EECS 117

Lecture 7: Electrostatics Review

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Existence of Charge

- Charge, like mass, is an intrinsic property of matter.
- In some sense it is even more fundamental since unlike mass, its charge not a function of an object's velocity (relativistic invariant).
- Also note that an object's mass and charge can be independent since we have positive and negative charges. The *net* charge is the quantity of integral importance. A feather can have more charge than a heavy lead ball for this reason.
- Even though the existence of charge has been known since antiquity, the systematic study only began with people like Franklin and Coulomb in the early 1730s.

Coulomb and Franklin

- In his honor, the unit of charge is the Coulomb. For historically incorrect reasons (much to the chagrin of Franklin), the charge of an electron was assigned a negative value! Additionally, a unit of charge of one C (Coulomb) is actually quite a bit of charge.
- The charge of an electron $q = -1.6 \times 10^{-19} C$. Or there are about 6.25×10^{18} electrons in a charge of $-1\mathrm{C}$
- Experimentally it has been shown that the smallest unit of charge is the charge of an electron (charge is quantized)

Coulomb's Law

- Experimentally it is observed that the electric force between two charges follows an inverse square law form

$$|F_e| = \frac{q_1 q_2}{4\pi\epsilon_0 R_{12}^2}$$

- Note the similarity to the gravitational force $F_g = G_0 \frac{m_1 m_2}{R_{12}^2}$
- But the electric force is generally much stronger

$$\left|\frac{F_e}{F_g}\right| = \frac{q_1 q_2}{m_1 m_2} \frac{1}{4\pi\epsilon_0 G_0} \approx 10^{42}$$

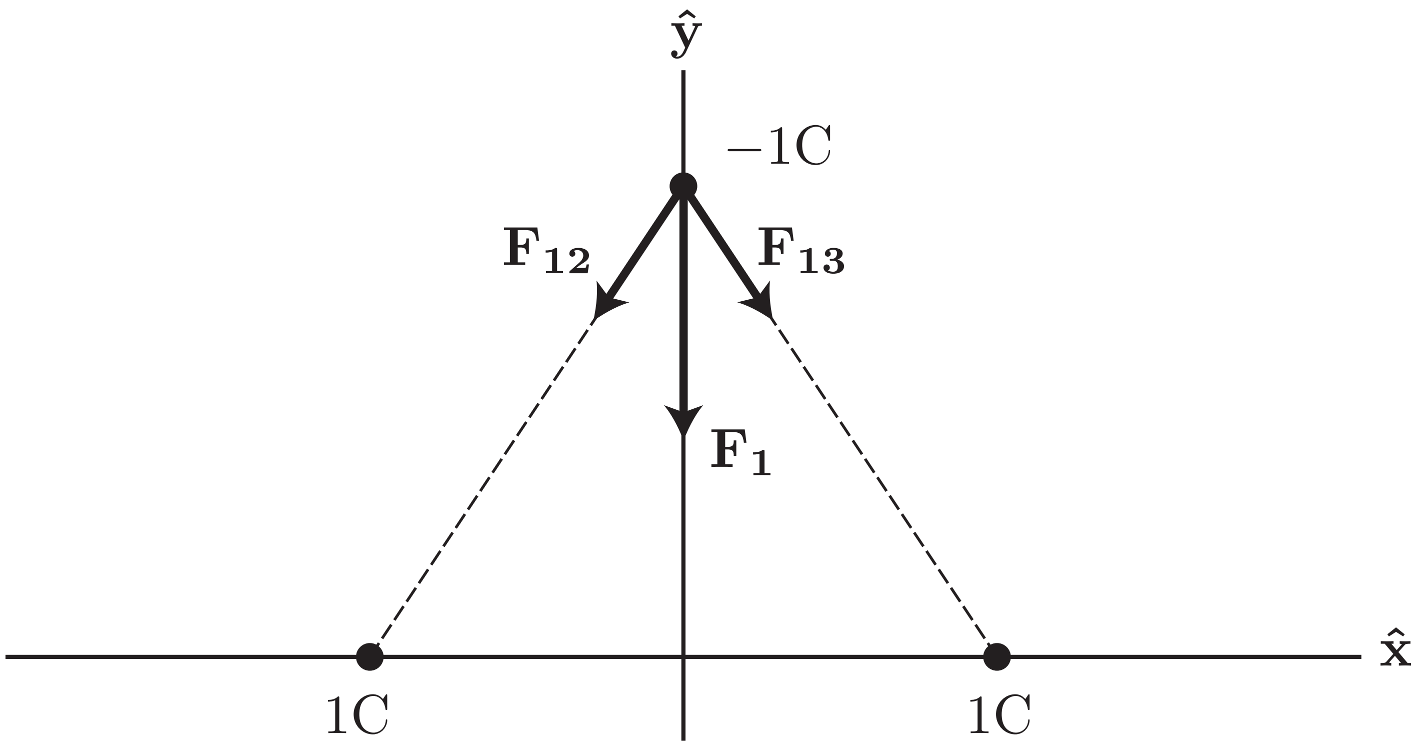
- But because we have negative and positive charges, electric forces are easily *shielded*. Gravitational forces are not, and hence very long range interaction is dominated by gravity

