# EECS 117

## *Lecture 20: Plane Waves*

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# Maxwell's Eq. in Source Free Regions

- In a source free region $\rho = 0$ and $\mathbf{J} = 0$

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

- Assume that $\mathbf{E}$ and $\mathbf{H}$ are uniform in the x-y plane so $\frac{\partial}{\partial x} = 0$ and $\frac{\partial}{\partial y} = 0$
- For this case the $\nabla \times \mathbf{E}$ simplifies

# Curl E for Plane Uniform Fields

- Writing out the curl of E in rectangular coordinates

$$\nabla \times \mathbf{E} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{pmatrix}$$

$$(\nabla \times \mathbf{E})_x = -\frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial dt}$$

$$(\nabla \times \mathbf{E})_y = -\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial dt}$$

$$(\nabla \times \mathbf{E})_z = 0 = -\mu \frac{\partial H_z}{\partial dt}$$

# Curl of H for Plane Uniform Fields

- Similarly, writing out the curl H in rectangular coordinates

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$-\frac{\partial H_y}{\partial z} = \epsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = \epsilon \frac{\partial E_y}{\partial t}$$

$$0 = \epsilon \frac{\partial E_z}{\partial t}$$

- Time variation in the $\hat{\mathrm{z}}$ direction is zero. Thus the fields are entirely transverse to the direction of propagation. We call such fields *TEM* “waves”

# Polarized TEM Fields

- For simplicity assume $E_y = 0$. We say the field polarized in the $\hat{\mathbf{x}}$-direction. This implies that $H_x = 0$ and $H_y \neq 0$

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$-\frac{\partial H_y}{\partial z} = \epsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu \frac{\partial^2 H_y}{\partial z \partial t}$$

$$\frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

# One Dimensional Wave Eq.

- We finally have it, a one-dimensional wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

- Notice similarity between this equation and the wave equation we derived for voltages and currents along a transmission line
- As before, the wave velocity is $v = \frac{1}{\sqrt{\mu\epsilon}}$
- The general solution to this equation is

$$E_x(z,t) = f_1(t - \frac{z}{v}) + f_2(t + \frac{z}{v})$$

# Wave Solution

- Let's review why this is the general solution

$$\frac{\partial E_x}{\partial t} = f_1' + f_2' \qquad \frac{\partial E_x}{\partial z} = -\frac{1}{v} f_1' + \frac{1}{v} f_2'$$

$$\frac{\partial^2 E_x}{\partial t^2} = f_1'' + f_2'' \qquad \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2} f_1'' + \frac{1}{v^2} f_2''$$

- A point on the wavefront is defined by $(t - z/v) = c$ where $c$ is a constant. The velocity of this point is therefore $v$

$$1 - \frac{1}{v}\frac{\partial z}{\partial t} = 0$$

$$\frac{\partial z}{\partial t} = v$$

# Wave Velocity

- We have thus shown that the velocity of this wave moves is

$$v = c = \frac{1}{\sqrt{\mu\epsilon}}$$

- In free-space, $c \approx 3 \times 10^8 \mathrm{m/s}$, the measured speed of light
- In a medium with relative permittivity $\epsilon_r$ and relative permeability $\mu_r$, the speed moves with effective velocity

$$v = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

- This fact alone convinced Maxwell that light is an EM wave

# Sinusoidal Plane Waves

- For time-harmonic fields, the equations simplify

$$\frac{dE_x}{dz} = -j\omega\mu H_y$$

$$\frac{dH_y}{dz} = j\omega\epsilon E_x$$

- This gives a one-dimensional Helmholtz equation

$$\frac{d^2 E_x}{d^2 z} = -\omega^2 \mu\epsilon E_x$$

# Solution of Helmholtz' Eq.

- The solution is now a simple exponential

$$E_x = C_1 e^{-jkz} + C_2 e^{jkz}$$

- The wave number is given by $k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v}$
- We can recover a traveling wave solution

$$E_x(z,t) = \Re\left(E_x e^{j\omega t}\right)$$

$$E_x(z,t) = \Re\left(C_1 e^{j(\omega t - kz)} + C_2 e^{j(\omega t + kz)}\right)$$

$$E_x(z,t) = C_1 \cos\left(\omega t - kz\right) + C_2 \cos\left(\omega t + kz\right)$$

- The wave has spatial variation $\lambda = \frac{2\pi}{k} = \frac{2\pi v}{\omega} = \frac{v}{f}$

