EECS 117

Lecture 22: Poynting's Theorem and Normal Incidence

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EM Power Carried by Plane Wave

- In a lossless medium, we have found that

$$E_x = E_0 \cos(\omega t - \beta z)$$

$$H_y = \frac{E_0}{\eta_0} \cos(\omega t - \beta z)$$

- where $\beta = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\mu/\epsilon}$
- The Poynting vector $\mathbf{S}$ is easily calculated

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{\mathbf{z}} \frac{E_0^2}{\eta_0} \cos^2(\omega t - \beta z)$$

$$\mathbf{S} = \hat{\mathbf{z}} \frac{E_0^2}{2\eta_0} \left(1 + \cos\left(2(\omega t - \beta z)\right)\right)$$

Average Power of Plane Wave

- If we average the Poynting vector over time, the magnitude is

$$S_{av} = \frac{E_0^2}{2\eta_0}$$

- This simple equation is very useful for estimating the electric field strength of a EM wave far from its source (where it can be approximated as a plane wave)
- The energy stored in the electric and magnetic fields are

$$w_e = \frac{1}{2}\epsilon|E_x|^2 = \frac{1}{2}\epsilon E_0^2 \cos^2(\omega t - \beta z)$$

$$w_m = \frac{1}{2}\mu|H_y|^2 = \frac{1}{2}\mu \frac{E_0^2}{\eta_0^2} \cos^2(\omega t - \beta z)$$

Plane Wave "Resonance"

- It's now clear that

$$w_m = \frac{1}{2}\mu\frac{E_0^2}{\frac{\mu}{\epsilon}}\cos^2(\omega t - \beta z) = w_e$$

- In other words, the stored magnetic energy is equal to the stored electric energy. In analogy with a LC circuit, we say that the wave is in resonance
- We can also show that

$$\frac{\partial}{\partial t}\int_V (w_m + w_e)dV = -\oint_S \mathbf{S}\cdot d\mathbf{S}$$

Example: Cell Phone Basestation

- A cell phone base station transmits $10\mathrm{kW}$ of power. Estimate the average electric field at a distance of $1\mathrm{m}$ from the antenna.
- Assuming that the medium around the antenna is lossless, the energy transmitted by the source at any given location from the source must be given by

$$P_t = \oint_{\mathrm{Surf}} \mathbf{S} \cdot d\mathbf{S}$$

- where Surf is a surface covering the source of radiation.
- Since we do not know the antenna radiation pattern, let's assume an isotropic source (equal radiation in all directions)

Example Cont.

- In that case, the average Poynting vector at a distance r from the source is given by

$$S = \frac{P_t}{4\pi r^2} = \frac{10^4}{4\pi}\frac{\mathrm{W}}{\mathrm{m}^2}$$

- This equation is simply derived by observing that the surface area of a sphere of radius r is given by $4\pi r^2$
- Using $S = \frac{1}{2}\frac{E_0^2}{\eta_0}$, we have

$$E_0 = \sqrt{2\eta_0 S} = \sqrt{2 \times 377 \times \frac{10^4}{4\pi}} = 775\frac{\mathrm{V}}{\mathrm{m}}$$

Example: Cell Phone Handset

- A cell phone handset transmits $1\mathrm{W}$ of power. What is the average electric field at a distance of $10\mathrm{cm}$ from the handset?

$$S = \frac{P_t}{4\pi r^2} = \frac{1}{4\pi .1^2}\frac{\mathrm{W}}{\mathrm{m}^2} = 77\frac{\mathrm{W}}{\mathrm{m}^2}$$

- We can see that the electric field near a handset is at a much lower level.
- What's a safe level?

