EECS 117

Lecture 6: Lossy Transmission Lines and the Smith Chart

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Dispersionless Line

- To find the conditions for the transmission line to be dispersionless in terms of the R, L, C, G, expand

$$\begin{aligned}
\gamma &= \sqrt{(j\omega L' + R')(j\omega C' + G')} \\
&= \sqrt{(j\omega)^2 LC(1 + \frac{R}{j\omega L} + \frac{G}{j\omega C} + \frac{RG}{(j\omega)^2 LC})} \\
&= \sqrt{(j\omega)^2 LC}\sqrt{\Box}
\end{aligned}$$

- Suppose that $R/L = G/C$ and simplify the $\Box$ term

$$\Box = 1 + \frac{2R}{j\omega L} + \frac{R^2}{(j\omega)^2 L^2}$$

Dispersionless Line (II)

- For $R/L = G/C$ the propagation constant simplifies

$$\square = \left(1 + \frac{R}{j\omega L}\right)^2 \qquad \gamma = -j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)$$

- Breaking γ into real and imaginary components

$$\gamma = R\sqrt{\frac{C}{L}} - j\omega\sqrt{LC} = \alpha + j\beta$$

- The attenauation constant α is independent of frequency. For low loss lines, $\alpha \approx -\frac{R}{Z_0}$ ✓
- The propagation constant β is a linear function of frequency ✓

Lossy Transmission Line Attenuation

- The power delivered into the line at a point z is now non-constant and decaying exponentially

$$P_{av}(z) = \frac{1}{2}\Re\left(v(z)i(z)^*\right) = \frac{|v^+|^2}{2|Z_0|^2} e^{-2\alpha z} \Re\left(Z_0\right)$$

- For instance, if $\alpha = .01\mathrm{m}^-1$, then a transmission line of length $\ell = 10\mathrm{m}$ will attenuate the signal by $10\log(e^{2\alpha\ell})$ or 2 dB. At $\ell = 100\mathrm{m}$ will attenuate the signal by $10\log(e^{2\alpha\ell})$ or 20 dB.

Lossy Transmission Line Impedance

- Using the same methods to calculate the impedance for the low-loss line, we arrive at the following line voltage/current

$$v(z) = v^+ e^{-\gamma z}(1 + \rho_L e^{2\gamma z}) = v^+ e^{-\gamma z}(1 + \rho_L(z))$$

$$i(z) = \frac{v^+}{Z_0} e^{-\gamma z}(1 - \rho_L(z))$$

- Where $\rho_L(z)$ is the complex reflection coefficient at position z and the load reflection coefficient is unaltered from before
- The input impedance is therefore

$$Z_{in}(z) = Z_0 \frac{e^{-\gamma z} + \rho_L e^{\gamma z}}{e^{-\gamma z} - \rho_L e^{\gamma z}}$$

Lossy T-Line Impedance (cont)

- Substituting the value of ρ_L we arrive at a similar equation (now a hyperbolic tangent)

$$Z_{in}(-\ell) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}$$

- For a short line, if $\gamma\delta\ell \ll 1$, we may safely assume that

$$Z_{in}(-\delta\ell) = Z_0 \tanh(\gamma\delta\ell) \approx Z_0\gamma\delta\ell$$

- Recall that $Z_0\gamma = \sqrt{Z'/Y'}\sqrt{Z'Y'}$
- As expected, input impedance is therefore the series impedance of the line (where $R = R'\delta\ell$ and $L = L'\delta\ell$)

$$Z_{in}(-\delta\ell) = Z'\delta\ell = R + j\omega L$$

Review of Resonance (I)

- We'd like to find the impedance of a series resonator near resonance $Z(\omega) = j\omega L + \frac{1}{j\omega C} + R$
- Recall the definition of the circuit Q

$$Q = \omega_0 \frac{\text{time average energy stored}}{\text{energy lost per cycle}}$$

