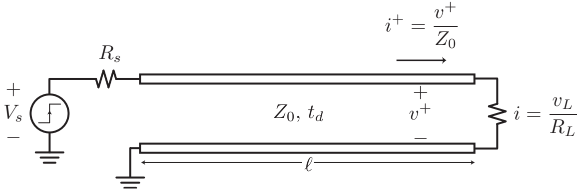
Energy Stored (II)

- Recall that $Z_0 = \sqrt{L'/C'}$. The total energy stored on the line at time $t_d = \ell/v$:

- And the power delivered onto the line in time t_d:

- As expected, the results match (conservation of energy).

Transmission Line Termination



- Consider a finite transmission line with a termination resistance
- At the load we know that Ohm's law is valid: $I_L = V_L/R_L$
- So at time $t = \ell/v$, our pulse reaches the load. Since the current on the T-line is $i^+ = v^+/Z_0 = V_s/(Z_0 + R_s)$ and the current at the load is V_L/R_L, a discontinuity is produced at the load.

Reflections

- Thus a reflected wave is created at discontinuity

- Solving for the forward and reflected waves

Reflection Coefficient

- And therefore the reflection from the load is given by

- The reflection coefficient is a very important concept for transmission lines: $-1 \leq \Gamma_L \leq 1$
- $\Gamma_L = -1$ for $R_L = 0$ (short)
- $\Gamma_L = +1$ for $R_L = \infty$ (open)
- $\Gamma_L = 0$ for $R_L = Z_0$ (match)
- Impedance match is the proper termination if we don't want any reflections

Propagation of Reflected Wave (I)

- If $\Gamma_L \neq 0$, a new reflected wave travels toward the source and unless $R_s = Z_0$, another reflection also occurs at source!
- To see this consider the wave arriving at the source. Recall that since the wave PDE is linear, a superposition of any number of solutins is also a solution.
- At the source end the boundary condition is as follows

- The new term v_2^+ is used to satisfy the boundary condition

Propagation of Reflected Wave (II)

- The current continuity requires $I_s = i_1^+ + i_1^- + i_2^+$
- Solve for v_2^+ in terms of known terms
- But $v_1^+ = \frac{Z_0}{R_s+Z_0} V_s$

Propagation of Reflected Wave (III)

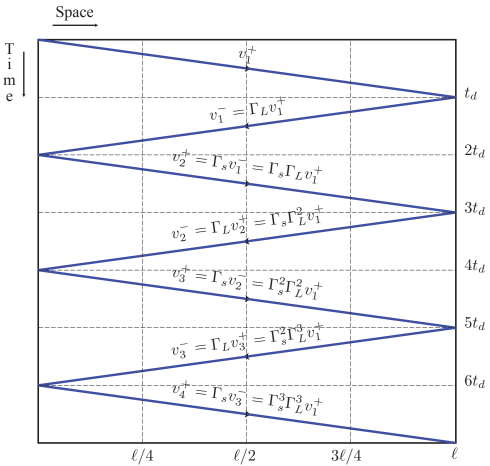
- So the source terms cancel out and

- The reflected wave bounces off the source impedance with a reflection coefficient given by the same equation as before

- The source appears as a short for the incoming wave
- Invoke superposition! The term v_1^+ took care of the source boundary condition so our new v_2^+ only needed to compensate for the v_1^- wave ... the reflected wave is only a function of v_1^-

Bounce Diagram

- We can track the multiple reflections with a "bounce diagram"