Complex Poynting Theorem

- Last lecture we derived the Poynting Theorem for general electric/magnetic fields. In this lecture we'd like to derive the Poynting Theorem for time-harmonic fields.
- We can't simply take our results from last lecture and simply transform $\frac{\partial}{\partial t} \rightarrow j\omega$. This is because the Poynting vector is a non-linear function of the fields.
- Let's start from the beginning

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J} = (j\omega\epsilon + \sigma)\mathbf{E}$$

Complex Poynting Theorem (II)

- Using our knowledge of circuit theory, $P = V \times I^*$, we compute the following quantity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^*$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot (-j\omega \mathbf{B}) - \mathbf{E} \cdot (j\omega \mathbf{D}^* + \mathbf{J}^*)$$

- Applying the Divergence Theorem

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) dV = \oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S}$$

$$\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = -\int_V \mathbf{E} \cdot \mathbf{J}^* dV + \int_V j\omega (\mathbf{E} \cdot \mathbf{D}^* - \mathbf{H}^* \cdot \mathbf{B}) dV$$

Complex Poynting Theorem (III)

- Let's define $\sigma_{\text{eff}} = \omega\epsilon'' + \sigma$, and $\epsilon = \epsilon'$. Since most materials are non-magnetic, we can ignore magnetic losses

$$\int_S (\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{S} = -\int_V \sigma\mathbf{E}\cdot\mathbf{D}^* dV - j\omega\int_V (\mu\mathbf{H}^*\cdot\mathbf{H} - \epsilon\mathbf{E}\cdot\mathbf{E}^*)\, dV$$

- Notice that the first volume integral is a real number whereas the second volume integral is imaginary

$$\Re\left(\oint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{dS}\right) = -2\int_V P_c dV$$

$$\Im\left(\oint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{dS}\right) = -4\omega\int_V (w_m - w_e) dV$$

Complex Poynting Vector

- Let's compute the average vector $\mathbf{S}$

$$\mathbf{S} = \Re\left(\mathbf{E}e^{j\omega t}\right) \times \Re\left(\mathbf{H}e^{j\omega t}\right)$$

- First observe that $\Re(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$, so that

$$\Re(\mathbf{G}) \times \Re(\mathbf{F}) = \frac{1}{2}(\mathbf{G} + \mathbf{G}^*) \times \frac{1}{2}(\mathbf{F} + \mathbf{F}^*)$$

$$= \frac{1}{4}(\mathbf{G} \times \mathbf{F} + \mathbf{G} \times \mathbf{F}^* + \mathbf{G}^* \times \mathbf{F} + \mathbf{G}^* \times \mathbf{F}^*)$$

$$= \frac{1}{4}\left[(\mathbf{G} \times \mathbf{F}^* + \mathbf{G}^* \times \mathbf{F}) + (\mathbf{G} \times \mathbf{F} + \mathbf{G}^* \times \mathbf{F}^*)\right]$$

$$= \frac{1}{2}\Re\left(\mathbf{G} \times \mathbf{F}^* + \mathbf{G} \times \mathbf{F}\right)$$

Average Complex Poynting Vector

- Finally, we have computed the complex Poynting vector with the time dependence

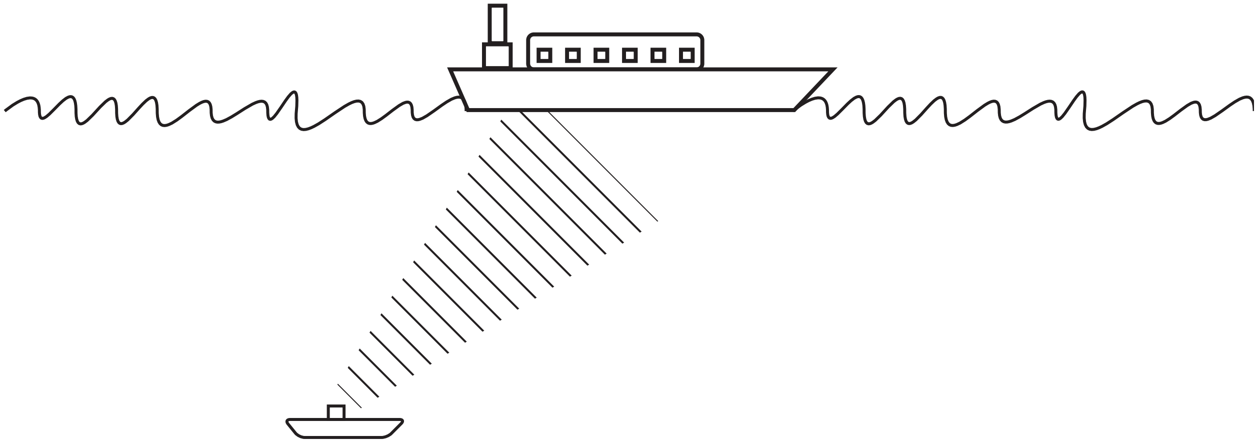
$$\mathbf{S} = \frac{1}{2}\Re\left(\mathbf{E}\times\mathbf{H}^* + \mathbf{E}\times\mathbf{H}e^{2j\omega t}\right)$$

