EECS 117

Lecture 9: Electrostatics and Poisson's Equation

Prof. Niknejad

University of California, Berkeley

Review of Divergence

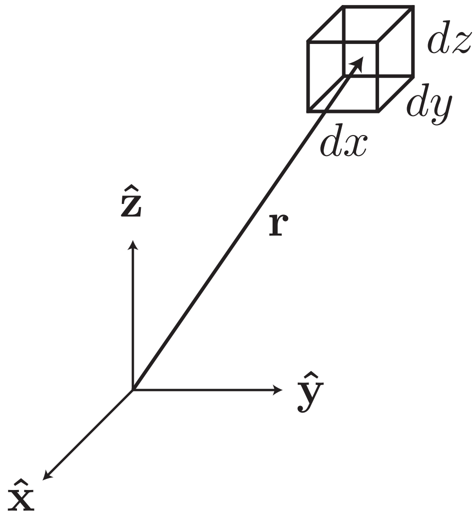
- If we try to measure the flux at a point, we are tempted to compute $\oint_S \mathbf{A} \cdot d\mathbf{S}$ is a small volume surrounding a point
- This flux will diminish, though, as we make the volume smaller and smaller. We might suspect that normalizing by volume will solve the problem
- In fact, the divergence of a vector is defined as follows

$$\mathrm{div}\mathbf{A} \triangleq \lim_{V \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{V}$$

- We usually write

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A}$$

Divergence in Rectangular Coor. (I)



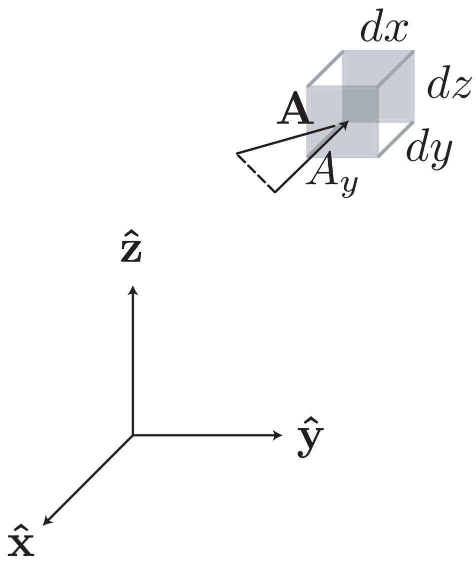
- Consider the flux crossing a small differential volume at position (x_0, y_0, z_0). We can break up the flux $\oint_S \mathbf{A} \cdot d\mathbf{S}$ into several components

$$= \int_{x-\Delta x/2} + \int_{x+\Delta x/2} + \int_{y-\Delta y/2} + \int_{y+\Delta y/2} + \int_{z-\Delta z/2} + \int_{z+\Delta z/2}$$

Divergence in Rectangular Coor. (II)

- Without loss of generality, consider the flux crossing the surfaces in the xz plane, or the third and fourth terms in the above calculation
- The flux for these terms is simply

$$= -A_y(x_0, y_0 - \frac{\Delta y}{2}, z_0)\Delta x \Delta z + A_y(x_0, y_0 + \frac{\Delta y}{2}, z_0)\Delta x \Delta z$$



Divergence in Rectangular Coor. (III)

- For a differential volume this becomes

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \Delta x \Delta z \Delta y \frac{\partial A_y}{\partial y}$$

- Thus the total flux is simply

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \Delta x \Delta y \Delta z \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

- For *rectangular* coordinates, we have therefore proved that

$$\text{div } A = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

Divergence in Rectangular Coor. (IV)

- In terms of the ∇ operator

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

- Since the dot product of the operator ∇ and the vector $\mathbf{A}$ gives

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- We can rewrite $\mathrm{div} A$ as simply $\nabla \cdot \mathbf{A}$

Gauss' Law (again)

- Applying the definition of divergence to the electric flux density D, we have

$$\lim_{V\to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{V} = \frac{q_{\text{inside}}}{V} = \rho = \nabla \cdot \mathbf{D}$$