Vector Nature of Force

- We know that force is a vector quantity. it has direction in addition to magnitude and the components of force obey vector properties. It points in a direction along the segment connecting the charge centers. The statement “like charges repel and opposite charges attract” is part of our everyday culture.
- Writing the force in vector form

$$\mathbf{F_e} = \hat{\mathbf{R}} \frac{q_1 q_2}{4\pi\epsilon_0 R^2} = \frac{\mathbf{R}}{|R|} \frac{q_1 q_2}{4\pi\epsilon_0 |R|^2} = \mathbf{R} \frac{q_1 q_2}{4\pi\epsilon_0 |R|^3}$$

Superposition of Forces



- Force on charges can be derived by vectorial summation
- Note that $\mathbf{F_1} = \mathbf{F_{12}} + \mathbf{F_{13}} = -|F_1|\hat{\mathbf{y}}$

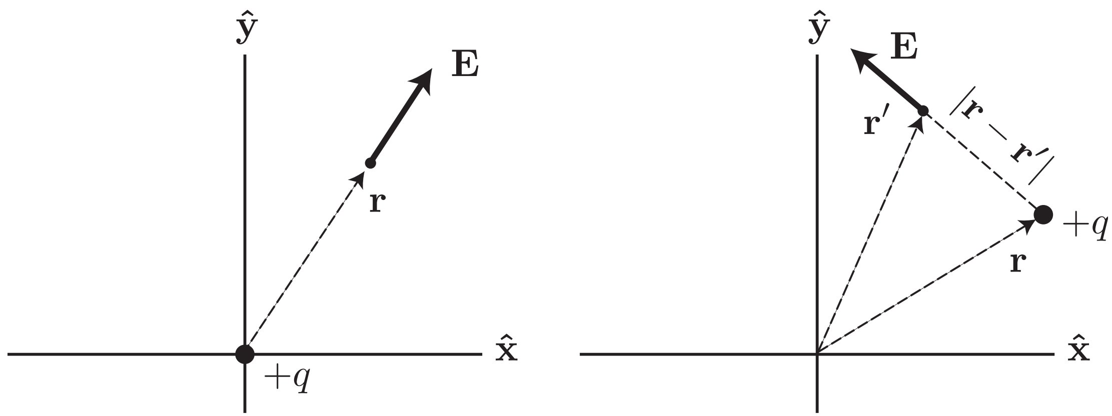
Electric Field

- A convenient way to think about electrostatics forces is to suppose the existence of a field. Since action at a distance runs counter to relativity, we suppose that something must "advertise" the existence of charge. We call that something the field

$$\mathbf{E} \triangleq \lim_{q \to 0} \frac{\mathbf{F}}{q}$$

- This is the force felt by a test charge of unit magnitude
- Why take limits? q has to be small enough as to not disturb the other charges in the volume in question. In reality we know that charge is quantized and $q > q_e$

Field of a Point Charge



- The field of a point charge q at the origin is therefore

$$\mathbf{E} = \hat{\mathbf{r}} \frac{q}{4\pi\epsilon_0 r^2}$$

- If q is not at origin, vectors make things easy

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \qquad \hat{\mathbf{R}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

Field Superposition

- For N charges, we simply sum the field due to each point charge

$$\mathbf{E}(\mathbf{r}) = \sum_i \frac{q_i \hat{\mathbf{R}}_\mathbf{i}}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r_i}|^2}$$

- Field lines are a convenient way to visualize the electric field. By convention, fields point away from positive charges and point into negative charges.
- We can use a program like Mathematica to plot field lines. But it's important to develop skills in sketching the field