- For a series resonator, $Q = \omega_0 L/R$. For a small frequency shift from resonance $\delta\omega \ll \omega_0$

$$Z(\omega_0 + \delta\omega) = j\omega_0 L + j\delta\omega L + \frac{1}{j\omega_0 C}\left(\frac{1}{1 + \frac{\delta\omega}{\omega_0}}\right) + R$$

Review of Resonance (II)

- Which can be simplified using the fact that $\omega_0 L = \frac{1}{\omega_0 C}$

$$Z(\omega_0 + \delta\omega) = j2\delta\omega L + R$$

- Using the definition of Q

$$Z(\omega_0 + \delta\omega) = R\left(1 + j2Q\frac{\delta\omega}{\omega_0}\right)$$

- For a parallel line, the same formula applies to the admittance

$$Y(\omega_0 + \delta\omega) = G\left(1 + j2Q\frac{\delta\omega}{\omega_0}\right)$$

- Where $Q = \omega_0 C/G$

$\lambda/2$ T-Line Resonators (Series)

- A shorted transmission line of length ℓ has input impedance of $Z_{in} = Z_0 \tanh(\gamma\ell)$
- For a low-loss line, Z_0 is almost real
- Expanding the $\tanh$ term into real and imaginary parts

$$\tanh(\alpha\ell + j\beta\ell) = \frac{\sinh(2\alpha\ell)}{\cos(2\beta\ell) + \cosh(2\alpha\ell)} + \frac{j\sin(2\beta\ell)}{\cos(2\beta\ell) + \cosh(2\alpha\ell)}$$

- Since $\lambda_0 f_0 = c$ and $\ell = \lambda_0/2$ (near the resonant frequency), we have $\beta\ell = 2\pi\ell/\lambda = 2\pi\ell f/c = \pi + 2\pi\delta f\ell/c = \pi + \pi\delta\omega/\omega_0$
- If the lines are low loss, then $\alpha\ell \ll 1$

$\lambda/2$ Series Resonance

- Simplifying the above relation we come to

$$Z_i n = Z_0 \left(\alpha \ell + j \frac{\pi \delta\omega}{\omega_0} \right)$$

- The above form for the input impedance of the series resonant T-line has the same form as that of the series LRC circuit
- We can define equivalent elements

$$R_{eq} = Z_0 \alpha \ell = Z_0 \alpha \lambda/2$$

$$L_{eq} = \frac{\pi Z_0}{2\omega_0} \qquad C_{eq} = \frac{2}{Z_0 \pi \omega_0}$$

$\lambda/2$ Series Resonance Q

- The equivalent Q factor is given by

$$Q = \frac{1}{\omega_0 R_{eq} C_{eq}} = \frac{\pi}{\alpha \lambda_0} = \frac{\beta_0}{2\alpha}$$

- For a low-loss line, this Q factor can be made very large. A good T-line might have a Q of 1000 or 10,000 or more
- It's difficult to build a lumped circuit resonator with such a high Q factor

$\lambda/4$ T-Line Resonators (Parallel)

- For a short-circuited $\lambda/4$ line

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell = Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell}$$

- Multiply numerator and denominator by $-j \cot \beta\ell$

$$Z_{in} = Z_0 \frac{1 - j \tanh \alpha\ell \cot \beta\ell}{\tanh \alpha\ell - j \cot \beta\ell}$$

- For $\ell = \lambda/4$ at $\omega = \omega_0$ and $\omega = \omega_0 + \delta\omega$

$$\beta\ell = \frac{\omega_0 \ell}{v} + \frac{\delta\omega \ell}{v} = \frac{\pi}{2} + \frac{\pi\delta\omega}{2\omega_0}$$

$\lambda/4$ T-Line Resonators (Parallel)