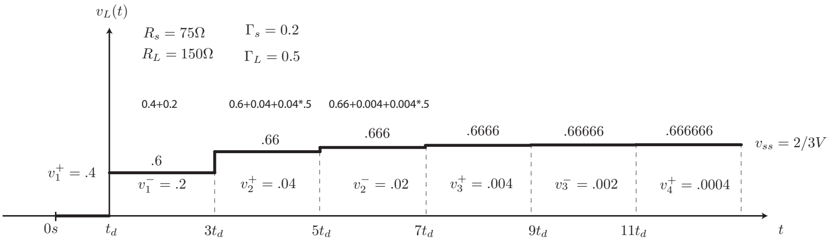


Freeze time

- If we freeze time and look at the line, using the bounce diagram we can figure out how many reflections have occurred
- For instance, at time $2.5t_d = 2.5\ell/v$ three waves have been excited (v_1^+,v_1^-, v_2^+), but v_2^+ has only travelled a distance of $\ell/2$
- To the left of $\ell/2$, the voltage is a summation of three components: $v = v_1^+ + v_1^- + v_2^+ = v_1^+(1 + \Gamma_L + \Gamma_L\Gamma_s)$.
- To the right of $\ell/2$, the voltage has only two components: $v = v_1^+ + v_1^- = v_1^+(1 + \Gamma_L)$.

Freeze Space

- We can also pick at arbitrary point on the line and plot the evolution of voltage as a function of time
  - For instance, at the load, assuming $R_L > Z_0$ and $R_S > Z_0$, so that $\Gamma_{s,L} > 0$, the voltage at the load will will increase with each new arrival of a reflection



Steady-State Voltage on Line (I)

- To find steady-state voltage on the line, we sum over all reflected waves:

- Or in terms of the first wave on the line

- Notice geometric sums of terms like $\Gamma_L^k \Gamma_s^k$ and $\Gamma_L^{k+1} \Gamma_s^k$. Let $x = \Gamma_L \Gamma_s$:

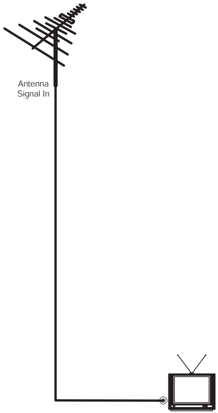
Steady-State Voltage on Line (II)

- The sums converge since $x < 1$
- Or more compactly
- Substituting for Γ_L and Γ_s gives

What Happened to the T-Line?

- For steady state, the equivalent circuit shows that the transmission line has disappeared.
- This happens because if we wait long enough, the effects of propagation delay do not matter
- Conversly, if the propagation speed were infinite, then the T-line would not matter
- But the presence of the T-line will be felt if we disconnect the source or load!
- That's because the T-line *stores* reactive energy in the capaciance and inductance
- Every real circuit behaves this way! Circuit theory is an abstraction

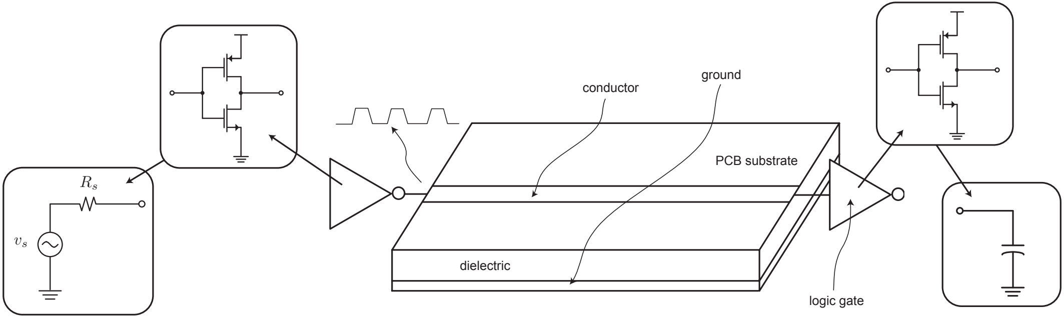
TV Ghosts



- If the cable from the antenna to the analog TV is very long, and the line is not properly terminated, you'll see a "ghost" image, which is usually the first reflected wave making a round trip delay. There are more ghosts, but they are weaker.
- Note that the reflected signal is essentially forming an exact copy of the first signal but just arriving late.
- A fond childhood memory !

