EECS 117

Lecture 26: TE and TM Waves

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TE Waves

- TE means that $e_z = 0$ but $h_z \neq 0$. If $k_c \neq 0$, we can use our solutions directly

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial h_z}{\partial x} \qquad H_y = \frac{-j\beta}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} \qquad E_y = \frac{-j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

- Since $k_c \neq 0$, we find h_z from the Helmholtz's Eq.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_z = 0$$

TE Wave Helmholtz Eq.

- Since $H_z = h_z(x,y)e^{-j\beta z}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \underbrace{-\beta^2 + k^2}_{k_c^2}\right) h_z = 0$$

- Solving the above equation is sufficient to find all the fields.
- We can also define a wave impedance to simplify the computation

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta}$$

Wave Cutoff Frequency

- Since $\beta = \sqrt{k^2 - k_c^2}$, we see that the impedance is not constant as a function of frequency.
- In fact, for wave propagation we require β to be real, or $k > k_c$

$$\omega\sqrt{\mu\epsilon} > k_c$$

$$\omega > \frac{k_c}{\sqrt{\mu\epsilon}} = \omega_c$$

- For wave propagation, the frequency ω must be larger than the cutoff frequency ω_c
- Thus the waveguide acts like a high-pass filter

TM Waves

- Now the situation is the dual of the TE case, $e_z \neq 0$ but $h_z = 0$
- Our equations simplify down to

$$H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial e_z}{\partial y} \qquad H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial e_z}{\partial x} \qquad E_y = \frac{-j\beta}{k_c^2}\frac{\partial e_z}{\partial y}$$

- And for $k_c \neq 0$, our reduced Helmholtz's Eq. for E_z

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z = 0$$

TM Wave Impedance

- With e_z known, all the fields can be derived from the above equations
- The wave impedance is given by

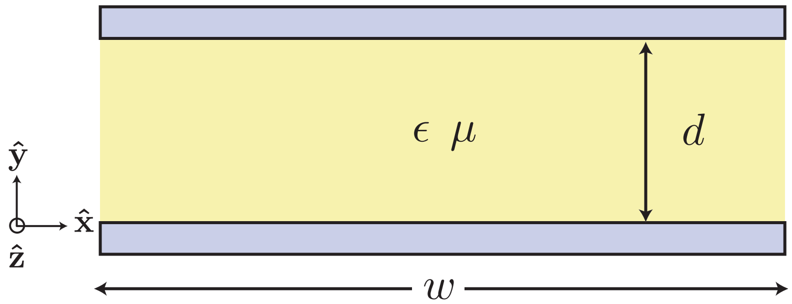
$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon}$$

- Since $\beta = \sqrt{k^2 - k_c^2}$, we see that the impedance is not constant as a function of frequency.
- The same high-pass cutoff behavior is also seen with the TM wave

TE/TM Wave General Solution

1. Solve the reduced Helmholtz eq. for e_z or h_z
2. Compute the transverse fields
3. Apply the boundary conditions to find k_c and any unknown constants
4. Compute $\beta = \sqrt{k^2 - k_c^2}$, so that $\gamma = j\beta$ and $Z_{TM} = \frac{\beta}{\omega\epsilon}$

Parallel Plate Waveguide



- Consider a simple parallel plate waveguide structure
- Let's begin by finding the properties of a TEM mode of propagation
- Last lecture we found that the TEM wave has an electrostatic solution in the transverse plane. We can thus solve this problem by solving Laplace's eq. in the region $0 \leq y \leq d$ and $0 \leq x \leq w$n

$$\nabla^2 \Phi = 0$$

Voltage Potential of TEM Mode

- The waveguide structure imposes the boundary conditions on the surface of the conductors

