EECS 117

Lecture 25: Field Theory of T-Lines and Waveguides

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Waveguides and Transmission Lines

- We started this course by studying transmission lines by using the concept of distributed circuits. Now we'd like to develop a field theory based approach to analyzing transmission lines.
- Transmission lines have one or more disconnected conductors. Waveguides, though, can consist of a single conductor. We'd also like to analyze waveguide structures, such as a hollow metal pipe, or a hollow rectangular structure. These are known as waveguides.
- These structures have a uniform cross-sectional area. We shall show that these structures can support wave propagation in the axial direction.

General Wave Propagation

- We shall assume that waves in the guide take the following form

$$\mathbf{E}(x,y,z) = [\mathbf{e}(x,y) + \hat{\mathbf{z}} e_z(x,y)]\, e^{-j\beta z} = \mathcal{E} e^{-j\beta z}$$

$$\mathbf{H}(x,y,z) = [\mathbf{h}(x,y) + \hat{\mathbf{z}} h_z(x,y)]\, e^{-j\beta z}$$

- It's important to note that we have broken the wave into two components, a part in the plane of the cross-section, or the transverse component $\mathbf{e}(x,y)$, and component in the direction of wave propagation, an axial component, $e_z(x,y)$.
- Recall that TEM plane waves have no components in the direction of propagation.

Maxwell's Equations

- Naturally, the fields in the waveguide or T-line have to satisfy Maxwell's equations. In particular

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

- Recall that $\nabla \times (\mathbf{F}f) = \nabla \mathbf{f} \times \mathbf{F} + f\nabla \times \mathbf{F}$

$$\nabla \times \mathbf{E} = -j\beta e^{-j\beta z}\hat{\mathbf{z}} \times \mathcal{E} + e^{-j\beta z}\nabla \times \mathcal{E}$$

- Note that $\nabla \times (\hat{\mathbf{z}}e_z(x,y))$ does not have a $\hat{\mathrm{z}}$-component whereas $\nabla \times \mathrm{e}$ has only a $\hat{\mathrm{z}}$-component

$$\nabla \times \mathbf{e} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix} = -\hat{\mathbf{x}}\frac{\partial E_y}{\partial z} + \hat{\mathbf{y}}\frac{\partial E_x}{\partial z} + \hat{\mathbf{z}}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$$

Curl of z-Component

- Since E_x and E_y have only (x, y) dependence

$$\nabla \times \mathbf{e} = \hat{\mathbf{z}}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$$

- Taking the curl of the $\hat{z}$-component generates only a transverse component

$$\nabla \times \hat{\mathbf{z}}\mathbf{e}_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & e_z \end{vmatrix} = \hat{\mathbf{x}}\frac{\partial e_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial e_z}{\partial x}$$

Curl of E

- Collecting terms we see that the curl of E has two terms, an axial term and a transverse term

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} = (-j\beta e^{-j\beta z})\underbrace{(\hat{\mathbf{z}} \times \mathbf{e})}_{\text{t}-\text{plane}} +$$

$$e^{-j\beta z}\left[\underbrace{\hat{\mathbf{z}}}_{\text{axial}}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) + \underbrace{\hat{\mathbf{x}}\frac{\partial e_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial e_z}{\partial x}}_{\text{t}-\text{plane}}\right]$$

The Field Component Equations

- Note that $\hat{\mathbf{z}} \times (E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}) = E_x \hat{\mathbf{y}} - E_y \hat{\mathbf{x}}$, so the x-component of the curl equation gives

(1)
$$j\beta E_y + \frac{\partial e_z}{\partial y} = -j\omega\mu H_x$$

- and the y-component gives

(2)
$$j\beta E_x + \frac{\partial e_z}{\partial x} = j\omega\mu H_y$$

- The z-component defines the third of our important equations

(3)
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu h_z(x, y)$$

The Curl of H

- Note that $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$, and so a set of similar equations can be derived without any extra math

(4) $$j\beta H_y + \frac{\partial h_z}{\partial y} = j\omega\epsilon E_x$$

(5) $$j\beta H_x + \frac{\partial h_z}{\partial x} = -j\omega\epsilon E_y$$

(6) $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon e_z(x, y)$$

Hx = f(non-Transverse Components)

