# EECS 117

## *Lecture 2: Transmission Line Discontinuities*

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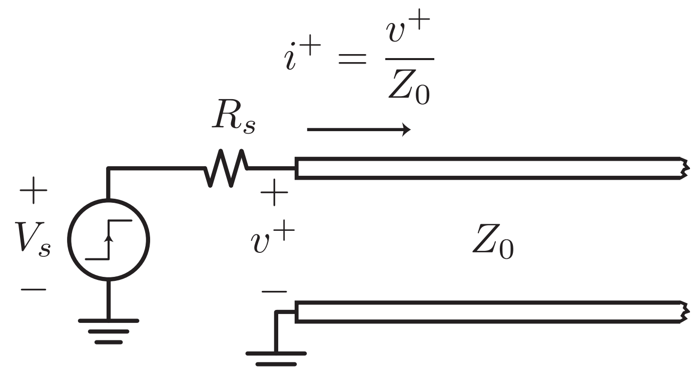
# Energy to "Charge" Transmission Line




- The power flow into the line is given by

$$P^+_{line} = i^+(0,t)v^+(0,t) = \frac{\left(v^+(0,t)\right)^2}{Z_0}$$

- Or in terms of the source voltage

$$P^+_{line} = \left(\frac{Z_0}{Z_0 + R_s}\right)^2 \frac{V_s^2}{Z_0} = \frac{Z_0}{(Z_0 + R_s)^2} V_s^2$$

# Energy Stored in Inds and Caps (I)

- But where is the power going? The line is lossless!
- Energy stored by a cap/ind is $\frac{1}{2}CV^2$/$\frac{1}{2}LI^2$
- At time $t_d$, a length of $\ell = vt_d$ has been "charged":

$$\frac{1}{2}CV^2 = \frac{1}{2}\ell C' \left(\frac{Z_0}{Z_0 + R_s}\right)^2 V_s^2$$

$$\frac{1}{2}LI^2 = \frac{1}{2}\ell L' \left(\frac{V_s}{Z_0 + R_s}\right)^2$$

- The total energy is thus

$$\frac{1}{2}LI^2 + \frac{1}{2}CV^2 = \frac{1}{2}\frac{\ell V_s^2}{(Z_0 + R_s)^2}\left(L' + C'Z_0^2\right)$$

# Energy Stored (II)

- Recall that $Z_0 = \sqrt{L'/C'}$. The total energy stored on the line at time $t_d = \ell/v$:

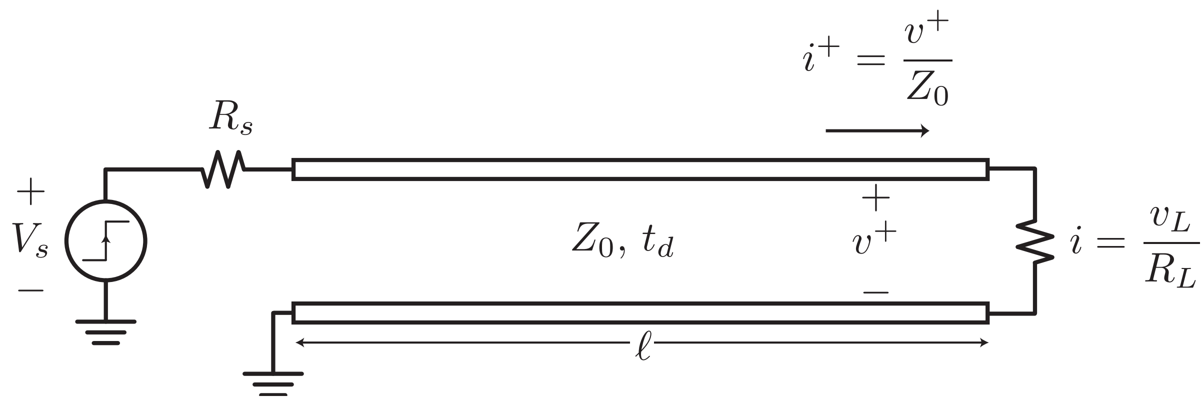
$$E_{line}(\ell/v) = \ell L' \frac{V_s^2}{(Z_0 + R_s)^2}$$