- Therefore we have the important result that at any given point

$$\nabla \cdot \mathbf{D} = \rho(x, y, z)$$

- This is the analog to the equivalent statement that we derived last lecture

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV$$

Divergence Theorem

- The *Divergence Theorem* is a direct proof of this relationship between the volume integral of the divergence and the surface integral (applies to any vector function A)

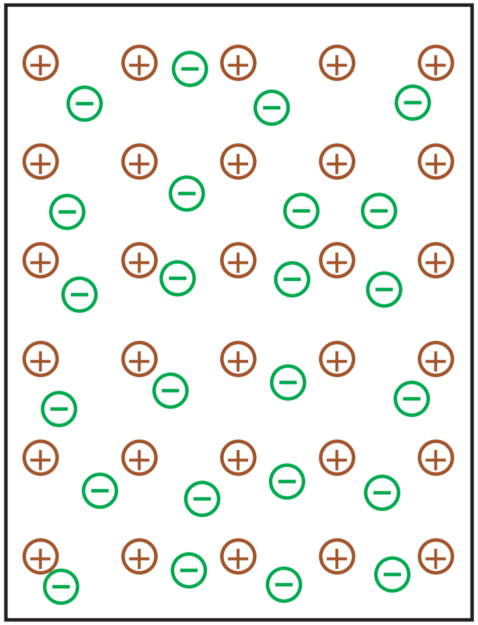
$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

- Here V is any bounded volume and S is the closed surface on the boundary of the volume
- Application of this theorem to the electric flux density D immediately gives us the differential form of Gauss' law from the integral form

Perfect Conductors (I)

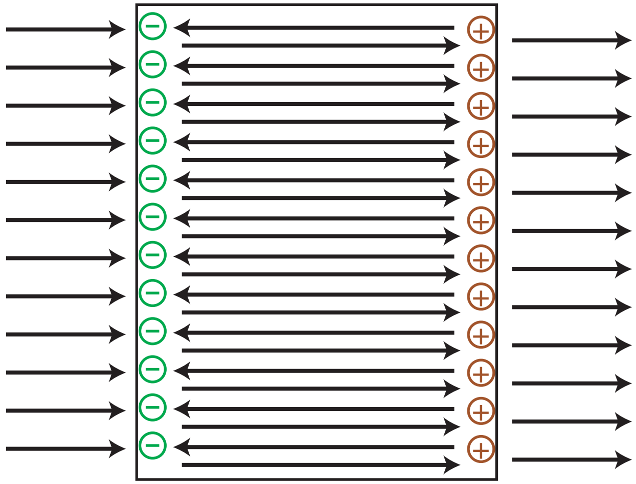
Perfect conductors are idealized materials with zero resistivity. Common metals such as gold, copper, aluminum are examples of good conductors.

A fuzzy picture of a perfect conductor is a material with many mobile charges that easily respond to an external fields. These carriers behave as if they are in vacuum and they can move unimpeded through the metal. These mobile charges, though, cannot leave the material due to a large potential barrier (related to the work function of the material).



Perfect Conductors (II)

One may argue that the electric field is zero under static equilibrium based on the following argument. Under static equilibrium charges cannot move. So we might argue that if an external field is applied, then all the charges in a perfect conductor would move to the boundary where they would remain due to the potential barrier. But this argument ignores the fact that the external field can be canceled by an *internal* field due to the rearrangement of charges (see figure below).



Perfect Conductors (III)

- Thus we have to invoke Gauss' law to further prove that in fact there can be no net charge in the body of a conductor. Because if net charge exists in the body, Gauss' law applied to a small sphere surrounding the charge would require a field in the body, which we already hypothesized to be zero. Therefore our argument is consistent.
- In conclusion, we are convinced that a perfect metal should have zero electric field in the body and no net charge in the body. Thus, we expect that if any net charge is to be found in the material, it would have to be on the boundary.