- Taking the average value, the complex exponential vanishes, so that

$$\mathbf{S_{av}} = \frac{1}{2}\Re\left(\mathbf{E}\times\mathbf{H}^*\right)$$

- We have thus justified that the quantity $\mathbf{S} = \mathbf{E}\times\mathbf{H}^*$ represents the complex power stored in the field.

Example: Submarine Communication



- Consider a submarine at a depth of $z = 100\ \mathrm{m}$. We would like to communicate with this submarine using a VLF $f = 3\ \mathrm{kHz}$. The conductivity of sea water is $\sigma = 4\ \mathrm{Sm}^{-1}$, $\epsilon_r = 81$, and $\mu \approx 1$.

Ocean Water Conductivity

- Note that we are forced to use low frequencies due to the conductivity of the ocean water. The loss conductive tangent

$$\tan \delta_c = \frac{\sigma}{\omega\epsilon} \sim 10^5 \gg 1$$

- Thus the ocean is a *good* conductor even at $3\ \mathrm{kHz}$
- The propagation loss and constant are thus equal

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \approx 0.2$$

Ocean Water Wave Propagation

- The wavelength in seawater is much smaller than in air ($\lambda_0 = 100 \text{ km}$ in air)

$$\lambda = \frac{2\pi}{\beta} = 29 \text{ m}$$

- Thus the phase velocity of the wave is also much smaller

$$v_p = f\lambda \approx 9 \times 10^4 \text{ m/s}$$

- The skin-depth, or the depth at which the wave is attenuated to about 37 % of its value, is given by

$$\delta = \frac{1}{\alpha} = 4.6 \text{ m}$$

Ocean Water Fields

- The wave impedance is complex with a phase of 45°

$$|\eta_c| = \sqrt{\frac{\mu\omega}{\sigma}} \approx 8 \times 10^{-2}\ \Omega$$

$$\angle\eta_c = e^{j45^\circ}$$

- Notice that $\eta_c \ll \eta_0$, the ocean water thus generates a very large magnetic field for wave propagation

$$H = \frac{E_0}{\eta_c} e^{-\alpha z} \cos(6\pi \times 10^3 t - \beta z - \phi_\eta)$$

- Where ϕ_η is the angle of the complex wave impedance, 45° in this particular case

Ship to Submarine Communication

- Now let's compute the required transmission power if the receiver at the depth of $z = 100\ \mathrm{m}$ is capable of receiving a signal of at least $1\ \mu\mathrm{V/m}$
- Side-note: the receiver sensitivity is set by the noise power at the input of the receiver. If the signal is too small, it's swamped by the noise.

$$E_0 e^{-\alpha z} \geq E_{\min} = 1\ \mu\mathrm{V/m}$$

- This requires $E_0 = 2.8\ \mathrm{kV/m}$, and a corresponding magnetic field of $H_0 = E_0/\eta_c = 37\ \mathrm{kA/m}$