Field Due to Charge Distribution

- It is often convenient to define a charge density ρ. We know that this is an essentially fictitious concept (due to the granularity of charge), but for any mildly macroscopic system, it's probably OK
- In a diff volume dV', the charge is $dq = \rho(\mathbf{r}')dV'$
- Therefore

$$d\mathbf{E} = \hat{\mathrm{R}}\frac{\rho(\mathbf{r}')dV'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}$$

- Summing (integrating) the fields

$$\mathbf{E} = \int_{V'} \hat{\mathrm{R}}\frac{\rho(\mathbf{r}')dV'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}$$

Electric Potential

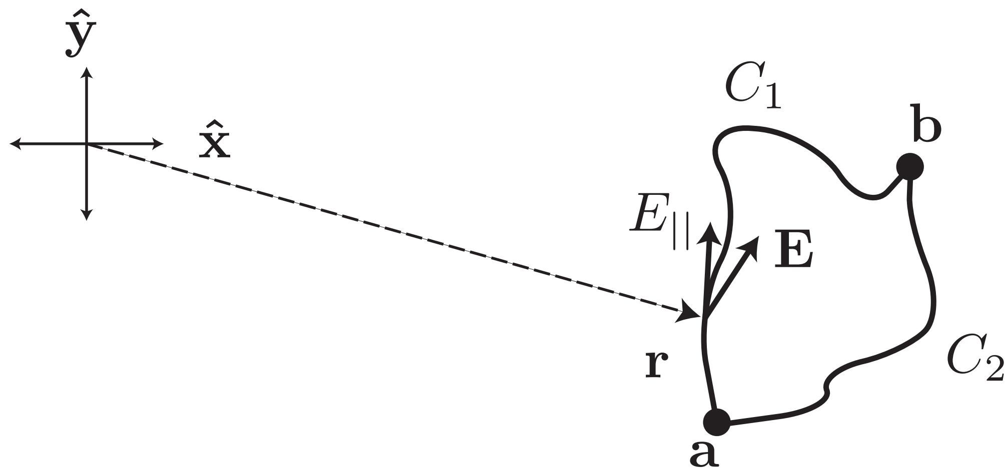
- We always approach problems with a dual energy/force perspective
- From an energy perspective, the work done in moving a charge against the field is simply

$$W = -\int_a^b \mathbf{F} \cdot d\ell = -\int_a^b q\mathbf{E} \cdot d\ell$$

- Let Φ be the energy normalized to charge

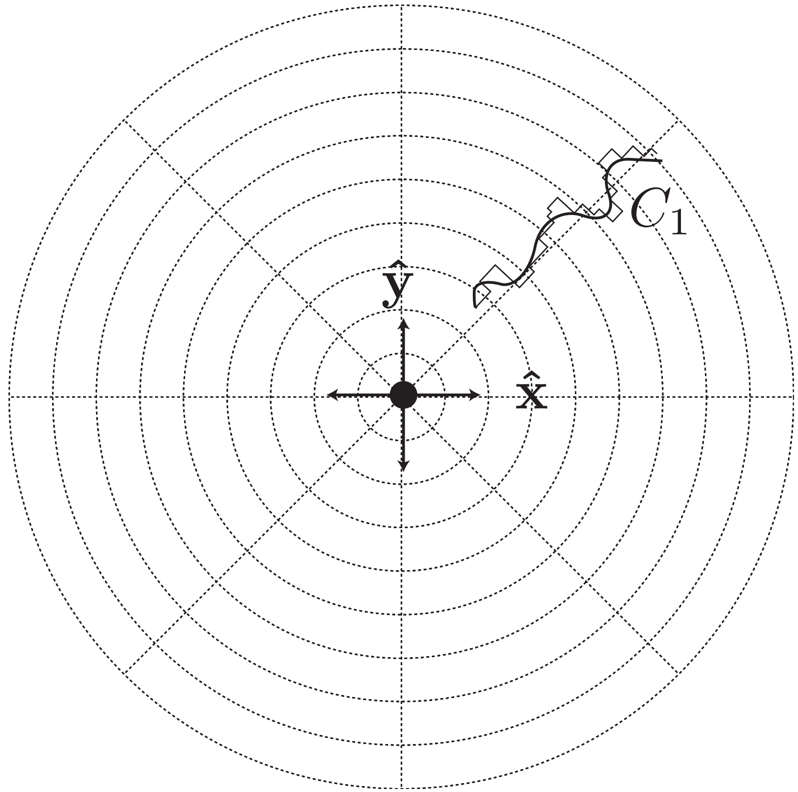
$$\Phi \triangleq \frac{W}{q}$$

Electric Potential



- Important question: Does Φ depend on path?
- For instance, we can pick path C_1 to integrate the function or path C_2
- If it's path independent, then the line integral will only be a function of endpoints
- Furthermore we can then define a potential function $\Phi(\mathbf{r})$