Perfect Conductors (IV)

- We could in fact define a perfect conductor as a material with zero electric field inside the material. This is an alternative way to define a perfect conductor without making any assumptions about conductivity (which we have not yet really explored)
- A perfect conductor is also an equipotential material under static conditions, or the potential is everywhere constant on the surface of a perfect conductor
- This is easy to prove since if $E \equiv 0$ in the material, then $\int \mathbf{E} \cdot d\ell$ is likewise zero between any two points in the material body.

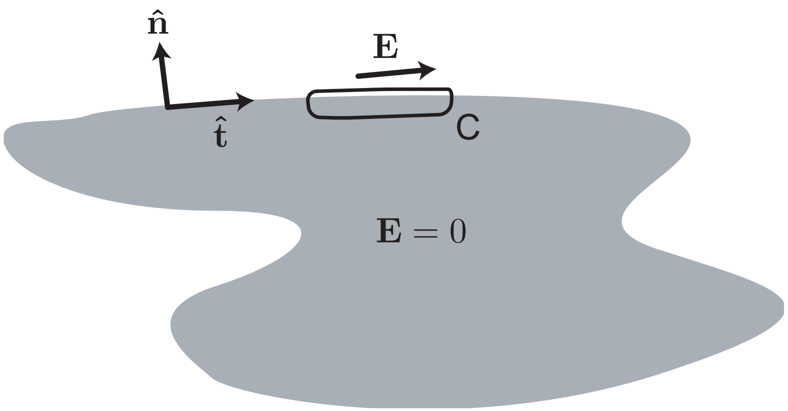
Tangential Boundary Conditions (I)

- It's easy to show that the electric field must cross the surface of a perfect conductor at a normal angle. Since

$$\int_C \mathbf{E} \cdot d\ell \equiv 0$$

For any path C, choose a path that partially crosses into the conductor. Since $E = 0$ inside the conductor, the only contribution to the integral are the side walls and E_t, or the tangential component along the path.

In the limit, we can make the path smaller and smaller until it is tangent to the surface and imperceptibly penetrates the material so the side wall contributions vanish.



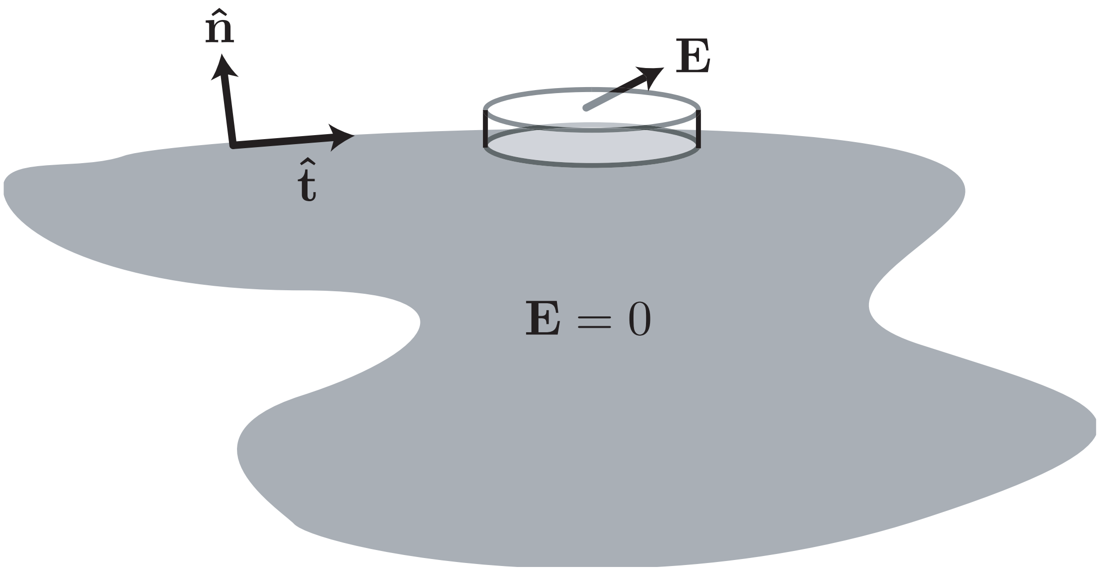
Tangential Boundary Conditions (II)

$$\int_C \mathbf{E} \cdot d\ell = E_t \Delta\ell \equiv 0$$

- We are thus led to conclude that $\boxed{E_t \equiv 0}$ at the surface
- Another argument is that since $\mathbf{E} = -\nabla\phi$ and furthermore since the surface is an equipotential, then clearly E_t is zero since there can be no change in potential along the surface