# Magnetic Field of Plane Wave

- We have the following relation

$$H_y = -\frac{1}{j\omega\mu}\frac{dE_x}{dz} = -\frac{1}{j\omega\mu}\left(-jkC_1 e^{-jkz} + C_2 jk e^{jkz}\right)$$

$$H_y = \frac{k}{\mu\omega}\left(C_1 e^{-jkz} - C_2 jk e^{jkz}\right)$$

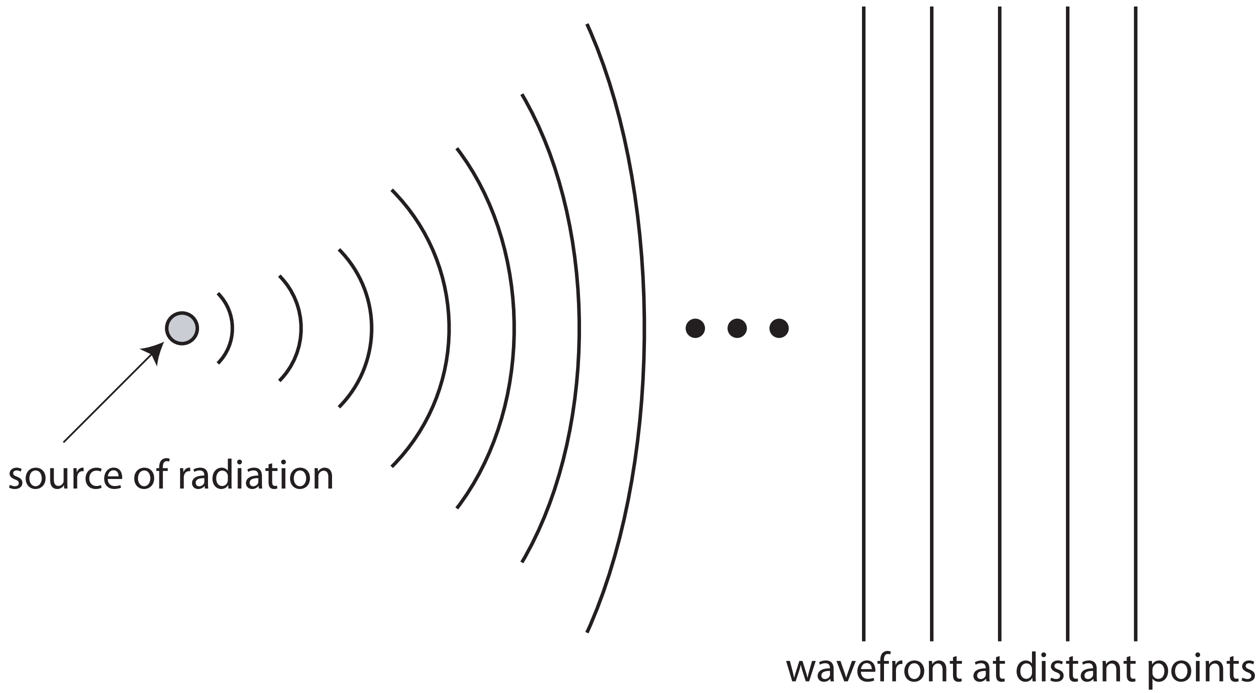
- By definition, $k = \omega\sqrt{\mu\epsilon}$

$$H_y = \sqrt{\frac{\epsilon}{\mu}}\left(C_1 e^{-jkz} - C_2 e^{jkz}\right)$$

- The ratio $E_x^+$ and $H_y^+$ has units of impedance and is given by the constant $\eta = \sqrt{\mu/\epsilon}$. $\eta$ is known as the impedance of free space

# Plane Waves

- Plane waves are the simplest wave solution of Maxwell's Eq. They seem to be a gross oversimplification but they nicely approximate real waves that are distant from their source



# Wave Equation in 3D

- We can derive the wave equation directly in a coordinate free manner using vector analysis

$$\nabla \times \nabla \times \mathbf{E} = \nabla \times -\mu \frac{\partial \mathbf{H}}{\partial \mathbf{t}} = \mu \frac{\partial \left( \nabla \times \mathbf{H} \right)}{\partial t}$$

- Substitution from Maxwell's eq.

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

# Wave Eq. in 3D (cont)

- Using the identity $\nabla \times \nabla \times \mathbf{E} = -\nabla^{\mathbf{2}}\mathbf{E} + \nabla (\nabla \cdot \mathbf{E})$
- Since $\nabla \cdot \mathbf{E} = 0$ in charge free regions

$$\nabla^{\mathbf{2}}\mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- In Phasor form we have $k^2 = \omega^2 \mu\epsilon$

$$\nabla^{\mathbf{2}}\mathbf{E} = -k^2\mathbf{E}$$

- Now it's trivial to get a 1-D version of this equation

$$\nabla^2 E_x = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} \qquad \frac{\partial^2 E_x}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

# Penetration of Waves into Conductors

- Inside a good conductor $\mathbf{J} = \sigma \mathbf{E}$
- In the time-harmonic case, this implies the lack of free charges

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} = (\sigma + j\omega\epsilon)\,\mathbf{E}$$

- Since $\nabla \cdot \nabla \times \mathbf{H} \equiv 0$, we have

$$(\sigma + j\omega\epsilon)\,\nabla \cdot \mathbf{E} \equiv 0$$

- Which in turn implies that $\rho = 0$
- For a good conductor the conductive currents completely outweighs the displacement current, e.g. $\sigma \gg \omega\epsilon$