- And the power delivered onto the line in time $t_d$:

$$P_{line} \times \frac{\ell}{v} = \frac{\frac{l}{v} Z_0 V_s^2}{(Z_0 + R_s)^2} = \ell \sqrt{\frac{L'}{C'}} \sqrt{L'C'} \frac{V_s^2}{(Z_0 + R_s)^2}$$

- As expected, the results match (conservation of energy).

# Transmission Line Termination




- Consider a finite transmission line with a termination resistance
- At the load we know that Ohm's law is valid: $I_L = V_L/R_L$
- So at time $t = \ell/v$, our pulse reaches the load. Since the current on the T-line is $i^+ = v^+/Z_0 = V_s/(Z_0 + R_s)$ and the current at the load is $V_L/R_L$, a discontinuity is produced at the load.

# Reflections

- Thus a reflected wave is created at discontinuity

$$V_L(t) = v^+(\ell, t) + v^-(\ell, t)$$

$$I_L(t) = \frac{1}{Z_0} v^+(\ell, t) - \frac{1}{Z_0} v^-(\ell, t) = V_L(t)/R_L$$

- Solving for the forward and reflected waves

$$2v^+(\ell, t) = V_L(t)(1 + Z_0/R_L)$$

$$2v^-(\ell, t) = V_L(t)(1 - Z_0/R_L)$$

# Reflection Coefficient

- And therefore the reflection from the load is given by

$$\Gamma_L = \frac{V^-(\ell,t)}{V^+(\ell,t)} = \frac{R_L - Z_0}{R_L + Z_0}$$

- Reflection coefficient is a very important concept for transmisslin lines: $-1 \leq \Gamma_L \leq 1$
- $\Gamma_L = -1$ for $R_L = 0$ (short)
- $\Gamma_L = +1$ for $R_L = \infty$ (open)
- $\Gamma_L = 0$ for $R_L = Z_0$ (match)
- Impedance match is the proper termination if we don't want any reflections

# Propagation of Reflected Wave (I)

- If $\Gamma_L \neq 0$, a new reflected wave travels toward the source and unless $R_s = Z_0$, another reflection also occurs at source!
- To see this consider the wave arriving at the source. Recall that since the wave PDE is linear, a superposition of any number of solutins is also a solution.
- At the source end the boundary condition is as follows

$$V_s - I_s R_s = v_1^+ + v_1^- + v_2^+$$

- The new term $v_2^+$ is used to satisfy the boundary condition

# Propagation of Reflected Wave (II)

- The current continuity requires $I_s = i_1^+ + i_1^- + i_2^+$

$$V_s = (v_1^+ - v_1^- + v_2^+)\frac{R_s}{Z_0} + v_1^+ + v_1^- + v_2^+$$

- Solve for $v_2^+$ in terms of known terms

$$V_s = \left(1 + \frac{R_s}{Z_0}\right)(v_1^+ + v_2^+) + \left(1 - \frac{R_s}{Z_0}\right)v_1^- +$$

- But $v_1^+ = \frac{Z_0}{R_s+Z_0}V_s$

$$V_s = \frac{R_s + Z_0}{Z_0}\frac{Z_0}{R_s + Z_0}V_s + \left(1 - \frac{R_s}{Z_0}\right)v_1^- + \left(1 + \frac{R_s}{Z_0}\right)v_2^+$$