Normal Boundary Conditions (I)

- Consider a “pill-box” hugging the surface of a perfect conductor as shown below



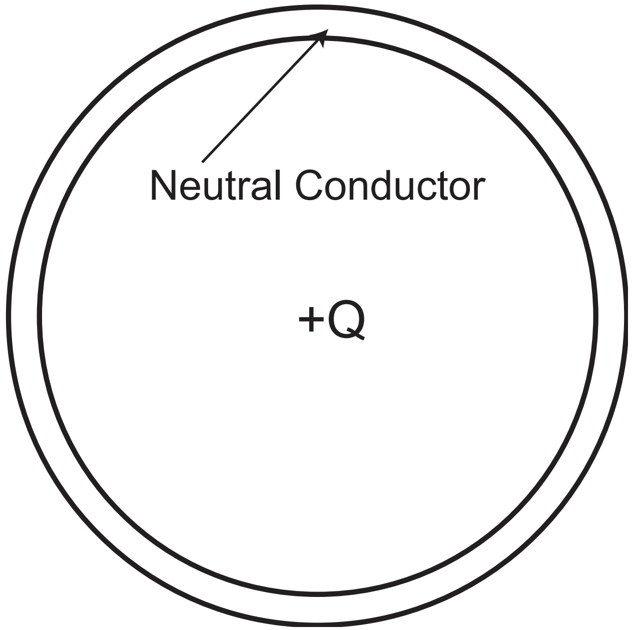
Normal Boundary Conditions (II)

- The electric flux density is computed for this surface. Like before, we'll make the volume smaller and smaller until the sidewall contribution goes to zero. Since the bottom is still in the conductor and the field is zero, this term will not contribute to the integral either. The only remaining contribution is the normal component of the top surface

$$\oint \mathbf{D} \cdot d\mathbf{S} = D_n dS = Q_{\text{inside}} = dS \rho_s$$

- We have thus shown that $\boxed{D_n = \rho_s}$

Example: Spherical Shell



- Consider a charge placed at the center of a spherical shell made of perfect conducting materials
- Let's find the surface charge density on the inner and outer surface of the conductor

Example (cont)