# Conductive vs. Displacement Current

- To see this, consider a good conductor with $\sigma \sim 10^7 \mathrm{S/m}$ up to very high mm-wave frequencies $f \sim 100\mathrm{GHz}$
- The displacement current is still only

$$\omega\epsilon \sim 10^{11} 10^{-11} \sim 1$$

- This is seven orders of magnitude smaller than the conductive current
- For all practical purposes, therefore, we drop the displacement current in the volume of good conductors

# Wave Equation inside Conductors

- Inside of good conductors, therefore, we have

$$\nabla \times \mathbf{H} = \sigma \mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla \left( \nabla \cdot \mathbf{E} \right) - \nabla^{\mathbf{2}} \mathbf{E} = -\nabla^{\mathbf{2}} \mathbf{E} = -j\omega\mu \nabla \times \mathbf{H}$$

$$\nabla^{\mathbf{2}} \mathbf{E} = j\omega\mu\sigma \mathbf{E}$$

- One can immediately conclude that $\mathbf{J}$ satisfies the same equation

$$\nabla^{\mathbf{2}} \mathbf{J} = j\omega\mu\sigma \mathbf{J}$$

- Applying the same logic to $\mathbf{H}$, we have

$$\nabla \times \nabla \times \mathbf{H} = \nabla \left( \nabla \cdot \mathbf{H} \right) - \nabla^{\mathbf{2}} \mathbf{H} = -\nabla^{\mathbf{2}} \mathbf{H} = (j\omega\epsilon + \sigma) \nabla \times \mathbf{E}$$

$$\nabla^{\mathbf{2}} \mathbf{H} = j\omega\mu\sigma \mathbf{H}$$

# Plane Waves in Conductors

- Let's solve the 1D Helmholtz equation once again for the conductor

$$\frac{d^2 E_z}{d^2 x} = j\omega\mu\sigma E_z = \tau^2 E_z$$

- We define $\tau^2 = j\omega\mu\sigma$ so that

$$\tau = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

- Or more simply, $\tau = (1+j)\sqrt{\pi f \mu\sigma} = \frac{1+j}{\delta}$
- The quantity $\delta = \frac{1}{\sqrt{\pi f \mu\sigma}}$ has units of meters and is an important number

# Solution for Fields

- The general solution for the plane wave is given by

$$E_z = C_1 e^{-\tau x} + C_2 e^{\tau x}$$

- Since $E_z$ must remain bounded, $C_2 \equiv 0$

$$E_z = E_0 e^{-\tau x} = \underbrace{E_x e^{-x/\delta}}_{\text{mag}} \underbrace{e^{-jx/\delta}}_{\text{phase}}$$

- Similarly the solution for the magnetic field and current follow the same form

$$H_y = H_0 e^{-x/\delta} e^{-jx/\delta}$$

$$J_z = J_0 e^{-x/\delta} e^{-jx/\delta}$$

# Penetration Depth

- The wave decays exponentially into the conductor. For this reason, $\delta$ is called the *penetration depth*, or more commonly, the *skin depth*. The fields drop to $1/e$ of their values after traveling one skin depth into the conductor. After several skin depths, the fields are essentially zero
- You may also say that the wave exists only on the "skin" of the conductor
- For a good conductor at $f = 1\mathrm{GHz}$

$$\delta = \frac{1}{\sqrt{\mu\sigma f\pi}} \sim 10^{-6}\mathrm{m}$$

- As the frequency is increased, $\delta \to 0$, or the fields completely vanish in the volume of the conductor

# Total Current in Conductor

- Why do fields decay in the volume of conductors?
- The induced fields cancel the incoming fields. As $\sigma \to \infty$, the fields decay to zero inside the conductor.
- The total surface current flowing in the conductor volume is given by

$$J_{sz} = \int_0^\infty J_z dx = \int_0^\infty J_0 e^{-(1+j)x/\delta} dx$$

$$J_{sz} = \frac{J_0 \delta}{1+j}$$

- At the surface of the conductor, $E_{z0} = \frac{J_0}{\sigma}$

# Internal Impedance of Conductors

- Thus we can define a surface impedance

$$Z_s = \frac{E_{z0}}{J_{sz}} = \frac{1+j}{\sigma\delta}$$

$$Z_s = R_s + j\omega L_i$$

- The real part of the impedance is a resistance

$$R_s = \frac{1}{\delta\sigma} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

- The imaginary part is inductive

$$\omega L_i = R_s$$

- So the phase of this impedance is always $\pi/4$

# Interpretation of Surface Impedance

- The resistance term is equivalent to the resistance of a conductor of thickness $\delta$
- The inductance of the surface impedance represents the “internal” inductance for a large plane conductor
- Note that as $\omega \to \infty$, $L_i \to 0$. The fields disappear from the volume of the conductor and the internal impedance is zero
- We commonly apply this surface impedance to conductors of finite width or even coaxial lines. It’s usually a pretty good approximation to make as long as the conductor width and thickness is much larger than $\delta$