Path Independence for Point Charges (I)



- Consider the field of a point charge. For a point charge it's relatively easy to show that Φ is independent of path.
- Decompose C_1 into radial and tangential components

Path Independence for Point Charges (II)

$$\int_{C_1} = \int_{C_1^1} + \int_{C_1^2} + \int_{C_1^3} + \cdots = \underbrace{\sum_i \int_{C_1^{R_i}}}_{radial} + \underbrace{\sum_j \int_{C_1^{T_j}}}_{tang}$$

- But $\sum_j \int_{C_1^{T_j}} = 0$ since $d\ell$ is normal to $\mathbf{E}$ since $\mathbf{E}$ is purely radial. Thus the integral is due only to the radial components of the path
- For another path C_2, the radial component is the same, so the integral is path independent

Path Independence in General

- Again, by superposition, we can decompose the field into components arising from individual charges

$$E = E_1 + E_2 + E_3 + \cdots$$

- Then the field is path independent for *any* distribution of charges. The value of the integral is thus only a function of the endpoints a and b

$$-\int_C \mathbf{E} \cdot d\ell = \Phi(b) - \Phi(a)$$

- Also, for any closed path in a static field

$$\oint \mathbf{E} \cdot d\ell \equiv 0$$

Potential of a Point Charge

- For a point charge at the origin, the potential between two points is given by

$$\int_C \mathbf{E} \cdot d\ell = \frac{q}{4\pi\epsilon_0} \int \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0} \left| \frac{-1}{r} \right]_a^b$$

- Let's take the reference at infinity to be zero $\Phi(\infty) = 0$

$$\Phi(r) = -\frac{q}{4\pi\epsilon_0} \left(-\frac{1}{r} \right) = \frac{q}{4\pi\epsilon_0 r}$$

- For more than one point charge, superposition applies

$$\Phi(r) = \sum_i \frac{q_i}{4\pi\epsilon_0 R_i}$$

Potential of a Charge Distribution

- For a continuous charge distribution, the charge in a volume dV' is given by $dq_i = \rho(\mathbf{r}')dV'$
- Applying what we have learned already

$$d\Phi(\mathbf{r}) = \frac{\rho(\mathbf{r}')dV'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}$$

- And integrating over the entire volume we arrive at

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')dV'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}$$

- Potential is much nicer to work with than the field since it's a scalar computation

Relation Between Potential and Field

- By definition of potential, we have

$$d\Phi = -\mathbf{E}\cdot d\ell = -\mathbf{E}\cdot(\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) = -(E_x dx + E_y dy + E_z dz)$$

- The total change in potential in terms of partials is given by

$$d\Phi = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}}dx + \frac{\partial \mathbf{\Phi}}{\partial \mathbf{y}}dy + \frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}}dz$$

- Equating components we see that $E_p = -\frac{\partial \mathbf{\Phi}}{\partial \mathbf{p}}$
- We can write this compactly in terms of ∇

$$\mathbf{E} = -\nabla\mathbf{\Phi}$$

- Note that Φ is a scalar but $\nabla\mathbf{\Phi}$ is a vector

Gradient of a Function

- We may think of $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial \mathbf{x}} + \hat{\mathbf{y}}\frac{\partial}{\partial \mathbf{y}} + \hat{\mathbf{z}}\frac{\partial}{\partial \mathbf{z}}$
- But *be careful*, this only applies to rectangular coordinates
- Consider polar coordinates for instance. Since $d\ell = dR\,\hat{\mathbf{R}} + R\,d\theta\,\hat{\theta}$, equating $-\mathbf{E}\cdot d\ell$ to $d\Phi$ we have

$$-\left(E_R dR + E_\theta R d\theta\right) = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{R}} dR + \frac{\partial \mathbf{\Phi}}{\partial \theta} d\theta$$

- Or in polar coordinates we have

$$\nabla = \frac{\partial}{\partial \mathbf{R}}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial}{\partial \theta}\hat{\theta}$$