- So $\cot\beta\ell = -tan\frac{\pi\delta\omega}{2\omega_0} \approx \frac{-\pi\delta\omega}{2\omega_0}$ and $\tanh\alpha\ell \approx \alpha\ell$

$$Z_{in} = Z_0 \frac{1 + j\alpha\ell\pi\delta\omega/2\omega_0}{\alpha\ell + j\pi\delta\omega/2\omega_0} \approx \frac{Z_0}{\alpha\ell + j\pi\delta\omega/2\omega_0}$$

- This has the same form for a parallel resonant RLC circuit

$$Z_{in} = \frac{1}{1/R + 2j\delta\omega C}$$

- The equivalent circuit elements are

$$R_{eq} = \frac{Z_0}{\alpha\ell} \qquad C_{eq} = \frac{\pi}{4\omega_0 Z_0} \qquad L_{eq} = \frac{1}{\omega_0^2 C_{eq}}$$

$\lambda/4$ T-Line Resonators Q Factor

- The quality factor is thus

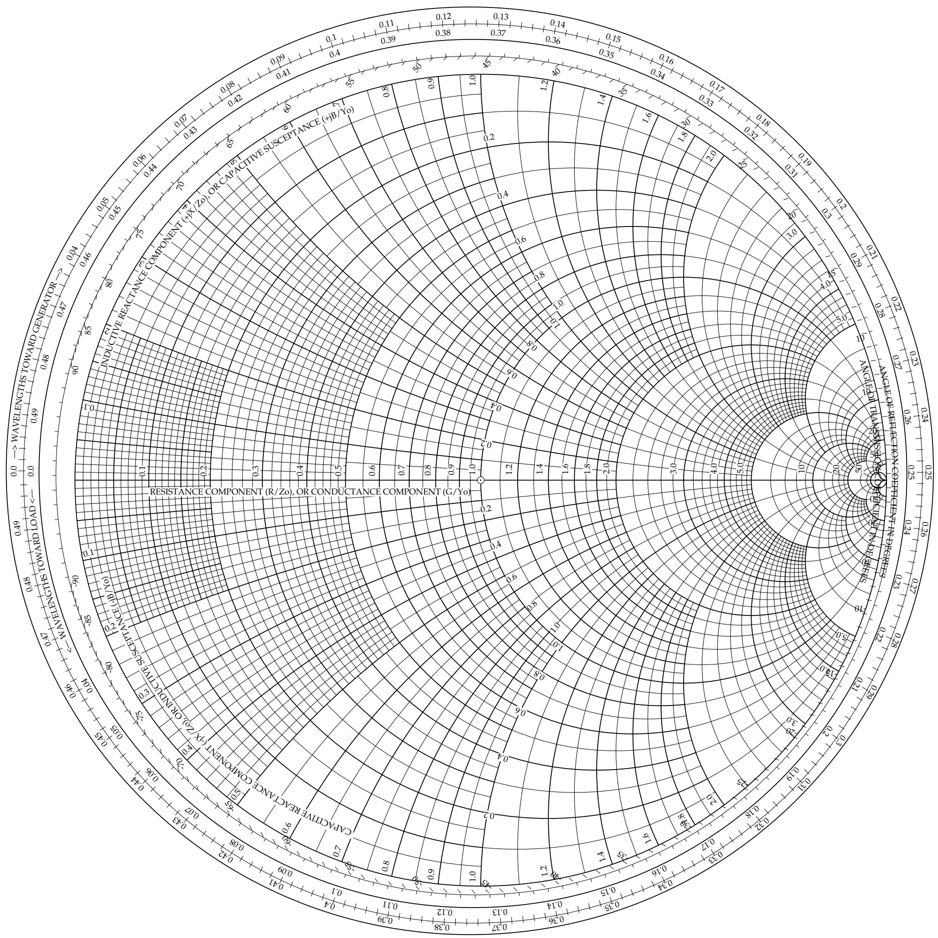
$$Q = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha}$$

The Smith Chart

- The Smith Chart is simply a graphical calculator for computing impedance as a function of reflection coefficient $z = f(\rho)$
- More importantly, many problems can be easily visualized with the Smith Chart
- This visualization leads to a insight about the behavior of transmission lines
- All the knowledge is coherently and compactly represented by the Smith Chart
- Why else study the Smith Chart? It's beautiful!
- There are deep mathematical connections in the Smith Chart. It's the tip of the iceberg! Study complex analysis to learn more.