# Propagation of Reflected Wave (III)

- So the source terms cancel out and

$$v_2^+ = \frac{R_s - Z_0}{Z_0 + R_s} v_1^- = \Gamma_s v_1^-$$

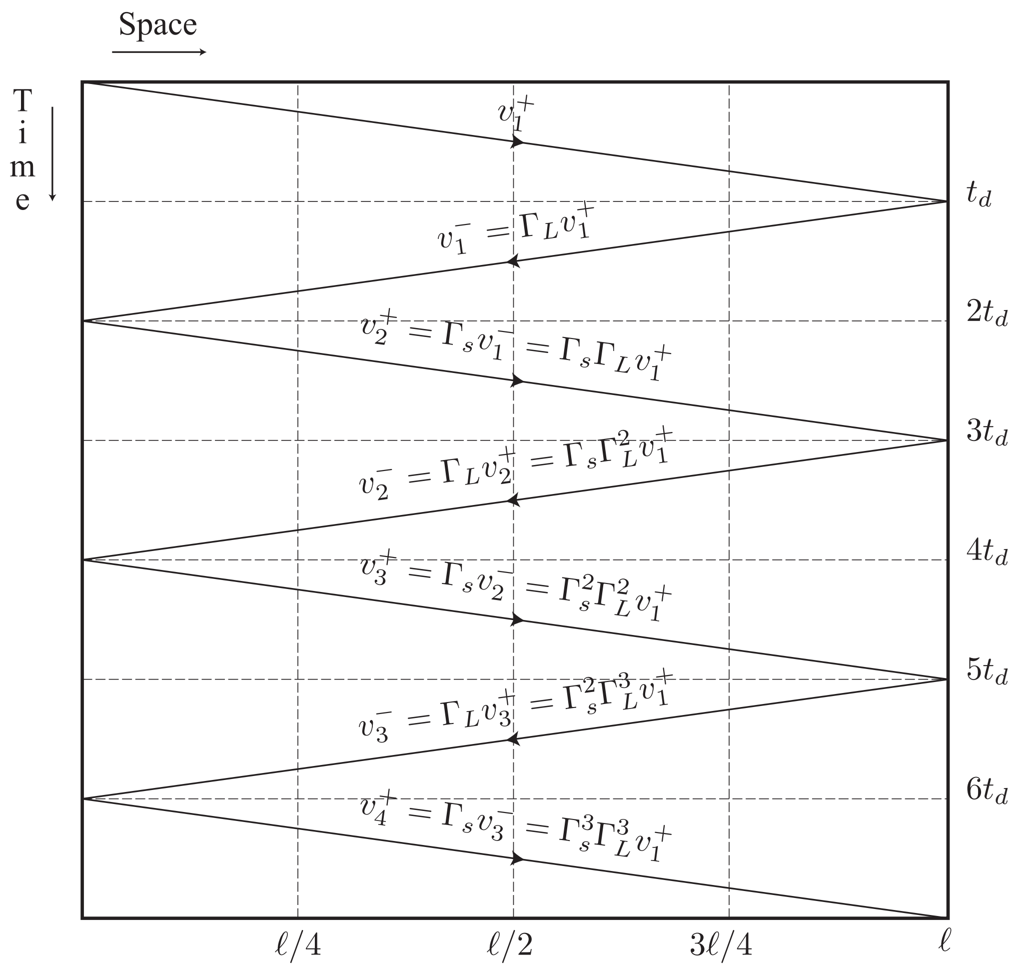
- The reflected wave bounces off the source impedance with a reflection coefficient given by the same equation as before

$$\Gamma(R) = \frac{R - Z_0}{R + Z_0}$$

- The source appears as a short for the incoming wave
- Invoke superposition! The term $v_1^+$ took care of the source boundary condition so our new $v_2^+$ only needed to compensate for the $v_1^-$ wave ... the reflected wave is only a function of $v_1^-$

# Bounce Diagram

- We can track the multiple reflections with a "bounce diagram"