Poynting Vector in Ocean Water

- This is a very large amount of power to generate at the source. The power density at the source is

$$S_{av} = \frac{1}{2}\Re(\mathbf{E} \times \mathbf{H}^*)$$

$$S_{av} = \frac{1}{2}(2.84 \times 37\cos(45^\circ)) = 37\ \mathrm{MW/m^2}$$

- At a depth of $100\ \mathrm{m}$, the power density drops to extremely small levels

$$S_{av}(100\mathrm{m}) = 4.6 \times 10^{-12}\ \mathrm{MW/m^2}$$

Reflections from a Perfect Conductor

- Consider a plane wave incident *normally* onto a conducting surface

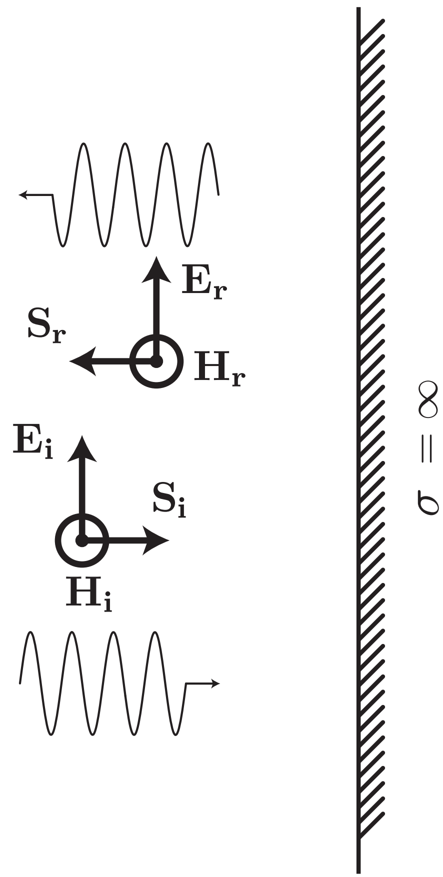
$$E_i = \hat{\mathbf{x}} E_{i0} e^{-j\beta_1 z}$$

$$H_i = \hat{\mathbf{y}} \frac{E_{i0}}{\eta_0} e^{-j\beta_1 z}$$

- The reflected wave (if any) has the following form

$$E_r = \hat{\mathbf{x}} E_{r0} e^{j\beta_1 z}$$

$$H_r = -\hat{\mathbf{y}} \frac{E_{r0}}{\eta_0} e^{j\beta_1 z}$$



Boundary Conditions at Interface

- The conductor forces the tangential electric field to vanish at the surface $z = 0$

$$\mathbf{E}(z = 0) = 0 = \hat{\mathbf{x}}(E_{i0} + E_{r0})$$

- This implies that the reflected wave has equal and opposite magnitude and phase

$$E_{r0} = -E_{i0}$$

- This is similar to wave reflection from a transmission line short-circuit load.

Total Field

- We can now write the total electric and magnetic field in region 1

$$\mathbf{E}(z) = \hat{\mathbf{x}} E_{i0}(e^{-j\beta_1 z} - e^{j\beta_1 z}) = -\hat{\mathbf{x}} E_{i0} j 2 \sin(\beta_1 z)$$

$$\mathbf{H}(z) = \hat{\mathbf{y}} \frac{E_{i0}}{\eta_0}(e^{-j\beta_1 z} + e^{j\beta_1 z}) = \hat{\mathbf{y}} \frac{E_{i0}}{\eta_0} 2 \cos(\beta_1 z)$$

- The net complex power carried by the wave

$$\mathbf{E} \times \mathbf{H}^* = -\hat{\mathbf{z}} \frac{E_{i0}^2}{\eta_0} 4j \sin(\beta_1 z) \cos(\beta_1 z)$$

- is reactive. That means that the average power is zero

$$\mathbf{S_{av}} = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{H}^*) = 0$$

