EECS 117

Lecture 19: Faraday's Law and Maxwell's Eq.

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Magnetic Energy of a Circuit

- Last lecture we derived that the total magnetic energy in a circuit is given by

$$E_m = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

- We would like to show that this implies that $M \leq \sqrt{L_1 L_2}$. Let's re-write the above into the following positive definite form

$$E_m = \frac{1}{2}L_1 \left(I_1 + \frac{M}{L_1} I_2 \right)^2 + \frac{1}{2}\left(L_2 - \frac{M^2}{L_1} \right) I_2^2$$

- An important observation is that regardless of the current I_1 or I_2, the magnetic energy is non-negative, so $E_m \geq 0$

Magnetic Energy is Always Positive

- Consider the current $I_2 = \frac{-L_1}{M} I_1$, which cancels the first term in E_m

$$E_m = \frac{1}{2}\left(L_2 - \frac{M^2}{L_1}\right) I_2^2 \geq 0$$

- Since $I_2^2 \geq 0$, we have

$$L_2 - \frac{M^2}{L_1} \geq 0$$

- Therefore it's now clear that this implies

$$L_1 L_2 \geq M^2$$

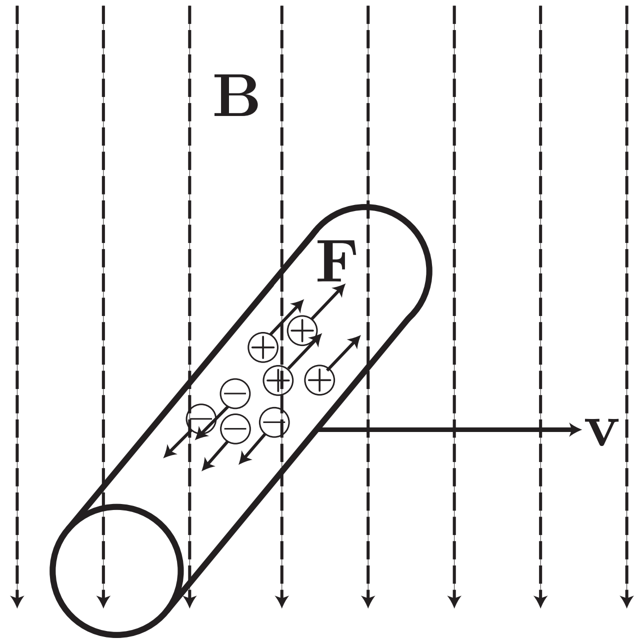
Coupling Coefficient

- Usually we express this inequality as

$$M = k\sqrt{L_1 L_2}$$

- Where k is the *coupling coefficient* between two circuits, with $|k| \leq 1$.
- If two circuits are perfectly coupled (all flux from circuit one crosses circuit 2), $k = 1$ (ideal transformer)
- Note that $M < 0$ implies that $k < 0$, which is totally reasonable as long as k lies on the unit interval $-1 \leq k \leq 1$
- Negative coupling just means that the flux gets inverted before crossing the second circuit. This is easily achieved by winding the circuits with opposite orientation.

Motion in Magnetic Field



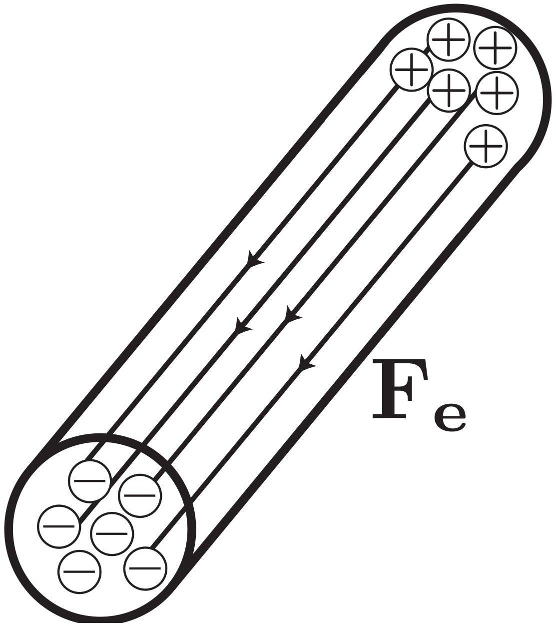
- Consider moving a bar in a constant magnetic field
- The conductors therefore feel a force