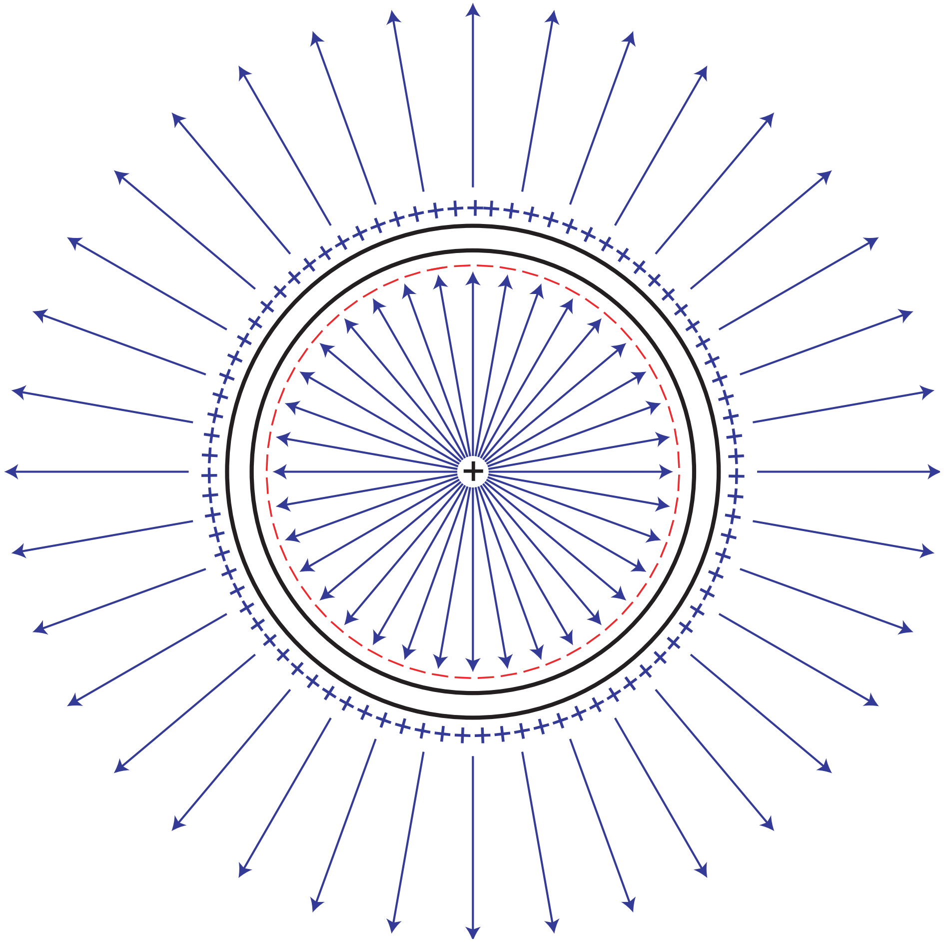
- Let's apply Gauss' law for the dashed sphere lying inside the conductor. Since $\mathbf{E} \equiv \mathbf{0}$ on this surface, the charge inside bus likewise be zero $Q_{\text{inside}} = 0$ which implies that there exists a uniform charge density (by symmetry) of $\rho_{\text{inner}} = -Q/S_i$ where $S_i = 4\pi a^2$
- If we now consider a larger sphere of radius $r > b$, since the sphere is neutral, the net charge is just the isolated charge Q. Thus

$$D_r 4\pi r^2 = Q + \underbrace{\rho_{\text{inside}} S_1}_{-Q} + \rho_{\text{outside}} S_2 = Q$$

- Thus $\rho_{\text{outside}} = Q/S_2 = \frac{Q}{4\pi b^2}$

Example: Field Sketch

The fields have been sketched in the figure below. Radial symmetry is preserved and an induced negative and positive charge density appear at the shell surfaces

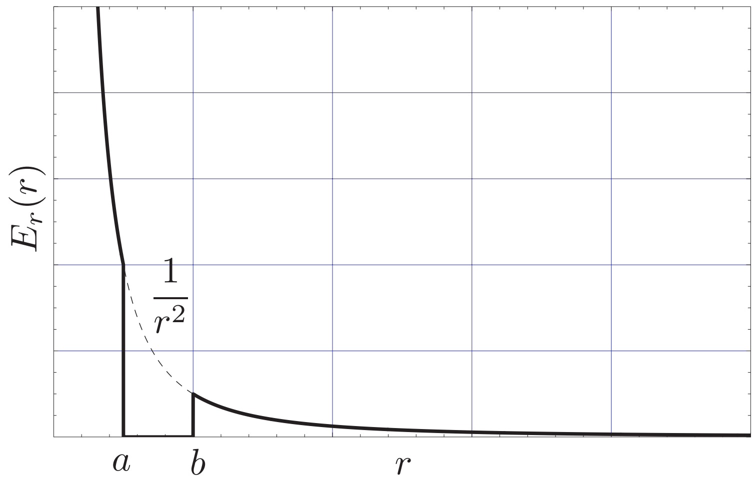


Example: Electric Field Summary

- We have found the following relations to hold

$$
\begin{aligned}
D_r &= \frac{Q}{4\pi r^2} & r < a \\
D_r &= 0 & a \le r \le b \\
D_r &= \frac{Q}{4\pi r^2} & r > b
\end{aligned}
$$