PCB Interconnect

- Suppose $\ell = 3\text{cm}$, $v = 3 \times 10^8\text{m/s}$, so that $t_p = \ell/v = 10^{-10}\text{s} = 100\text{ps}$
- On a time scale $t < 100\text{ps}$, the voltages on interconnect act like transmission lines!
  - Fast digital circuits need to consider T-line effects



Example: Open Line (I)

- Source impedance is $Z_0/4$, so $\Gamma_s = -0.6$, load is open so $\Gamma_L = 1$
- As before a positive going wave is launched $v_1^+ = Z_0/(Z_0 + Z_0/4)V_s = 0.8V_s$
- Upon reaching the load, a reflected wave of of equal amplitude is generated and the load voltage overshoots $v_L = v_1^+ + v_1^- = 1.6\text{V}$
- Note that the current reflection is negative of the voltage

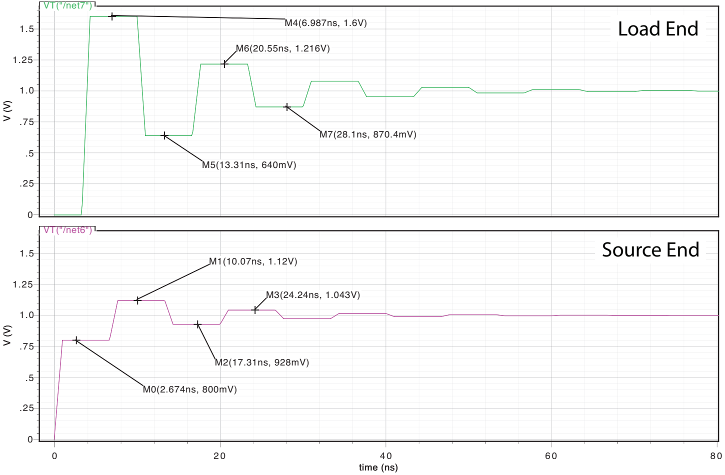
- This means that the sum of the currents at the load is zero (open)

Example: Open Line (II)

- At the source a new reflection is created $v_2^+ = \Gamma_L \Gamma_s v_1^+$, and note $\Gamma_s < 0$, so $v_2^+ = -.6 \times 0.8 = -0.48$.
- At a time $3t_p$, the line charged initially to $v_1^+ + v_1^-$ drops in value

- So the voltage on the line undershoots (< 1 times V_s)
- And on the next cycle $5t_p$ the load voltage again overshoots
- We observe ringing with frequency $2t_p$

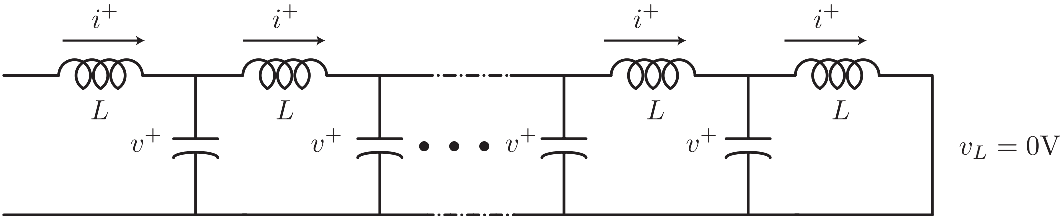
Example: Open Line Ringing



- Observed waveform as a function of time. Risetime due to SPICE $t_{step} = 20\text{ps}$.

Physical Intuition: Shorted Line (I)

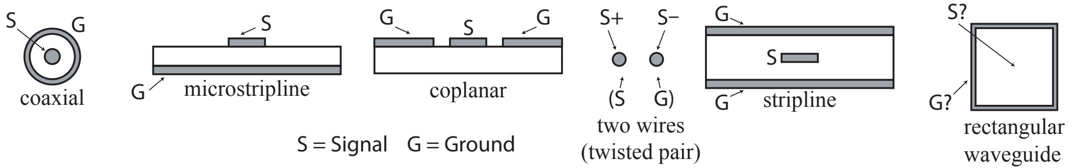
- The intitial step charges the "first" capacitor through the "first" inductor since the line is uncharged
- There is a delay since on the rising edge of the step, the inductor is an open
- Each successive capacitor is charged by "its" inductor in a uniform fashion ... this is the forward wave v_1^+



Physical Intuition: Shorted Line (II)