- We can now reduce these (6) equations into (4) equations if we take e_z and h_z as known components. Since

$$j\beta H_x = -\frac{\partial h_z}{\partial x} - j\omega\epsilon E_y$$

and

$$E_y = \left(-\frac{\partial e_z}{\partial y} - j\omega\mu H_x\right)\frac{1}{j\beta}$$

- substituting E_y into the above equation

$$j\beta H_x = -\frac{\partial h_z}{\partial x} - \frac{j\omega\epsilon}{j\beta}\left(-\frac{\partial e_z}{\partial y} - j\omega\mu H_x\right)$$

$$j\beta H_x = -\frac{\partial h_z}{\partial x} + \frac{\omega\epsilon}{\beta}\frac{\partial e_z}{\partial y} + j\frac{\omega^2\mu\epsilon}{\beta}H_x$$

Hx = f(z) (cont)

- Collecting terms we have and $k^2 = \omega^2\mu\epsilon$

$$\left(j\beta - j\frac{k^2}{\beta}\right) H_x = \left(-\frac{\partial h_z}{\partial x} + \frac{\omega\epsilon}{\beta}\frac{\partial e_z}{\partial y}\right)$$

- Let $k_c^2 = k^2 - \beta^2$ and simplify

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial e_z}{\partial y} - \beta\frac{\partial h_z}{\partial x}\right) \tag{7}$$

- In the above eq. we have found the transverse component x in terms of the the axial components of the fields

Hy = f(z)

- We can also solve for H_y in terms of e_z and h_z

$$j\beta H_y = j\omega\epsilon E_x - \frac{\partial h_z}{\partial y} \qquad jE_x = -\frac{1}{\beta}\frac{\partial e_z}{\partial x} + \frac{j\omega\mu}{\beta}H_y$$

$$j\beta H_y = \omega\epsilon\left(-\frac{1}{\beta}\frac{\partial e_z}{\partial x} + \frac{j\omega\mu}{\beta}H_y\right) - \frac{\partial h_z}{\partial y}$$

- Collecting terms

$$\left(j\beta^2 - j\frac{k^2}{\beta}\right)H_y = -\frac{\omega\epsilon}{\beta}\frac{\partial e_z}{\partial x} - \beta\frac{\partial h_z}{\partial y}$$

(8) $$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial e_z}{\partial x} - \beta\frac{\partial h_z}{\partial y}\right)$$

$\mathbf{E = f(z)}$

- In a similar fashion, we can also derive the following equations

(9)
$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial e_z}{\partial x} + \omega\mu\frac{\partial h_z}{\partial y}\right)$$

(10)
$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial e_z}{\partial y} + \omega\mu\frac{\partial h_z}{\partial x}\right)$$

- Notice that we now have found a functional relation between all the transverse fileds in terms of the axial components of the fields

TEM, TE, and TM Waves

- We can classify all solutions for the field components into 3 classes of waves.
- TEM waves, which we have already studied, have no z-component. In other words $e_z = 0$ and $h_z = 0$
- TE waves, or transverse electric waves, has a transverse electric field, so while $e_z = 0$, $h_z \neq 0$ (also known as magnetic waves)
- TM waves, or transverse magnetic waves, has a transverse magnetic field, so while $h_z = 0$, $e_z \neq 0$ (also known as electric waves)

TEM Waves (again)

- From our equations (7) - (10), we see that if e_z and h_z are zero, then all the fields are zero unless $k_c = 0$
- This can be seen by working directly with equations (1) and (5)

$$j\beta E_y = -j\omega\mu H_x$$

$$j\beta H_x = -j\omega\epsilon E_y$$

$$j\beta E_y = \frac{-\omega\mu}{\beta}(-j\omega\epsilon E_y)$$

$$\beta^2 E_y = \omega^2\mu\epsilon E_y = k^2 E_y$$

- Thus $\beta^2 = k^2$, or $k_c = 0$

TEM Helmholtz Equation

- The Helmholtz Eq. $(\nabla^2 + k^2)\mathbf{E}$ simplifies for the TEM case. Take the x-component