- A plot is shown below



Example: Potential

- Let's compute the potential from the fields. Take the reference point at ∞ to be zero and integrate along a radial path. For $r > b$

$$\phi = -\int_{\infty}^{r} E_r dr = \frac{Q}{4\pi\epsilon r}$$

- Likewise for $a \leq r \leq b$, since E_r is zero for the path inside the conducting sphere, the potential remains constant at $\frac{Q}{4\pi\epsilon_0 b}$ until we exit the conductor
- Finally, once outside the conductor for $r < a$ we continue integrating

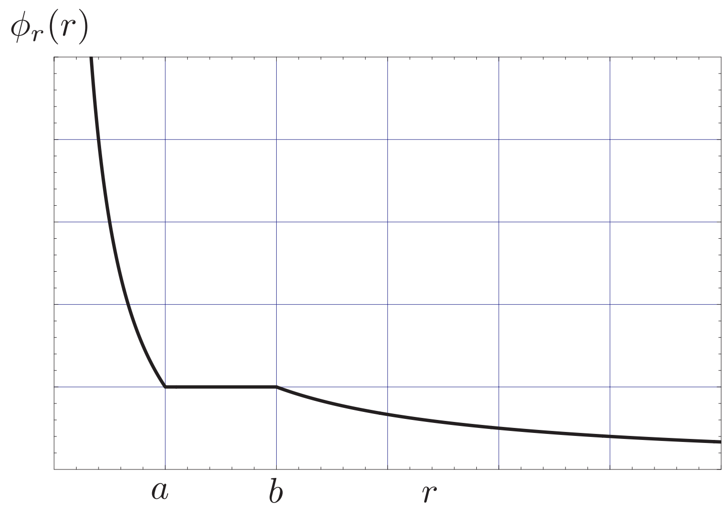
$$\phi = \frac{Q}{4\pi\epsilon b} - \int_{a}^{r} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

Example: Potential (cont)

- Thus we have

$$\phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} + \frac{1}{r} - \frac{1}{a}\right)$$

- A plot of the potential is shown below. Note that potential on the conductor is non-zero and its value depends on the amount of charge at the center



Grounded Shell

- What if we now ground the shell? That means that $\phi = 0$ on the surface of the conductor. The work done in moving a point charge from infinity to the surface of the shell is thus zero. Choose a radial path

$$\phi = -\int_{\infty}^{r} E_r dr = 0$$

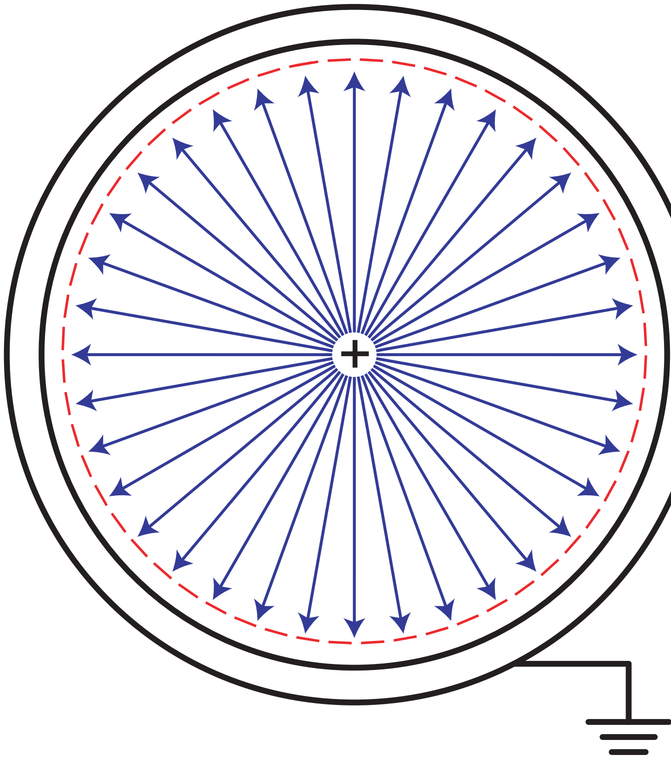
- Since the function E_r monotonic, $E_r = 0$ everywhere outside the sphere! A grounded spherical shell acts like a good shield.
- If $E_r(b) = 0$, then $\rho_s = 0$ at the outer surface as well.