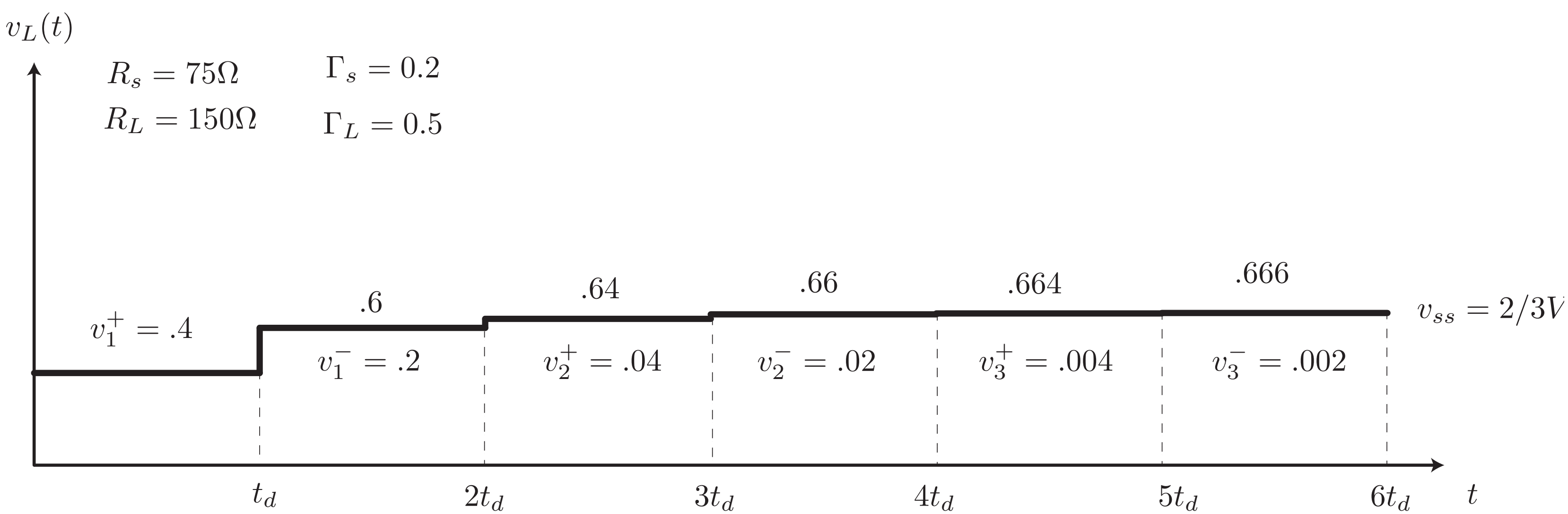


# Freeze time

- If we freeze time and look at the line, using the bounce diagram we can figure out how many reflections have occurred
- For instance, at time $2.5t_d = 2.5\ell/v$ three waves have been excited ($v_1^+$,$v_1^-$, $v_2^+$), but $v_2^+$ has only travelled a distance of $\ell/2$
- To the left of $\ell/2$, the voltage is a summation of three components: $v = v_1^+ + v_1^- + v_2^+ = v_1^+(1 + \Gamma_L + \Gamma_L\Gamma_s)$.
- To the right of $\ell/2$, the voltage has only two components: $v = v_1^+ + v_1^- = v_1^+(1 + \Gamma_L)$.

# Freeze Space

- We can also pick at arbitrary point on the line and plot the evolution of voltage as a function of time
  - For instance, at the load, assuming $R_L > Z_0$ and $R_S > Z_0$, so that $\Gamma_{s,L} > 0$, the voltage at the load will will increase with each new arrival of a reflection



# Steady-State Voltage on Line (I)

- To find steady-state voltage on the line, we sum over all reflected waves:

$$v_{ss} = v_1^+ + v_1^- + v_2^+ + v_2^- + v_3^+ + v_3^- + v_4^+ + v_4^- + \cdots$$

- Or in terms of the first wave on the line

$$v_{ss} = v_1^+(1 + \Gamma_L + \Gamma_L\Gamma_s + \Gamma_L^2\Gamma_s + \Gamma_L^2\Gamma_s^2 + \Gamma_L^3\Gamma_s^2 + \Gamma_L^3\Gamma_s^3 + \cdots$$