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0$$

- Since the z-component of the field is a complex exponential

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 + k^2\right) E_x = 0$$

- Since $k^2 = \beta^2$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x = 0$$

TEM has Static Transverse Fields

- The same result applies to E_y so that we have

$$\nabla_t^2 \mathbf{e}(x, y) = 0$$

  where $\nabla_t = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$

- This is a two-dimensional Laplace equation. Recall that static fields satisfy Laplace's Eq. So it appears that our wave is a static field in the transverse plane!
- Therefore, applying our knowledge of electrostatics, we have

$$\mathbf{e}(x, y) = -\nabla_t \Phi(x, y)$$

- Where Φ is a scalar potential. This can also be seen by taking the curl of $\mathbf{e}$. If it's a static field, the curl must be identically zero

Static E Field

- Notice that

$$\nabla_t \times \mathbf{e} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & 0 \end{vmatrix} = \hat{\mathbf{z}} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -j\omega\mu h_z = 0$$

- Since $h_z = 0$, the curl of $\mathbf{e}$ is zero and thus the field behaves statically. Since $\nabla \cdot \mathbf{D} = \nabla_t \cdot \epsilon \mathbf{e} = 0$, we also have

$$\nabla_t^2 \Phi(x, y) = 0$$

- And thus we can also define a *unique* potential function in the transverse plane

$$V_{12} = -\int_1^2 \mathbf{e}(x, y) \cdot d\ell$$

Ampère's Law (I)

- By Ampère's Law (extended to include displacement current)

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J}$$

- Or in integral form

$$\oint_C \mathbf{H} \cdot d\ell = \int_S \mathbf{J} \cdot d\mathbf{S} + j\omega \int_S \mathbf{D} \cdot d\mathbf{S}$$

- But since $e_z = 0$, the surface integral term of displacement current vanishes and we have

$$\oint_C \mathbf{H} \cdot d\ell = \int_S \mathbf{J} \cdot d\mathbf{S} = I$$

- Which is an equation satisfied by *static* magnetic fields.

Faraday's Law

- It's easy to show that the electric fields also behave statically by using Faraday's law

$$\nabla \times \mathbf{E} = -j\omega B$$

or

$$\oint_C \mathbf{E} \cdot d\ell = -j\omega \int_S \mathbf{B} \cdot dS = 0$$

- The RHS is zero since $h_z = 0$ for TEM waves. Thus

$$\oint_C \mathbf{E} \cdot d\ell = 0$$

- and a unique potential can be defined.

TEM Wave Impedance

- By equation (4) we have

$$j\beta H_y = j\omega\epsilon E_x$$

- Thus the TEM wave impedance can be defined as

$$Z_{TEM} = \frac{E_x}{H_y} = \frac{\beta}{\omega\epsilon} = \frac{k}{\omega\epsilon} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

- From equation (5) we have

$$-j\beta H_x = j\omega\epsilon E_y$$

$$Z_{TEM} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

TEM Fields (last)

- Since $H_x = -E_y/\eta$ and $H_y = E_x/\eta$, we have

$$\mathbf{h}(x, y) = \frac{\hat{\mathbf{z}} \times \mathbf{e}(x, y)}{Z_{TEM}}$$

- Thus we only need to compute the electric field to find all the fields in the problem. This is exactly what we found when we studied uniform plane waves

TEM General Solution

1. Begin by solving Laplace's Eq. in the transverse plane (2D problem)
2. Apply boundary conditions to resolve some of the unknown constants
3. Compute the fields $\mathbf{e}$ and $\mathbf{h}$
4. Compute V and I (voltage and currents)
5. The propagation constant $\gamma = j\beta = j\omega\sqrt{\mu\epsilon}$ and the impedance is given by $Z_0 = V/I$

TEM Waves in Hollow Waveguides?

- What do our equations tell us about wave propagation in a hollow waveguide, such as a metal pipe?
- If TEM waves travel inside a such a structure, the transverse components must be solutions to the static 2D fields.
- But if we have a metal conductor surrounding a region, we have already proven in electrostatics that the only solution is a zero field, which are of no interest to us.
- Thus TEM waves *cannot* travel in such waveguides!
- We can "see" through a metal pipe, so what's going on?
- There must be other types of waves traveling through it.