An Impedance Smith Chart

- Without further ado, here it is!



Generalized Reflection Coefficient

- In sinusoidal steady-state, the voltage on the line is a T-line

$$v(z) = v^+(z) + v^-(z) = V^+(e^{-\gamma z} + \rho_L e^{\gamma z})$$

- Recall that we can define the reflection coefficient anywhere by taking the ratio of the reflected wave to the forward wave

$$\rho(z) = \frac{v^-(z)}{v^+(z)} = \frac{\rho_L e^{\gamma z}}{e^{-\gamma z}} = \rho_L e^{2\gamma z}$$

- Therefore the impedance on the line ...

$$Z(z) = \frac{v^+ e^{-\gamma z}(1 + \rho_L e^{2\gamma z})}{\frac{v^+}{Z_0} e^{-\gamma z}(1 - \rho_L e^{2\gamma z})}$$

Normalized Impedance

- ...can be expressed in terms of $\rho(z)$

$$Z(z) = Z_0 \frac{1+\rho(z)}{1-\rho(z)}$$

- It is extremely fruitful to work with normalized impedance values $z = Z/Z_0$

$$z(z) = \frac{Z(z)}{Z_0} = \frac{1+\rho(z)}{1-\rho(z)}$$

- Let the normalized impedance be written as $z = r + jx$ (note small case)
- The reflection coefficient is “normalized” by default since for passive loads $|\rho| \leq 1$. Let $\rho = u + jv$

Dissection of the Transformation

- Now simply equate the $\Re$ and $\Im$ components in the above equaiton

$$r + jx = \frac{(1+u)+jv}{(1-u)-jv} = \frac{((1+u+jv)(1-u+jv)}{(1-u)^2+v^2}$$

- To obtain the relationship between the (r,x) plane and the (u,v) plane

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

$$x = \frac{v(1-u)+v(1+u)}{(1-u)^2+v^2}$$

- The above equations can be simplified and put into a nice form

Completing Your Squares...

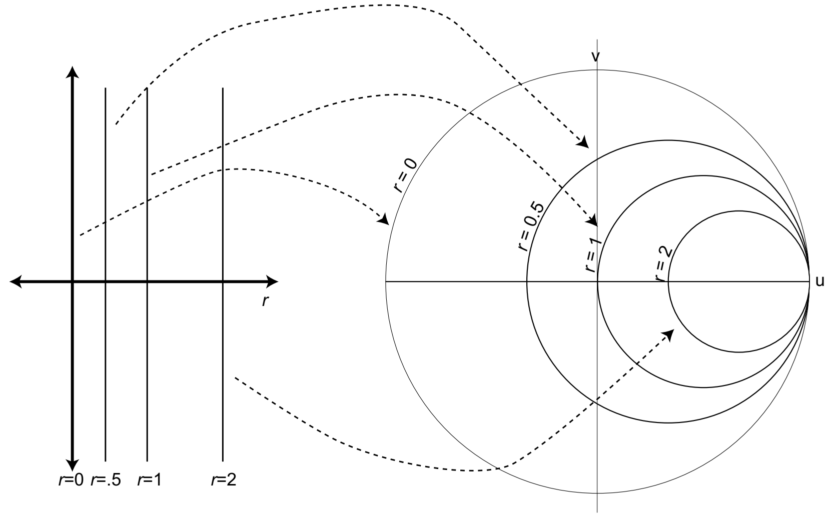
- If you remember your high school algebra, you can derive the following equivalent equations