$$\Phi(x, 0) = 0$$

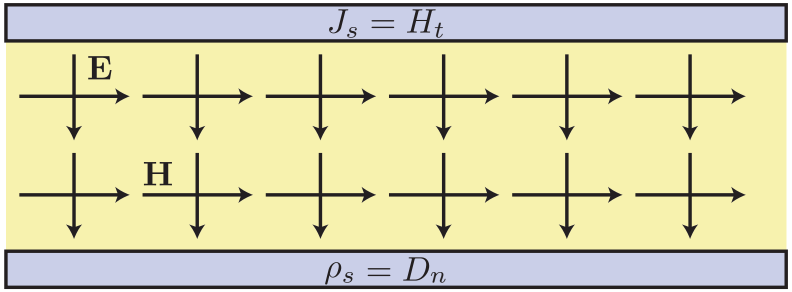
$$\Phi(x, d) = V_0$$

- Neglecting fringing fields for simplicity, we have

$$\Phi(x, y) = Ay + B$$

- The first boundary condition requires that $B \equiv 0$ and the second one can be used to solve for $A = V_0/d$.

Transverse Fields of TEM Mode



- The electric field is now computed from the potential

$$\mathbf{e}(x,y) = -\nabla_t \Phi = -\left(\frac{\partial \Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \Phi}{\partial y}\hat{\mathbf{y}}\right) = -\hat{\mathbf{y}}\frac{V_0}{d}$$

$$\mathbf{E} = \mathbf{e}(x,y)e^{-j\beta z} = -\hat{\mathbf{y}}\frac{V_0}{d}e^{-jkz}$$

$$\mathbf{H} = \frac{\hat{\mathbf{z}} \times \mathbf{E}}{Z_{TEM}} = \hat{\mathbf{x}}\frac{V_0}{d\eta}e^{-jkz}$$

Guide Voltages and Currents

- The E and H fields are shown above. Notice that the fields diverge on charge

$$\rho_n = \hat{\mathbf{n}} \cdot \mathbf{D} = \epsilon \frac{V_0}{d} e^{-jkz}$$

- This charge is traveling at the speed of light and giving rise to a current

$$I = \rho_n w c = w \frac{1}{\sqrt{\epsilon\mu}} \epsilon \frac{V_0}{\eta d} e^{-jkz} = \frac{w V_0}{\eta d} e^{-jkz}$$

Guide Currents

- We should also be able to find the guide current from Ampère's law

$$I = \oint_{C_b} \mathbf{H} \cdot d\ell = wH_x = \frac{wV_0}{\eta d} e^{-jkz}$$

- This matches our previous calculation. A third way to calculate the current is to observe that $J_s = H_t$

$$I = \int_0^w \mathbf{J_s} \cdot \hat{\mathbf{z}} dx = \frac{wV_0}{\eta d} e^{-jkz}$$

- The line characteristic impedance is the ratio of voltage to current

$$Z_0 = \frac{V}{I} = V_0 \frac{\eta d}{wV_0} = \eta \frac{d}{w}$$

Guide Impedance and Phase Velocity

- The guide impedance is thus only a function of the geometry of the guide. Likewise, the phase velocity

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

- The phase velocity is constant and independent of the geometry.

TM Mode of Parallel Plate Guide

- For TM modes, recall that $h_z = 0$ but $e_z \neq 0$
- We begin by solving the reduced Helmholtz Eq. for e_z

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z(x,y) = 0$$

- where $k_c^2 = k^2 - \beta^2$. As before, we take $\frac{\partial}{\partial x} = 0$ for simplicity

$$\left(\frac{\partial^2}{\partial y^2} + k_c^2\right) e_z(x,y) = 0$$

- The general solution of this simple equation is

$$e_z(x,y) = A \sin k_c y + B \cos k_c y$$

TM Mode Boundary Conditions

- Even though $e_z \neq 0$ inside the guide, at the boundary of the conductors, the tangential field, and hence e_z must be zero.
- This implies that $B = 0$ in the general solution. Also, applying the boundary condition at $y = d$

$$e_z(x, y = d) = 0 = A \sin k_c d$$

- This is only true in general if $k_c = 0$. But we have already seen that this corresponds to a TEM wave. We are now interested in TM waves so the argument of the sine term must be a multiple of $n\pi$ for $n = 1, 2, 3, \ldots$

$$k_c d = n\pi \rightarrow k_c = \frac{n\pi}{d}$$

Axial Fields in Guide

- The propagation constant is thus related to the geometry of the guide (unlike the TEM case)