Standing Wave

- The reflected wave interferes with the incident wave to create a standing wave

$$E(z,t) = \Im(E(z)e^{j\omega t}) = \Im(E_{i0} j 2 \sin(\beta_1 z) e^{j\omega t})$$

$$E(z,t) = 2E_{i0} \sin(\beta_1 z) \cos(\omega t)$$

$$H(z,t) = \frac{2E_{i0}}{\eta_1} \cos(\beta_1 z) \sin(\omega t)$$

- Note that the E and H fields are in time quadrature (90° phase difference)
- The instantaneous power is given by

$$S = \frac{4E_{i0}^2}{\eta_1} \underbrace{\sin(\beta_1 z)\cos(\beta_1 z)}_{2\sin(2\beta_1 z)} \underbrace{\cos(\omega t)\sin(\omega t)}_{2\sin(2\omega t)}$$

Standing Wave Power

- The electric and magnetic powers are readily calculated

$$w_e = \frac{1}{2}\epsilon_1 |E_1|^2 = 2\epsilon_1 |E_{i0}|^2 \sin^2(\beta_1 z)\cos^2(\omega t)$$

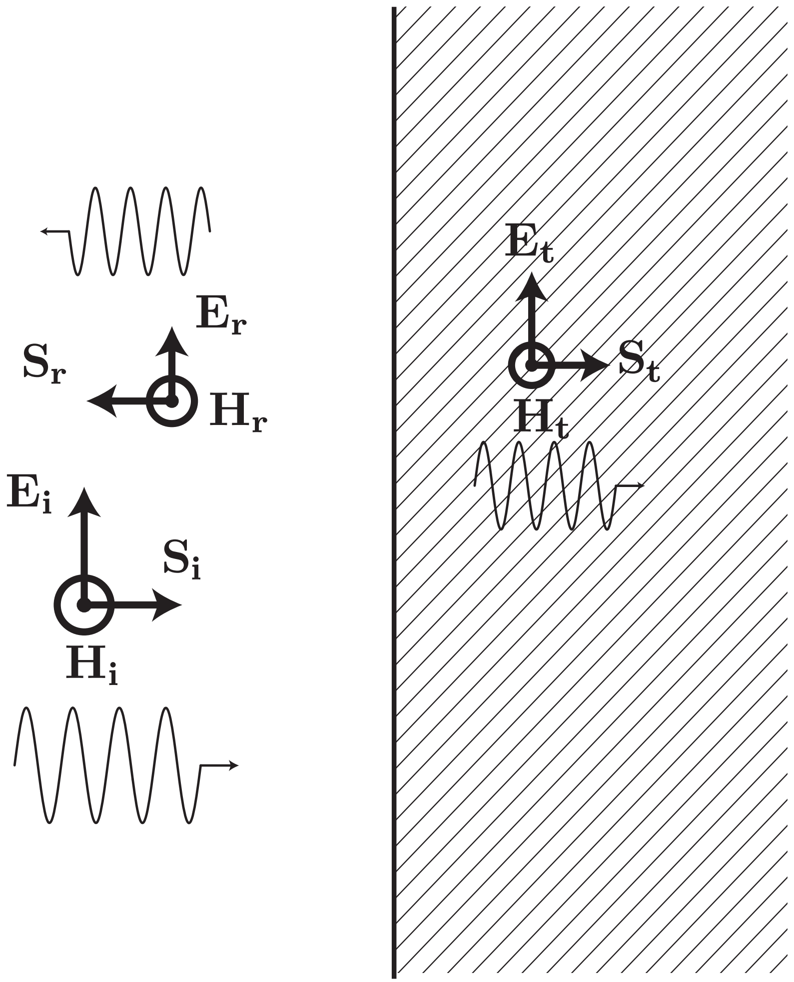
$$w_m = \frac{1}{2}\mu_1 |H_1|^2 = 2\epsilon_1 |E_{i0}|^2 \cos^2(\beta_1 z)\sin^2(\omega t)$$

- Note that the magnetic field at the boundary of the conductor is supported (or equivalently *induces*) a surface current

$$J_s = \hat{\mathbf{n}}\times\mathbf{H} = -\hat{\mathbf{x}}\frac{2E_{i0}}{\eta_1}\ \mathrm{A/m}$$

- If the material is a good conductor, but lossy, then this causes power loss at the conductor surface.

