# EECS 117

## *Lecture 23: Oblique Incidence and Reflection*

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# Review of TEM Waves

- We found that $\mathbf{E}(z) = \hat{\mathbf{x}} E_{i0} e^{-j\beta z}$ is a solution to Maxwell's eq. But clearly this wave should propagate in any direction and the physics should not change. We need a more general formulation.
- Consdier the following "plane wave"

$$\mathbf{E}(x,y,z) = \mathbf{E_0} e^{-j\beta_x x - j\beta_y y - j\beta_z z}$$

- This function also satisfies Maxwell's wave eq. In the time-harmonic case, this is the Helmholtz eq.

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

- where $k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}$

# Conditions Imposed by Helmholtz

- Each component of the vector must satisfy the scalar Helmholtz eq.

$$\nabla^2 E_x + k^2 E_x = 0$$

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x + k^2 E_x = 0$$

- Carrying out the simple derivatives

$$-\beta_x^2 - \beta_y^2 - \beta_z^2 + k^2 = 0$$

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = k^2$$

- Define $\mathbf{k} = \hat{\mathbf{x}}\beta_x + \hat{\mathbf{y}}\beta_y + \hat{\mathbf{z}}\beta_z$ as the propagation vector

# Propagation Vector

- The propagation vector can be written as a scalar times a unit vector

$$\mathbf{k} = k\hat{\mathbf{a}}_n$$

- The magnitude $k$ is given by $k = \omega\sqrt{\mu\epsilon}$
- As we'll show, the vector direction $\hat{\mathbf{a}}_n$ defines the direction of propagation for the plane wave
- Using the defined relations, we now have

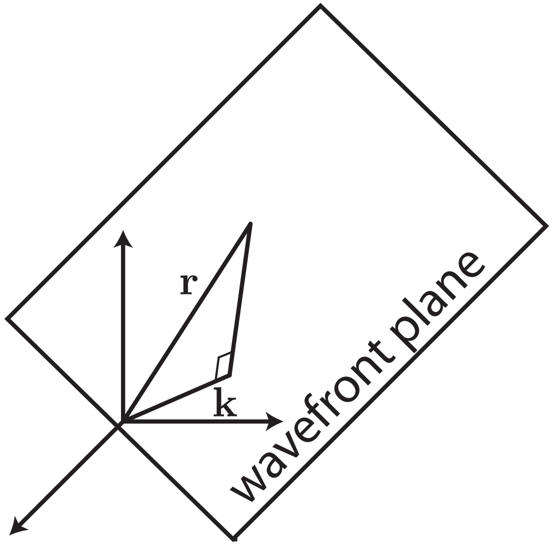
$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0}e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\beta_x = \mathbf{k}\cdot\hat{\mathbf{x}} = k\hat{\mathbf{a}}_n\cdot\hat{\mathbf{x}}$$

$$\beta_y = \mathbf{k}\cdot\hat{\mathbf{y}} = k\hat{\mathbf{a}}_n\cdot\hat{\mathbf{y}}$$

$$\beta_z = \mathbf{k}\cdot\hat{\mathbf{z}} = k\hat{\mathbf{a}}_n\cdot\hat{\mathbf{z}}$$

# Wavefront




- Recall that a wavefront is a surface of constant phase for the wave
- Then $\hat{\mathbf{a}}_n \cdot \mathbf{R} = \mathrm{constant}$ defines the surface of constant phase. But this surface does indeed define a plane surface. Thus we have a plane wave. Is it TEM?

# E is a "Normal" Wave

- Since our wave propagations in a source free region, $\nabla \cdot \mathbf{E} = 0$. Or

$$\mathbf{E_0} \cdot \nabla \left( e^{-jk\hat{\mathbf{a}}_n \cdot \mathbf{r}} \right) = 0$$

$$\nabla \left( e^{-jk\hat{\mathbf{a}}_n \cdot \mathbf{r}} \right) = \left( \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) e^{-j(\beta_x x + \beta_y y + \beta_z z)}$$

$$= -j(\beta_x \hat{\mathbf{x}} + \beta_y \hat{\mathbf{y}} + \beta_z \hat{\mathbf{z}}) e^{-j(\beta_x x + \beta_y y + \beta_z z)}$$

- So we have

$$-jk(\mathbf{E_0} \cdot \hat{\mathbf{a}}_n) e^{-jk\hat{\mathbf{a}}_n \cdot \mathbf{r}} = 0$$

- This implies that $\hat{\mathbf{a}}_n \cdot \mathbf{E_0} = 0$, or that the wave is polarized transverse to the direction of propagation