- Notice geometric sums of terms like $\Gamma_L^k\Gamma_s^k$ and $\Gamma_L^{k+1}\Gamma_s^k$. Let $x = \Gamma_L\Gamma_s$:

$$v_{ss} = v_1^+(1 + x + x^2 + \cdots + \Gamma_L(1 + x + x^2 + \cdots))$$

# Steady-State Voltage on Line (II)

- The sums converge since $x < 1$

$$v_{ss} = v_1^+ \left( \frac{1}{1 - \Gamma_L \Gamma_s} + \frac{\Gamma_L}{1 - \Gamma_L \Gamma_s} \right)$$

- Or more compactly

$$v_{ss} = v_1^+ \left( \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_s} \right)$$

- Substituting for $\Gamma_L$ and $\Gamma_s$ gives

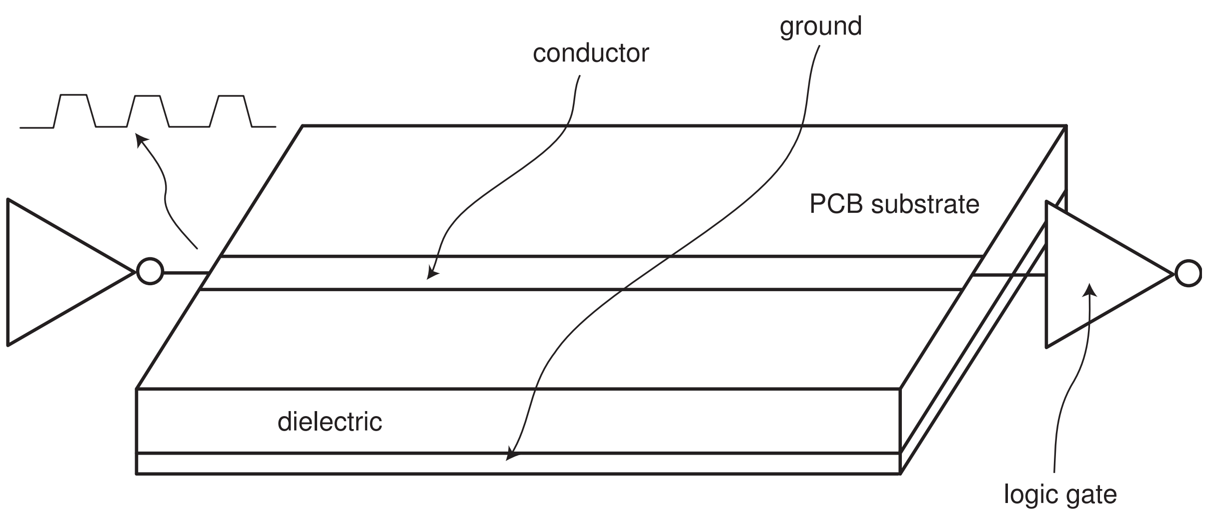
$$v_{ss} = V_s \frac{R_L}{R_L + R_s}$$

# What Happend to the T-Line?

- For steady state, the equivalent circuit shows that the transmission line has disappeared.
- This happens because if we wait long enough, the effects of propagation delay do not matter
- Conversly, if the propagation speed were infinite, then the T-line would not matter
- But the presence of the T-line will be felt if we disconnect the source or load!
- That's because the T-line *stores* reactive energy in the capaciance and inductance
- Every real circuit behaves this way! Circuit theory is an abstraction

# PCB Interconnect

- Suppose $\ell = 3\mathrm{cm}$, $v = 3 \times 10^8 \mathrm{m/s}$, so that $t_p = \ell / v = 10^{-10}\mathrm{s} = 100\mathrm{ps}$
- On a time scale $t < 100\mathrm{ps}$, the voltages on interconnect act like transmission lines!
  - Fast digital circuits need to consider T-line effects