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2}$$

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

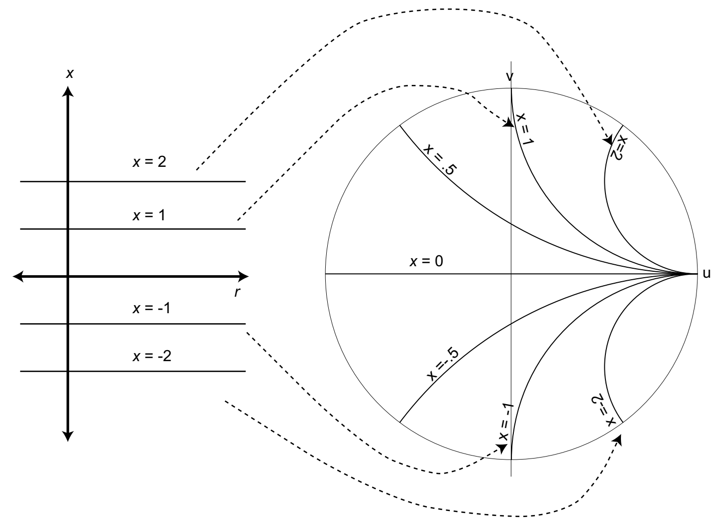
- These are circles in the (u,v) plane! Circles are good!
- We see that vertical and horizontal lines in the (r,x) plane (complex impedance plane) are transformed to circles in the (u,v) plane (complex reflection coefficient)

Resistance Transformations



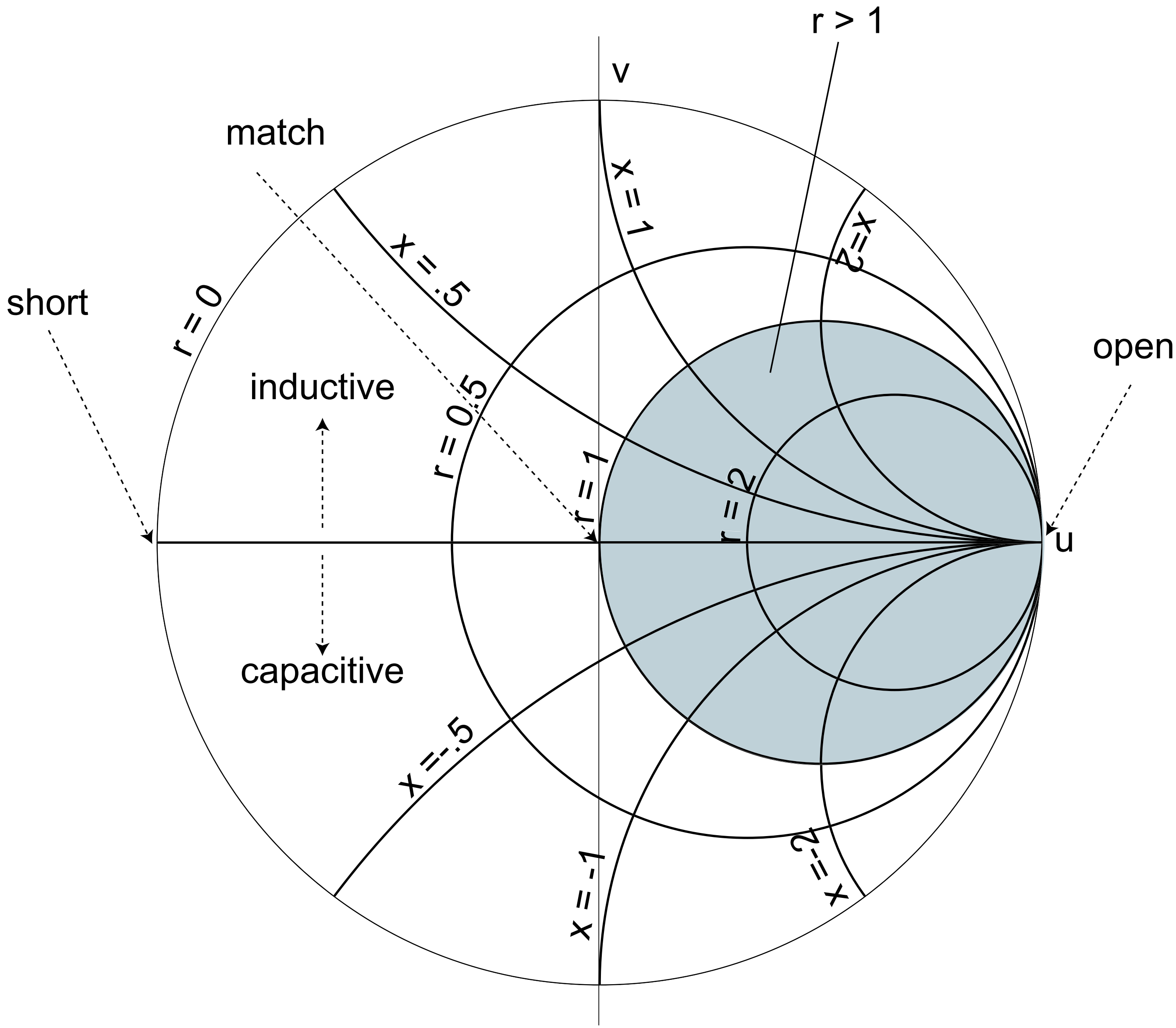
- $r = 0$ maps to $u^2 + v^2 = 1$ (unit circle)
- $r = 1$ maps to $(u - 1/2)^2 + v^2 = (1/2)^2$ (matched real part)
- $r = .5$ maps to $(u - 1/3)^2 + v^2 = (2/3)^2$ (load R less than Z_0)
- $r = 2$ maps to $(u - 2/3)^2 + v^2 = (1/3)^2$ (load R greater than Z_0)