# H is also a "Normal" Wave

- Since $\mathbf{H}(\mathbf{r}) = \frac{1}{-j\omega\mu}\nabla \times \mathbf{E}$, we can calculate the direction of the $H$ field
- Recall that $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla \mathbf{f} \times \mathbf{F}$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{j\omega\mu}\nabla \left(e^{-j\mathbf{k}\cdot\mathbf{r}}\right) \times \mathbf{E_0}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{j\omega\mu}\mathbf{E_0} \times \left(-j\mathbf{k}e^{-j\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathbf{H}(\mathbf{r}) = \frac{k}{\omega\mu}\hat{\mathbf{a}}_n \times \mathbf{E}(\mathbf{r}) = \frac{1}{\eta}\hat{\mathbf{a}}_n \times \mathbf{E}(\mathbf{r})$$

$$\eta = \frac{\mu\omega}{k} = \frac{\mu\omega}{\omega\sqrt{\epsilon\mu}} = \sqrt{\mu/\epsilon}$$

# TEM Waves

- So we have done it. We proved that the equations

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0} e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\eta}\hat{\mathbf{a}}_n \times \mathbf{E}(\mathbf{r})$$

- describe *plane waves* where $\mathbf{E}$ is perpendicular to the direction of propagation and the vector $\mathbf{H}$ is perpendicular to both the direction of propagation and the vector $\mathbf{E}$
- These are the simplest general wave solutions to Maxwell's equations.

# Wave Polarization

- Now we can be more explicit when we say that a wave is linearly polarized. We simply mean that the vector $\mathbf{E}$ lies along a line. But what if we take the superposition of two linearly polarized waves with a $90^\circ$ time lag

$$\mathbf{E}(z) = \hat{\mathbf{x}} E_1(z) + \hat{\mathbf{y}} E_2(z)$$

- The first wave is $\hat{\mathrm{x}}$-polarized and the second wave is $\hat{\mathrm{y}}$-polarized. The wave propagates in the $\hat{\mathrm{z}}$ direction
- In the time-harmonic domain, a phase lag corresponds to multiplication by $-j$

$$\mathbf{E}(z) = \hat{\mathbf{x}} E_{10} e^{-j\beta z} - j \hat{\mathbf{y}} E_{20} e^{-j\beta z}$$

# Elliptical Polarization

- In time domain, the waveform is described by the following equation

$$\mathbf{E}(z,t) = \Re\left(\mathbf{E}(z)e^{j\omega t}\right)$$

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}E_{10}\cos(\omega t - \beta z) + \hat{\mathbf{y}}E_{20}\sin(\omega t - \beta z)$$

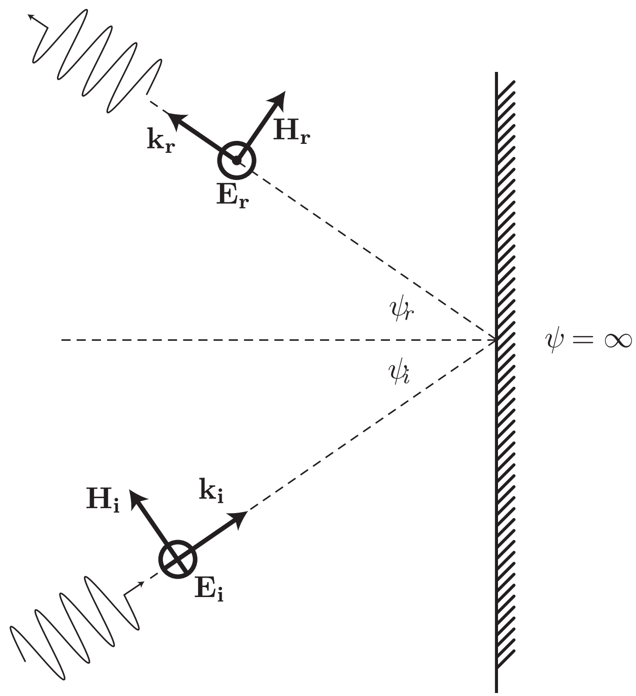
- At a paricular point in space, say $z = 0$, we have

$$\mathbf{E}(0,t) = \hat{\mathbf{x}}E_{10}\cos(\omega t) + \hat{\mathbf{y}}E_{20}\sin(\omega t)$$

- Thus the wave rotates along an elliptical path in the phase front!
- We can thus create waves that rotate in one direction or the other by simply adding two linearly polarized waves with the right phase

# Oblique Inc. on a Cond. Boundary

- Let the $\hat{x}$-$\hat{y}$ plane define the plane of incidence.
- Consider the polarization of a wave impinging obliquely on the boundary. We can identify two polarizations, perpendicular to the plane and parallel to the plane of incidence. Let's solve these problems separately.
- Any other polarized wave can always be decomposed into these two cases




A perpendicularly polarized wave.