Normal Incidence on a Dielectric



- Consider an incident wave onto a dielectric region.

Normal Incidence on a Dielectric

- We have the incident and possibly reflected waves

$$\mathbf{E_i} = \hat{\mathbf{x}} E_{i0} e^{-j\beta_1 z} \qquad \mathbf{E_r} = \hat{\mathbf{x}} E_{r0} e^{j\beta_1 z}$$

$$\mathbf{H_i} = \hat{\mathbf{y}} \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z} \qquad \mathbf{H_r} = -\hat{\mathbf{y}} \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}$$

- But we must also allow the possibility of a transmitted wave into region 2

$$E_t = \hat{\mathbf{x}} E_{t0} e^{-j\beta_2 z}$$

$$H_t = \hat{\mathbf{y}} \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$

Dielectric Boundary Conditions

- At the interface of the two dielectrics, assuming no interface charge, we have

$$E_{t1} = E_{t2}$$

$$E_{i0} + E_{r0} = E_{t0}$$

$$H_{t1} = H_{t2}$$

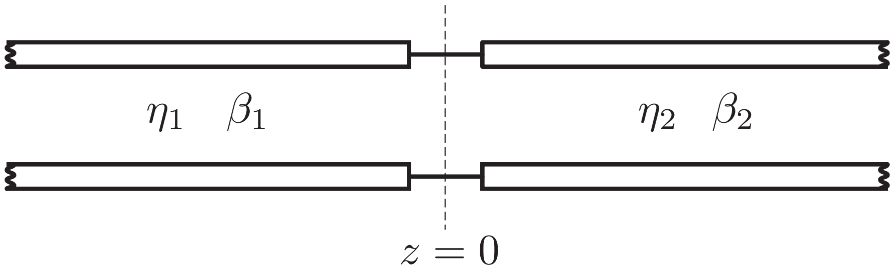
$$\frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

- We have met these equations before. The solution is

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0}$$

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}$$

Transmission Line Analogy



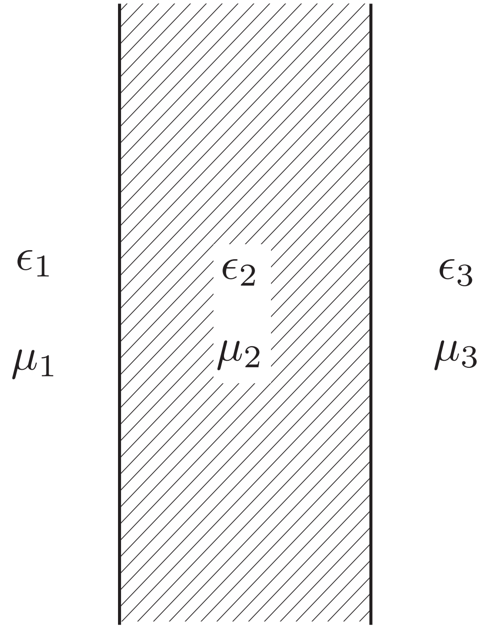
- These equations are identical to the case of the interface of two transmission lines
- The reflection and transmission coefficients are thus identical

$$\rho = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\tau = \frac{E_{t0}}{E_{i0}} = 1 + \rho$$

Harder Example

- Consider three dielectric materials. Instead of solving the problem the *long* way, let's use the transmission line analogy.
- First solve the problem at the interface of region 2 and 3. Region 3 acts like a load to region 2. Now transform this load impedance by the length of region 2 to present an equivalent load to region 1.



Glass Coating

- A very practical example is the case of minimizing reflections for eyeglasses. Due to the impedance mismatch, light normally reflects at the interface of air and glass. One method to reduce this reflection is to coat the glass with a material to eliminate the reflections.
- From our transmission line analogy, we know that this coating is acting like a quarter wave transmission line with

$$\eta = \sqrt{\eta_0 \ \eta_{\mathrm{glass}}}$$