Reactance Transformations

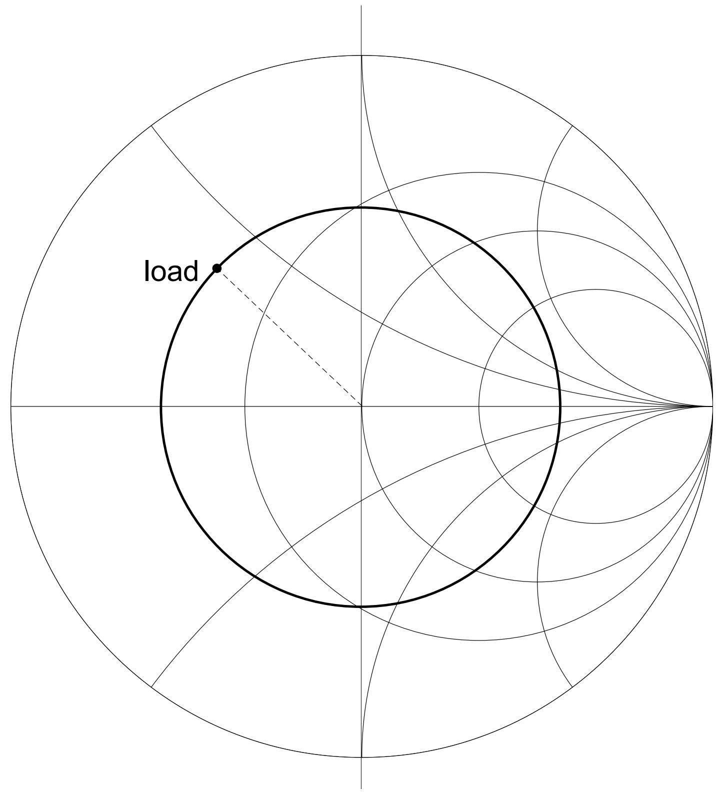


- $x = \pm 1$ maps to $(u-1)^2 + (v \mp 1)^2 = 1$
- $x = \pm 2$ maps to $(u-1)^2 + (v \mp 1/2)^2 = (1/2)^2$
- $x = \pm 1/2$ maps to $(u-1)^2 + (v \mp 2)^2 = 2^2$
- Inductive reactance maps to upper half of unit circle
- Capacitive reactance maps to lower half of unit circle