# Example: Open Line (I)

- Source impedance is $Z_0/4$, so $\Gamma_s = -0.6$, load is open so $\Gamma_L = 1$
- As before a positive going wave is launched $v_1^+$
- Upon reaching the load, a reflected wave of of equal amplitude is generated and the load voltage overshoots $v_L = v_1^+ + v_1^- = 1.6\text{V}$
- Note that the current reflection is negative of the voltage

$$\Gamma_i = \frac{i^-}{i^+} = -\frac{v^-}{v^+} = -\Gamma_v$$

- This means that the sum of the currents at load is zero (open)

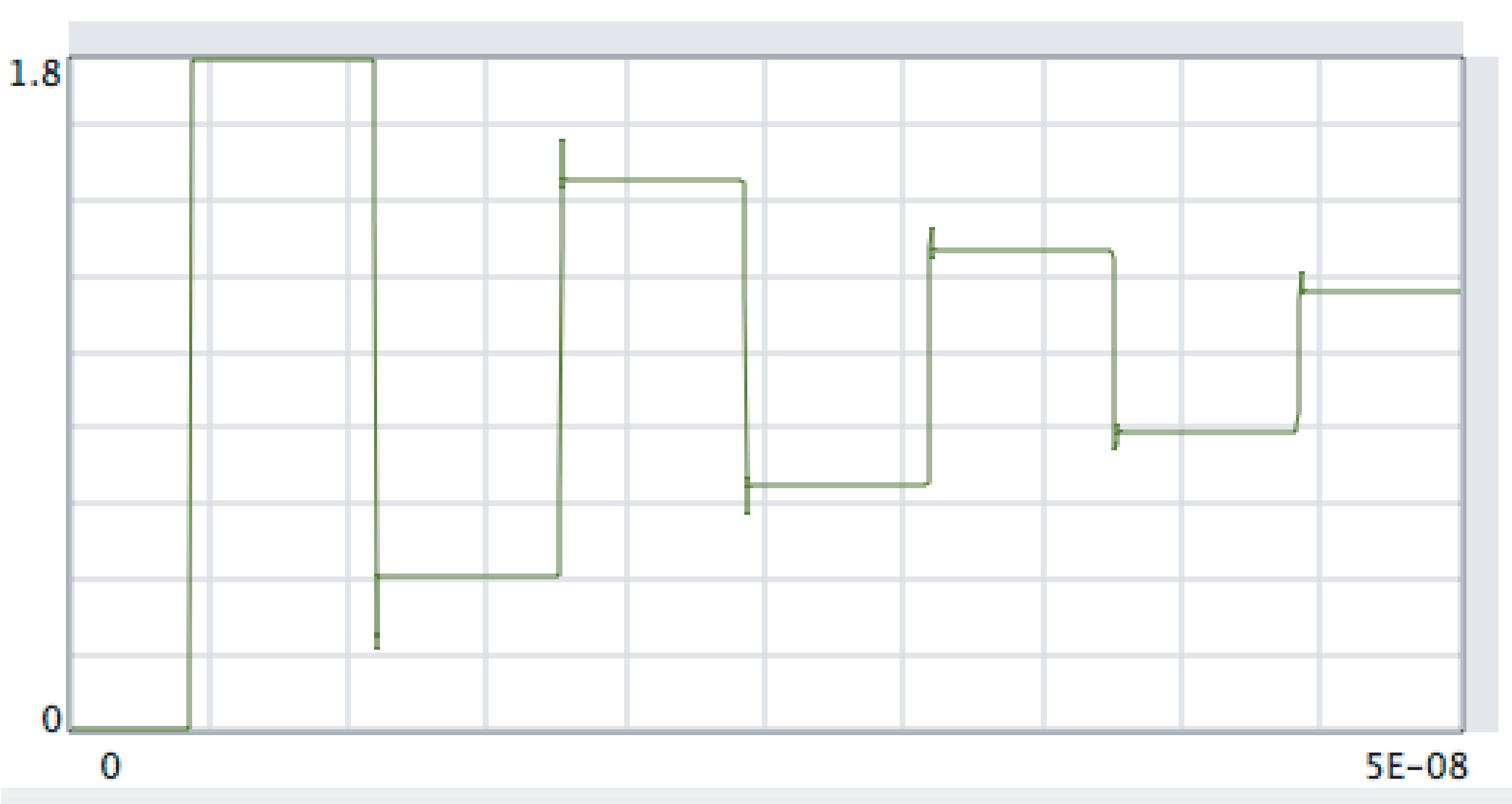
# Example: Open Line (II)

