EECS 117

Lecture 21: Wave Propagation in Lossy Media and Poynting's Theorem

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Time-Harmonic Wave Equation

- Start by taking the curl of Faraday's Eq.

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\nabla \times \mathbf{B}$$

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\sigma\mathbf{E} + j\omega\epsilon\mathbf{E})$$

- In a source free region, $\nabla \cdot \mathbf{E} = 0$, and thus

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$$

- We thus have Helmholtz' equation

$$\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0$$

- Where $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = \alpha + j\beta$

Lossy Materials

- In addition to conductive losses σ, materials can also have dielectric and magnetic losses
- A lossy dielectric is characterized by a complex permittivity $\epsilon = \epsilon_r + j\epsilon_i$, where ϵ_i arises due to the phase lag between the field and the polarization. Likewise $\mu = \mu_r + j\mu_i$.
- Most materials we study are weakly magnetic and thus $\mu \approx \mu_r$.
- For now assume that ϵ,μ, and σ are real scalar quantities
- Thus

$$\gamma = \sqrt{(-\omega^2 \epsilon\mu)(1 + \frac{\sigma}{j\omega\epsilon})}$$

Propagation Constant and Loss

- Let's compute the real and imaginary part of γ

$$\gamma = j\omega\sqrt{\epsilon\mu}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{\frac{1}{2}}$$

- Consider $(1 - jh) = re^{-j\theta}$, so that

$$y = \sqrt{1 - jh} = \sqrt{r}e^{-j\theta/2}$$

- Note that $\tan\theta = -h$, and $r = \sqrt{1 + h^2}$. Finally

$$\cos\frac{\theta}{2} = \sqrt{\frac{1 + \cos\theta}{2}} = \sqrt{\frac{1 + \frac{1}{r}}{2}} = \sqrt{\frac{r + 1}{2r}}$$

Propagation Constant and Loss (cont)

- Similarly

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{r-1}{2r}}$$

$$y = \sqrt{r}e^{-j\theta/2} = \sqrt{\frac{r+1}{2}} - j\sqrt{\frac{r-1}{2}} = a + jb$$

- Using the above manipulations, we can now break γ into real and imaginary components

$$\gamma = j\omega\sqrt{\mu\epsilon}(a+jb) = -\omega\sqrt{\mu\epsilon}b + j\omega\sqrt{\mu\epsilon}a = \alpha + j\beta$$

$$\alpha = -\omega\sqrt{\mu\epsilon}\left(-\frac{\sqrt{r-1}}{\sqrt{2}}\right)$$

Propagation Constant and Loss (final)

- We have now finally shown that

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}$$

- It's easy to show that the imaginary part of ϵ can be lumped into an effective conductivity term
- In practice, most materials are either *low loss*, such that $\frac{\sigma_{\text{eff}}}{\omega\epsilon} \ll 1$, or *good conductors*, such that $\frac{\sigma_{\text{eff}}}{\omega\epsilon} \gg 1$
- In these extreme cases, simplified versions of the above equations are applicable

Effective Dielectric Constant

- We can also lump the conductivity into an effective dielectric constant

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + j\omega\epsilon \mathbf{E} = j\omega\epsilon_{\text{eff}} \mathbf{E}$$

  where $\epsilon_{\text{eff}} = \epsilon - j\sigma/\omega$. In the *low loss* case, this is a good way to include the losses

- When ϵ or μ become complex, the wave impedance is no longer real and the electric and magnetic field fall out of phase. Since $H = E/\eta_c$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_{\text{eff}}}} = \sqrt{\frac{\mu}{\epsilon - j\sigma/\omega}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}$$

Propagation in Low Loss Materials

- If $\frac{\sigma}{\omega\epsilon} \ll 1$, then our equations simplify

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right)$$

- To first order, the propagation constant is unchanged by the losses ($\sigma_{\text{eff}} = \sigma + \omega\epsilon''$)

$$\beta = \omega\sqrt{\mu\epsilon} \qquad \alpha = \frac{1}{2}\sigma_{\text{eff}}\sqrt{\frac{\mu}{\epsilon}}$$

- A more accurate expression can be obtained with a 1st order expansion of $(1+x)^n$

$$\beta = \omega\sqrt{\mu\epsilon}\left(1 + \frac{1}{8}\left(\frac{\sigma_{\text{eff}}}{\omega\epsilon'}\right)^2\right)$$

Propagation in Conductors

- As we saw in the previous lecture, this approximation is valid when $\frac{\sigma}{\omega\epsilon} \gg 1$

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu\sigma} = \omega\mu\sigma e^{j45^\circ}$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

- The phase velocity is given by $v_p = \omega/\beta$

$$v_p = \sqrt{\frac{2\omega}{\mu\sigma}}$$

- This is a function of frequency! This is a very dispersive medium.

Waves in Conductors

- The wavelength is given by

$$\lambda = \frac{v_p}{f} = 2\sqrt{\frac{\pi}{f\mu\sigma}}$$

- Example: Take $\sigma = 10^7\ \mathrm{S/m}$ and $f = 100\ \mathrm{MHz}$. Using the above equations

$$\lambda = 10^{-4}\ \mathrm{m}$$

$$v_p = 10^4\ \mathrm{m/s}$$

- Note that $\lambda_0 = 3\ \mathrm{m}$ in free-space, and thus the wave is very much smaller and much slower moving in the conductor

Energy Storage and Loss in Fields

- We have learned that the power density of electric and magnetic fields is given by

$$w_m = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} = \frac{1}{2}\epsilon E^2$$

$$w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = \frac{1}{2}\mu H^2$$

- Also, the power loss per unit volume due to Joule heating in a conductor is given by

$$p_{\text{loss}} = \mathbf{E}\cdot\mathbf{J}$$

- Using $\mathbf{J} = \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$, this can be expressed as

$$\mathbf{E}\cdot\mathbf{J} = \mathbf{E}\cdot\nabla\times\mathbf{H} - \frac{\partial}{\partial t}(\nabla\times\mathbf{D})$$

Poynting Vector

- We will demonstrate that the Poynting vector $\mathbf{E} \times \mathbf{H}$ plays an important role in the energy of an EM field.