# Perpendicular Polarization

- Let the angle of incidence and relfection we given by $\theta_i$ and $\theta_r$. Let the boundary consists of a perfect conductor

$$\mathbf{E_i} = \hat{\mathbf{y}} E_{i0} e^{-j\mathbf{k_1}\cdot\mathbf{r}}$$

- where $\mathbf{k_1} = k_1 \hat{\mathbf{a}}_{ni}$ and $\hat{\mathbf{a}}_{ni} = \hat{\mathbf{x}} \sin\theta_i + \hat{\mathbf{z}} \cos\theta_i$

$$\mathbf{E_i} = \hat{\mathbf{y}} E_{i0} e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H_i} = \frac{1}{\eta_i} \mathbf{a}_n \times \mathbf{E_i}$$

- For the reflected wave, similarly, we have $\hat{\mathbf{a}}_{nr} = \hat{\mathbf{x}} \sin\theta_r - \hat{\mathbf{z}} \cos\theta_r$ so that

$$\mathbf{E_r} = \hat{\mathbf{y}} E_{r0} e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$$

# Conductive Boundary Condition

- The conductor enforces the zero tangential field boundary condition. Since all of $\mathbf{E}$ is tangential in this case, at $z = 0$ we have

$$\mathbf{E_1}(x,0) = \mathbf{E_i}(x,0) + \mathbf{E_r}(x,0) = 0$$

- Substituting the relations we have

$$\hat{\mathbf{y}}\left(E_{i0}e^{-jk_1 x\sin\theta_i} + E_{r0}e^{-jk_1 x\sin\theta_r}\right) = 0$$

- For this equation to hold for any value of $x$ and $\theta$, the following conditions must hold

$$E_{r0} = -E_{i0} \qquad\qquad \theta_i = \theta_r$$

# Snell's Law

- We have found that the angle of incidence is equal to the angle of reflection (Snell's law)
- The total field, therefore, takes on an interesting form. The reflected wave is simply

$$\mathbf{E_r} = -\hat{\mathbf{y}} E_{i0} e^{-jk_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\mathbf{H_r} = \frac{1}{\eta_1}\hat{\mathbf{a}}_{nr} \times \mathbf{E_r}$$

$$\mathbf{H_r} = \frac{E_{i0}}{\eta_1}(-\hat{\mathbf{x}}\cos\theta_i - \hat{\mathbf{z}}\sin\theta_i) e^{-jk_1(x\sin\theta_i - z\cos\theta_i)}$$

- The total field is thus

$$\mathbf{E_1} = \mathbf{E_i} + \mathbf{E_r} = \hat{\mathbf{y}} E_{i0}\left(e^{-jk_1 z\cos\theta_i} - e^{jk_1 z\cos\theta_i}\right) e^{-jk_1 x\sin\theta_i}$$

# The Total Field

- Simplifying the expression for the total field

$$\mathbf{E_1} = -\hat{\mathbf{y}} 2j E_{i0} \underbrace{\sin(k_1 z \cos\theta_i)}_{\text{standing wave}} \underbrace{e^{-jk_1 x \sin\theta_i}}_{\text{prop. wave}}$$

$$\mathbf{H_1} = \frac{-2E_{i0}}{\eta_1} \Big( \quad \hat{\mathbf{x}} \quad \cos\theta_i \cos(k_1 z \cos\theta_i) e^{-jk_1 x \sin\theta_i} +$$

$$\hat{\mathbf{z}} \quad j \sin\theta_i \sin(k_1 z \cos\theta_i) e^{-jk_1 x \sin\theta_i} \Big)$$

# Important Observations

- In the $\hat{\mathbf{z}}$-direction, $E_{1y}$ and $H_{1x}$ maintain standing wave patterns (no average power propagates since $E$ and $H$ are $90^\circ$ out of phase. This matches our previous calculation for normal incidence
- Waves propagate in the $\hat{\mathbf{x}}$-direction with velocity $v_x = \omega/(k_1 \sin\theta_i)$
- Wave propagation in the $\hat{\mathbf{x}}$-direction is a non-uniform plane wave since its amplitude varies with $z$