- At source a new reflection is created $v_2^+ = \Gamma_L \Gamma_s v_1^+$, and note $\Gamma_s < 0$, so $v_2^+ = -.6 \times 0.8 = -0.48$.
- At a time $3t_p$, the line charged initially to $v_1^+ + v_1^-$ drops in value

$$v_L = v_1^+ + v_1^- + v_2^+ + v_2^- = 1.6 - 2 \times .48 = .64$$

- So the voltage on the line undershoots $< 1$
- And on the next cycle $5t_p$ the load voltage again overshoots
- We observe ringing with frequency $2t_p$

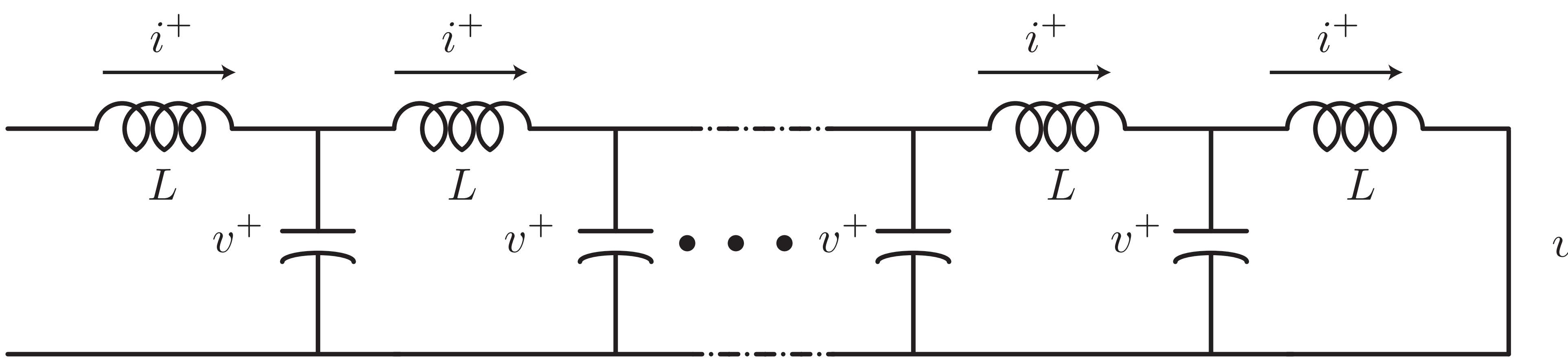
# Example: Open Line Ringing




- Observed waveform as a function of time.

# Physical Intuition: Shorted Line (I)

- The intitial step charges the “first” capacitor through the “first” inductor since the line is uncharged
- There is a delay since on the rising edge of the step, the inductor is an open
  - Each successive capacitor is charged by “its” inductor in a uniform fashion ... this is the forward wave $v_1^+$



# Physical Intuition: Shorted Line (II)

- The volage on the line goes up from left to right due to the delay in charging each inductor through the inductors
- The last inductor, though, does not have a capacitor to charge
- Thus the last inductor is discharged ... the extra charge comes by discharging the last capacitor
- As this capacitor discharges, so does it's neighboring capacitor to the left
- Again there is a delay in discharging the caps due to the inductors
- This discharging represents the backward wave $v_1^-$