Complete Smith Chart

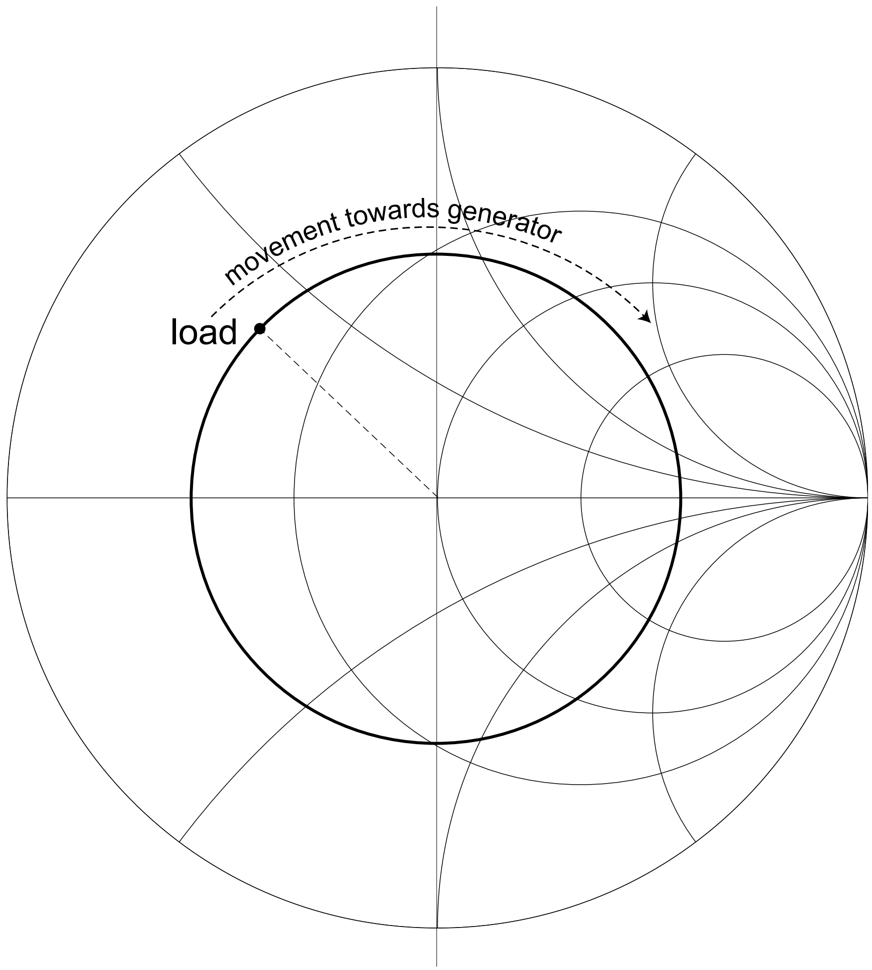


Load on Smith Chart



- First map z_L on the Smith Chart as ρ_L
- To read off the impedance on the T-line at any point on a lossless line, simply move on a circle of constant radius since $\rho(z) = \rho_L e^{2j\beta}$