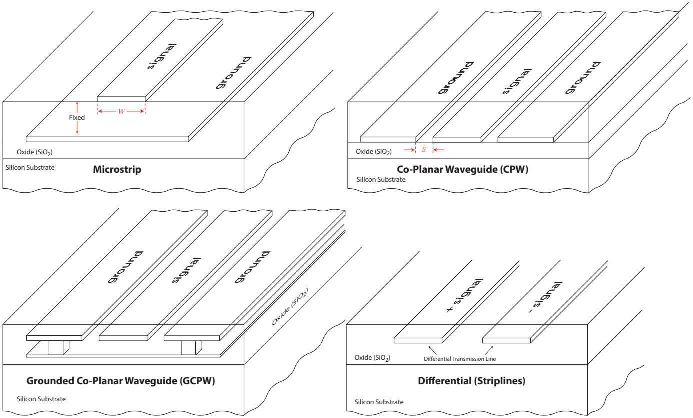
- The volage on the line goes up from left to right due to the delay in charging each inductor through the capacitors
- The last inductor, though, does not have a capacitor to charge
- Thus the last inductor is discharged ... the extra charge comes by discharging the last capacitor
- As this capacitor discharges, so does it's neighboring capacitor to the left
- Again there is a delay in discharging the caps due to the inductors
- This discharging represents the backward wave v_1^-

Transmission Line Menagerie



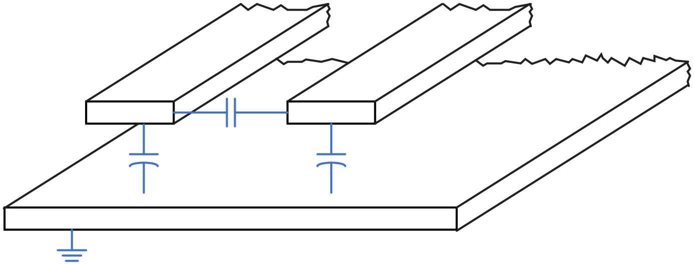
- T-Lines come in many shapes and sizes
- Coaxial usually 75Ω or 50Ω (cable TV, Internet)
- Microstrip lines are common on printed circuit boards (PCB) and integrated circuit (ICs)
- Coplanar also common on PCB and ICs
- Twisted pairs is almost a T-line, ubiquitous for phones/Ethernet

Planar PCB and Integrated Circuit T-Lines



- If we restrict the structures to planar configurations, these are the most popular structures. The "Silicon Substrate" is partially conductive whereas on a PCB a non-conductive dielectric layer would be used. The conductivity of silicon leads to additional loss and higher order modes that should be suppressed.

Propagation Modes

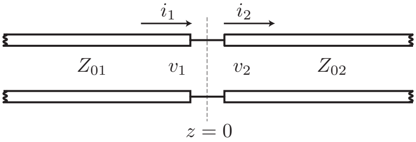


- Note the distinction between a single-ended T-line and a differential T-line. The differential T-line actually supports two modes of propagation, called even-mode and odd-mode (common-mode and differential-mode), and we'll discuss that later. In practice, all T-lines support both modes, it's just one mode is much more dominant.
- Notice that if a "balanced" line is constructed on top of a ground plane (or near a physical conductor), currents can also "return" through the ground plane. This is not the intended propagation mode !

Waveguides and Transmission Lines

- The transmission lines we've been considering propagate the "TEM" mode or Transverse Electro-Magnetic. Later we'll see that they can also propagation other modes
- Waveguides cannot propagate TEM, but propage TM (Transverse Magnetic) and TE (Transverse Electric)
- In general, *any* set of more than one lossless conductors with uniform cross-section can transmit TEM waves. Low loss conductors are commonly approximated as lossless.