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2}$$

- The axial fields are thus completely specified

$$e_z(x, y) = A_n \sin\left(\frac{n\pi y}{d}\right)$$

$$E_z(x, y, z) = A_n \sin\left(\frac{n\pi y}{d}\right) e^{-j\beta z}$$

Transverse TM Fields

- All the other fields are a function of E_z

$$H_x = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y} \qquad E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x} \qquad E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}$$

- So that $H_y = E_x = 0$ by inspection. The other components are

$$H_x = \frac{j\omega\epsilon}{k_c} A_n \cos\left(\frac{n\pi y}{d}\right) e^{-j\beta z}$$

$$E_y = \frac{-j\beta}{k_c} A_n \cos\left(\frac{n\pi y}{d}\right) e^{-j\beta z}$$

Cutoff Frequency

- As we have already noted, for wave propagation β must be real. Since $\beta = \sqrt{k^2 - k_c^2}$, we require

$$k > k_c$$

$$\omega\sqrt{\epsilon\mu} > k_c$$

$$\omega > \frac{k_c}{\sqrt{\mu\epsilon}} = \omega_c$$

$$\omega_c = \frac{n\pi}{d\sqrt{\mu\epsilon}}$$

- The guide acts like a high-pass filter for TM modes where the lowest propagation frequency for a particular mode n is given by

$$f_c = \frac{n}{2d\sqrt{\mu\epsilon}} = \frac{nc}{2d} = \frac{n}{\lambda_g}$$

TM Mode Velocity and Impedance

- The TM mode wave impedance is given by

$$Z_{TM} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\sqrt{\mu\epsilon}}{\omega\epsilon\sqrt{\mu\epsilon}} = \beta\sqrt{\mu\epsilon}k\epsilon = \frac{\beta\eta}{k}$$

- This is a purely real number for propagation modes $f > f_c$ and a purely imaginary impedance for cutoff modes
- The phase velocity is given by

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{k\sqrt{1-\left(\frac{k_c}{k}\right)^2}} = \frac{c}{\sqrt{1-\left(\frac{k_c}{k}\right)^2}} > c$$

- The phase velocity is faster than the speed of light! Does that bother you?

Phase Velocity

- It's important to remember that the phase velocity is a relationship between the spatial and time components of a wave in *steady-state*. It does not represent the wave evolution!
- Thus it's quite possible for the phase to advance faster than the time lag of "light" as long as this phase lag is a result of a steady-state process (you must wait an infinite amount of time!)
- The rate at which the wave evolves is given by the group velocity

$$v_g = \left(\frac{d\beta}{d\omega}\right)^{-1} \leq c$$

Power Flow

- Let's compute the average power flow along the guide for a TM mode. This is equal to the real part of the complex Poynting vector integrated over the guide

$$P_0 = \frac{1}{2}\Re \int_0^w \int_0^d \mathbf{E} \times \mathbf{H}^* \cdot \hat{\mathbf{z}} dy dx$$

$$\hat{\mathbf{z}} \cdot \mathbf{E} \times \mathbf{H}^* = E_y H_x^* = \frac{-j\omega\epsilon}{k_c} \left(A_n \cos \frac{n\pi y}{d} \right)^2 \frac{-j\beta}{k_c}$$

$$= \frac{\omega\epsilon\beta}{k_c^2} A_n^2 \cos^2 \frac{n\pi y}{d}$$

Power Flow (cont)

- Integrating the $\cos^2$ term produces a factor of $1/2$

$$P_0 = \frac{1}{4}\frac{w\omega\epsilon d}{k_c^2}|A_n|^2\Re(\beta)$$

- Therefore, as expected, if $f > f_c$, the power flow is non-zero but for cutoff modes, $f < f_c$, the average power flow is zero