# TE Waves

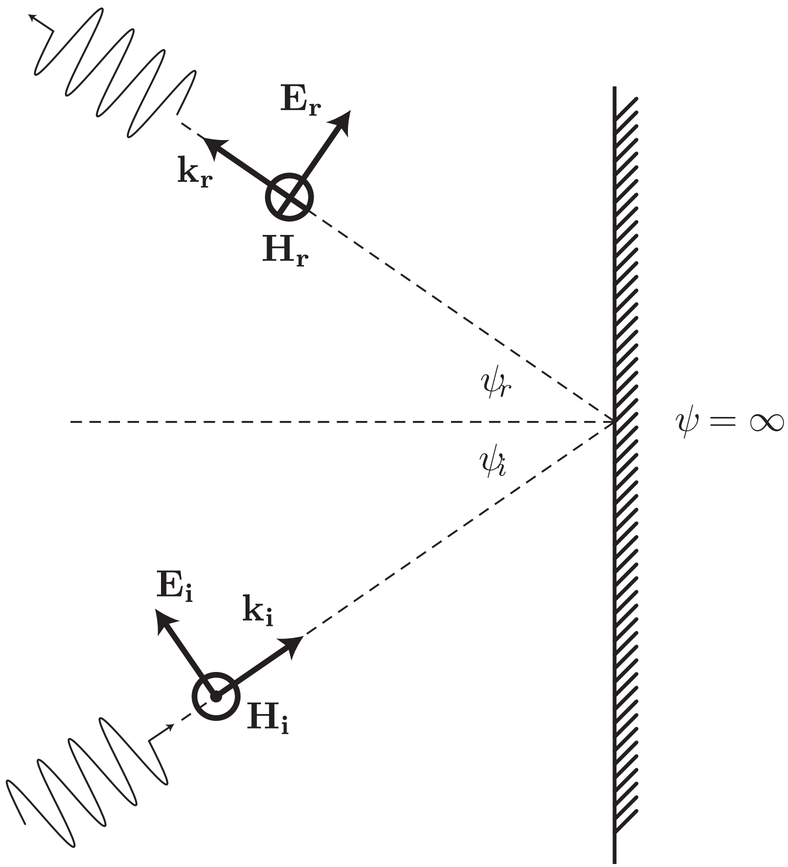
- Notice that on plane surfaces where $\mathbf{E} = 0$, we are free to place a conducting plane at that location without changing the fields outside of the region
- In particular, notice that $\mathbf{E} = 0$ when

$$\sin(k_1 z \cos\theta_i) = 0$$

$$k_1 z \cos\theta_i = \frac{2\pi}{\lambda_1} z \cos\theta_i = -m\pi$$

- This holds for $m = 1, 2, \ldots$
- So if we place a plane conductor at $z = -\frac{m\lambda_1}{2\cos\theta_i}$, there will be a "guided" wave traveling between the two planes in the $\hat{\mathbf{x}}$ direction
- Since $E_{1x} = 0$, this wave is a "TE" wave as $H_{1x} \neq 0$

# Parallel Polarization (I)




- Now the wave is polarized in the plane of incidence.
- The approach is similar to before but the tangential component of the electric field depends on the angle of incidence

# Parallel Polarization (II)

- Now consider an incident electric field that is in the plane of polarization

$$\mathbf{E_i} = \mathbf{E_{i0}}(\hat{\mathbf{x}} \cos\theta_i - \hat{\mathbf{z}} \sin\theta_i) e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{H_i} = \mathbf{y} \frac{E_{i0}}{\eta_1} e^{-j\mathbf{k}\cdot\mathbf{r}}$$

- Likewise, the reflected wave is expressed as

$$\mathbf{E_r} = \mathbf{E_{r0}}(\hat{\mathbf{x}} \cos\theta_i + \hat{\mathbf{z}} \sin\theta_i) e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{H_r} = -\mathbf{y} \frac{E_{r0}}{\eta_1} e^{-j\mathbf{k}\cdot\mathbf{r}}$$

- Note that $\mathbf{k_{i,r}} \cdot \mathbf{r} = x \sin\theta_{i,r} \pm z \cos\theta_{i,r}$

# Tangential Boundary Conditions

- Since the sum of the reflected and incident wave must have zero tangential component at the interface

$$E_{i0}\cos\theta_i e^{-jk_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-jk_1 x\sin\theta_r} = 0$$

- These equations must hold for all $\theta$. Thus $E_{r0} = -E_{i0}$ as before
- Thus we see that these equations can hold for all values of $x$ if and only if $\theta_i = \theta_r$

# The Total Field (Again)

- Using similar arguments as before, when we sum the fields to obtain the total field, we can observe a standing wave in the $\hat{\mathbf{z}}$ direction and wave propagation in the $\hat{\mathbf{x}}$ direction.
- Note that the magnetic field $\mathbf{H} = H\hat{\mathbf{y}}$ is always perpendicular to the direction of propagation but the electric field has a component in the $\hat{\mathbf{x}}$ direction.
- This type of wave is known as a $TM$ wave, or "transverse magnetic" wave