Cascade of T-Lines (I)

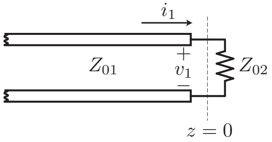


- Consider the junction between two transmission lines Z_{01} and Z_{02}
- At the interface $z = 0$, the boundary conditions are that the voltage/current has to be continuous

Cascade of T-Lines (II)

- Solve these equations in terms of v_1^+
- The reflection coefficient has the same form (easy to remember)

- The second line looks like a load impedance of value Z_{02}



Transmission Coefficient

- The wave launched on the new transmission line at the interface is given by

- This "transmitted" wave has a coefficient

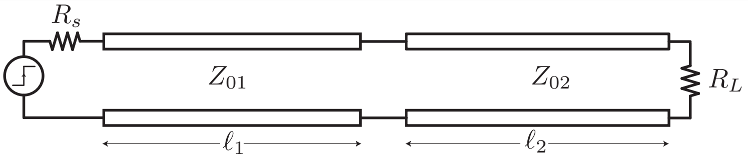
- Note the incoming wave carries a power

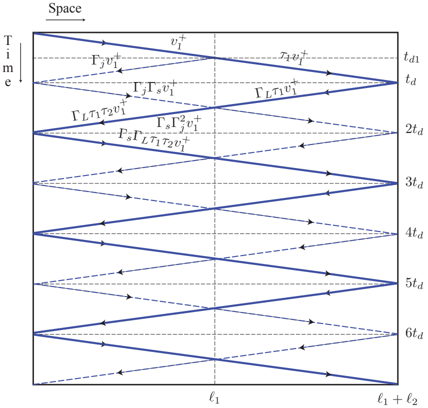
Conservation of Energy

- The reflected and transmitted waves likewise carry a power of
- By conservation of energy, it follows that
- You can verify that this relation holds!

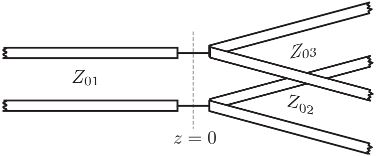
Bounce Diagram

- Consider the bounce diagram for the following arrangement:





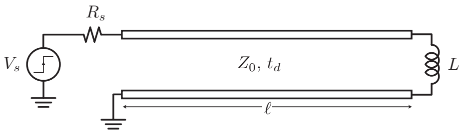
Junction of Parallel T-Lines



- Again invoke voltage/current continuity at the interface

- But $v_2^+ = v_3^+$, so the interface just looks like the case of two transmission lines Z_{01} and a new line with char. impedance $Z_{01}||Z_{02}$.

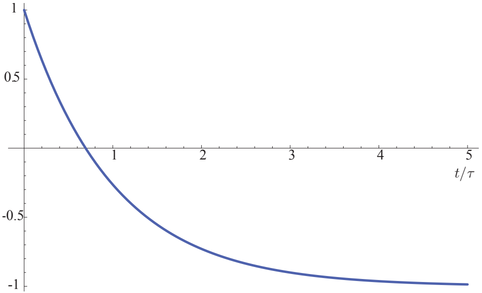
Reactive Terminations (I)



- Let's analyze the problem intuitively first
- When a pulse first "sees" the inductance at the load, it looks like an open so $\Gamma_0 = +1$
- As time progresses, the inductor looks more and more like a short! So $\Gamma_\infty = -1$

Reactive Terminations (II)

- So intuitively we might expect the reflection coefficient to look like this:



- The graph starts at $+1$ and ends at -1. In between we'll see that it goes through exponential decay (1st order ODE)

Reactive Terminations (III)

- Do equations confirm our intuition?

- And the voltage at the load is given by $v^+ + v^-$

- The right hand side is known, it's the incoming waveform

Solution for Reactive Term

- For the step response, the derivative term on the RHS is zero at the load
- So we have a simpler case $\frac{dv^+}{dt} = 0$
- We must solve the following equation
- For simplicity, assume at $t = 0$ the wave v^+ arrives at load

Laplace Domain Solution I

- In the Laplace domain
- Solve for reflection $V^{-}(s)$
- Break this into basic terms using partial fraction expansion

Laplace Domain Solution (II)

- Invert the equations to get back to time domain $t > 0$

- Note that $v^-(0) = v^+$ since initially the inductor is an open
- So the reflection coefficient is

- The reflection coefficient decays with time constant L/Z_0