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

$$\mathbf{E} \cdot \mathbf{J} = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{J}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

$$= \mathbf{H} \cdot (-\frac{\partial \mathbf{B}}{\partial t}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \left(\frac{\partial \mu \mathbf{H}}{\partial t} \right) = \frac{1}{2} \frac{\partial \mu \mathbf{H} \cdot \mathbf{H}}{\partial t} = \frac{1}{2} \frac{\partial \mu |\mathbf{H}|^2}{\partial t}$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \left(\frac{\partial \epsilon \mathbf{E}}{\partial t} \right) = \frac{1}{2} \frac{\partial \epsilon \mathbf{E} \cdot \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial \mu |\mathbf{E}|^2}{\partial t}$$

Poynting's Theorem

- Collecting terms we have shown that

$$\mathbf{E}\cdot\mathbf{J} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu|\mathbf{H}|^2\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon|\mathbf{E}|^2\right) - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

- Applying the Divergence Theorem

$$\int_V \mathbf{E}\cdot\mathbf{J}dV = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu|\mathbf{H}|^2 + \frac{1}{2}\epsilon|\mathbf{E}|^2\right) dV - \int_S \mathbf{E}\times\mathbf{H}dV$$

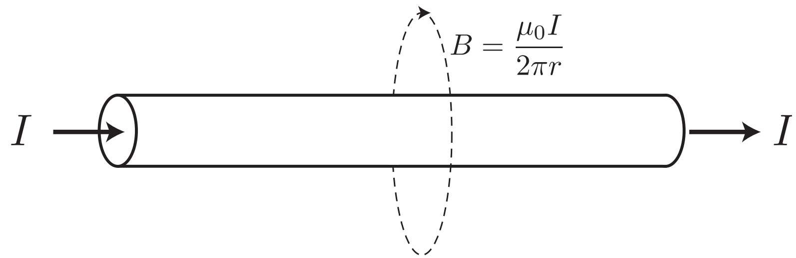
- The above equation can be re-stated as

power dissipated in volume V (heat)	$=$	rate of change of energy storage in volume V	$-$	a surface integral over the volume of $\mathbf{E}\times\mathbf{H}$

Interpretation of the Poynting Vector

- We now have a physical interpretation of the last term in the above equation. By the conservation of energy, it must be equal to the energy flow into or out of the volume
- We may be so bold, then, to interpret the vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ as the energy flow density of the field
- While this seems reasonable, it's important to note that the physical meaning is only attached to the integral of $\mathbf{S}$ and not to discrete points in space

Current Carrying Wire



- Consider the above wire carrying a uniform current I
- From circuit theory we know that the power loss in the wire is simply $I^2 R$. This is easily confirmed

$$P_L = \int_V \mathbf{E} \cdot \mathbf{J} dV = \int_V \frac{1}{\sigma} \mathbf{J} \cdot \mathbf{J} dV = \frac{1}{A^2 \sigma} int_V I^2 dV$$

$$P_L = \frac{A \cdot \ell}{A^2 \sigma} I^2 = \frac{\ell}{A \sigma} I^2$$

Energy Stored around a Wire Section

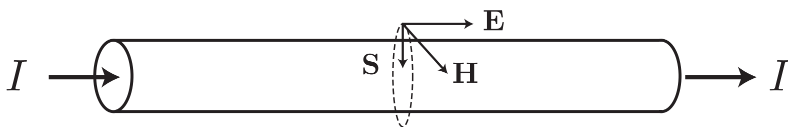
- Let's now apply Poynting's Theorem. Since the current is dc, we can neglect all time variation $\frac{\partial}{\partial t} = 0$ and thus the energy storage of the system is fixed in time.
- The magnetic field around the wire is simply given by

$$\mathbf{H} = \hat{\phi}\frac{I}{2\pi r}$$

- The electric field is proportional to the current density. At the surface of the wire

$$\mathbf{E} = \frac{1}{\sigma}\mathbf{J} = \frac{I}{\sigma A}\hat{\mathbf{z}}$$

Power Loss in Wire



- The Poynting vector at the surface thus points into the conductor

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{I}{\sigma A}\hat{\mathbf{z}} \times \hat{\phi}\frac{I}{2\pi r} = \frac{-\hat{\mathbf{r}} I^2}{2\pi r \sigma A}$$

- The power flow into the wire is thus given by

$$\int_s \mathbf{S} \cdot ds = \int_0^\ell \int_0^{2\pi} \frac{I^2}{2\pi r \sigma A} r d\theta dz = I^2 R$$

- This result confirms that the energy flowing into the wire from the field is heating up the wire.

Sources and Fields

- This result is surprising because it hints that the signal in a wire is carried by the fields, and not by the charges.
- In other words, if a signal propagates down a wire, the information is carried by the fields, and the current flow is impressed upon the conductor from the fields.
- We know that the sources of EM fields are charges and currents. But we also know that if the configuration of charges changes, the fields “carry” this information