Charged Spherical Shell

- To find the charge on the inner sphere, we can use the same argument as before unchanged so $\rho_{\text{inside}} = -Q/S_1$
- But now the material is not neutral! Where did the charge go?
- Imagine starting with the ungrounded case where the positive charge is induced on the outer surface. Then ground the sphere and we see that the charge flows out of the sphere into ground thus "charging" the material negative

Charged Shell Field Sketch

The fields have been sketched in the figure below. Radial symmetry is preserved and an induced negative charge density appears at inner shell surfaces. The field outside the shell is zero.



Poisson's Equation (aka Fish Eq.)

- We already have an equation for calculating $\mathbf{E}$ and ρ directly from a charge distribution
- If we know $\mathbf{E}$, then we can find ρ by simply taking the divergence: $\rho = \epsilon\nabla \cdot \mathbf{E}$. If we know ϕ is it possible to predict ρ? Sure:

$$\rho = \epsilon\nabla \cdot \mathbf{E} = -\epsilon\nabla \cdot \nabla\phi$$

- The operator $\nabla \cdot \nabla\mathbf{A}$ is a new operator and we call it the Laplacian ∇^2
- Thus we have Poisson's Equation $\boxed{\nabla^2\phi = -\frac{\rho}{\epsilon}}$
- In a charge free region we have $\nabla^2\phi = 0$. This is known as Laplace's equation

Laplacian in Rect. Coordinates

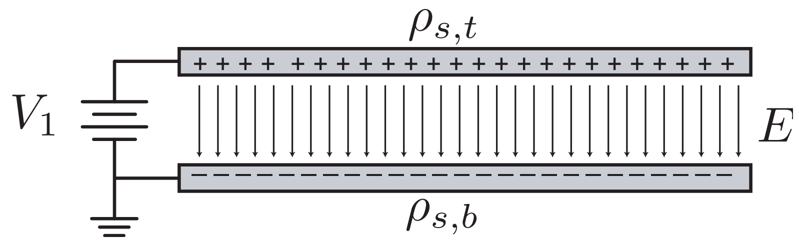
- Let's untangle the beast operator

$$\nabla \cdot \nabla \phi = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \right) \cdot \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{z}} \right)$$

- OK, all cross products involving mutual vectors die (e.g. $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} \equiv 0$) so we have

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Example: Parallel Plate Structure



- Consider a battery connected to a grounded parallel plate
- Suppose the plates are made of perfect conductors. If the plates are large, then we expect ϕ to be a function of z alone (ignore the "edge effects"). Thus the operator ∇^2 degenerates and inside the plates we have $0 \leq z \leq d$

$$\frac{\partial^2 \phi}{\partial z^2} = 0$$

Example (cont)

- Thus $\phi = az + b$. The boundary conditions on the top and bottom plate determine a and b uniquely. Since $\phi(z = 0) = 0$ we have $b = 0$. Also since $\phi(z = d) = V_1$, $a = V_1/d$

$$\phi(z) = \frac{V_1}{d} z$$

- And the field is easily computed

$$\mathbf{E} = -\nabla\phi = -\hat{\mathbf{z}} \frac{\partial}{\partial z} \left(\frac{V_1}{d} z \right) = -\hat{\mathbf{z}} \frac{V_1}{d}$$

Example (cont)

- The field is constant and points in the $-\hat{\mathbf{z}}$ direction. Since the charge density on the inner plate surface is equal to the normal component of the field there

$$\rho_{s,b} = \epsilon \hat{\mathbf{n}} \cdot \mathbf{E} = -\epsilon \frac{V_1}{d}$$

$$\rho_{s,t} = +\epsilon \frac{V_1}{d}$$