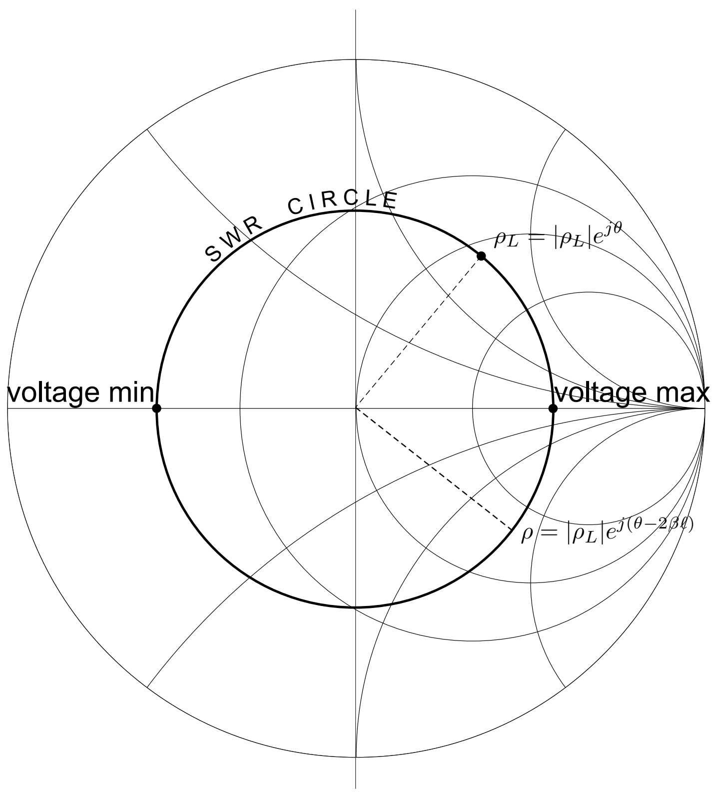
Motion Towards Generator



- Moving towards generator means $\rho(-\ell) = \rho_L e^{-2j\beta\ell}$, or clockwise motion
- For a lossy line, this corresponds to a spiral motion
- We're back to where we started when $2\beta\ell = 2\pi$, or $\ell = \lambda/2$
- Thus the impedance is periodic (as we know)

SWR Circle

Since SWR is a function of $|\rho|$, a circle at origin in (u,v) plane is called an SWR circle



- Recall the voltage max occurs when the reflected wave is in phase with the forward wave, so $\rho(z_{min}) = |\rho_L|$
- This corresponds to the intersection of the SWR circle with the positive real axis
- Likewise, the intersection with the negative real axis is the location of the voltge min

Example of Smith Chart Visualization

- Prove that if Z_L has an inductance reactance, then the position of the first voltage maximum occurs before the voltage minimum as we move towards the generator
- A visual proof is easy using Smith Chart
- On the Smith Chart start at any point in the upper half of the unit circle. Moving towards the generator corresponds to clockwise motion on a circle. Therefore we will always cross the positive real axis first and then the negative real axis.

Impedance Matching Example

- Single stub impedance matching is easy to do with the Smith Chart
- Simply find the intersection of the SWR circle with the $r = 1$ circle
- The match is at the center of the circle. Grab a reactance in series or shunt to move you there!

Series Stub Match

Admittance Chart

- Since $y = 1/z = \frac{1-\rho}{1+\rho}$, you can imagine that an Admittance Smith Chart looks very similar
- In fact everything is switched around a bit and you can buy or construct a combined admittance/impedance smith chart. You can also use an impedance chart for admittance if you simply map $x \rightarrow b$ and $r \rightarrow g$
- Be careful ... the caps are now on the top of the chart and the inductors on the bottom
- The short and open likewise swap positions

Admittance on Smith Chart

- Sometimes you may need to work with both impedances and admittances.
- This is easy on the Smith Chart due to the impedance inversion property of a $\lambda/4$ line

$$Z' = \frac{Z_0^2}{Z}$$

- If we normalize Z' we get y

$$\frac{Z'}{Z_0} = \frac{Z_0}{Z} = \frac{1}{z} = y$$

Admittance Conversion

- Thus if we simply rotate π degrees on the Smith Chart and read off the impedance, we're actually reading off the admittance!
- Rotating π degrees is easy. Simply draw a line through origin and z_L and read off the second point of intersection on the SWR circle

Shunt Stub Match

- Let's now solve the same matching problem with a shunt stub.
- To find the shunt stub value, simply convert the value of $z = 1 + jx$ to $y = 1 + jb$ and place a reactance of $-jb$ in shunt