$$\mathbf{F_m} = q\mathbf{v} \times \mathbf{B}$$

- This causes charge separation and thus the generation of an internal electric field that cancels the magnetic field

$$\mathbf{E} = \mathbf{v} \times \mathbf{B}$$

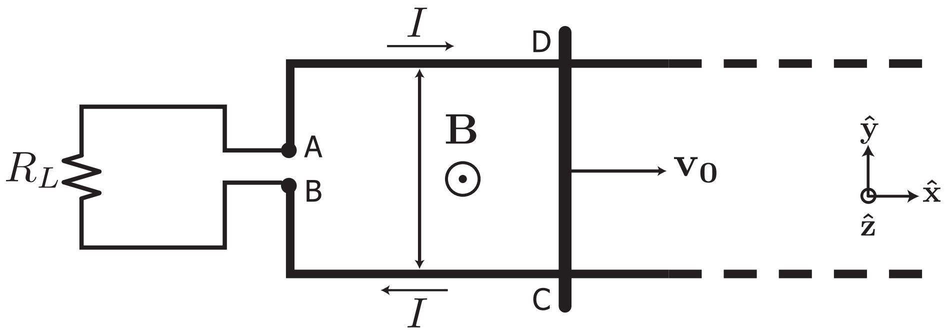
Motion (cont)



- The induced voltage in the bar

$$V_{ind} = -\int_1^2 \mathbf{E} \cdot d\ell = \int_1^2 (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{d}\ell$$

A Moving Metal Bar



- Consider the following "generator". A bar of length ℓ moves to the right with velocity v_0 (always making contact with the rest of the circuit)

$$V_{ind} = \int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{d}\ell = \int_C^D (\hat{\mathbf{x}} v_0 \times \hat{\mathbf{z}} B_0) \cdot \hat{\mathbf{y}} dy = -v_0 B_0 \ell$$

- Current I flows in a direction to decrease the flux (Lenz's Law)

Energy Dissipated by R

- This current flows through the resistor R_L where the energy of motion of the bar is converted to heat
- The load will dissipate energy

$$P_L = I^2 R_L = \frac{(v_0 B_0 \ell)^2}{R_L}$$

- This power comes from the mechanical work in moving the bar. The force experienced by a current carrying wire $\mathbf{dF} = I d\ell \times \mathbf{B}$

$$\mathbf{F_m} = I \int_C^D d\ell \times \mathbf{B} = I \int_C^D -\hat{\mathbf{y}} dy \times \hat{\mathbf{z}} B_0 = -I B_0 \ell$$

- Thus $P_{in} = -F_m v_0 = I B_0 \ell v_0 = P_L$

(im)Practical Example

- Let's say we do this experiment using the earth's magnetic field
- Use a bar with length $\ell = 1\ \mathrm{m}$, $B_0 = 0.5\ \mathrm{G}$
- To induce only $1\ \mathrm{V}$, we have to move the bar at a speed of

$$v_0 = \frac{V_{ind}}{B_0 \ell} = 2 \times 10^4\ \mathrm{m/s}$$

- The magnetic field on the surface of a neutron star is about $B_0 \approx 10^{12}\ \mathrm{G}$, or about $10^8\ \mathrm{T}$. Even moving at a speed of $v_0 = 1\ \mathrm{m/s}$, we generate

$$V_{ind} = 10^8\ \mathrm{V}$$

- Energy generation on a neutron star is easy!

Back to Faraday's Equation

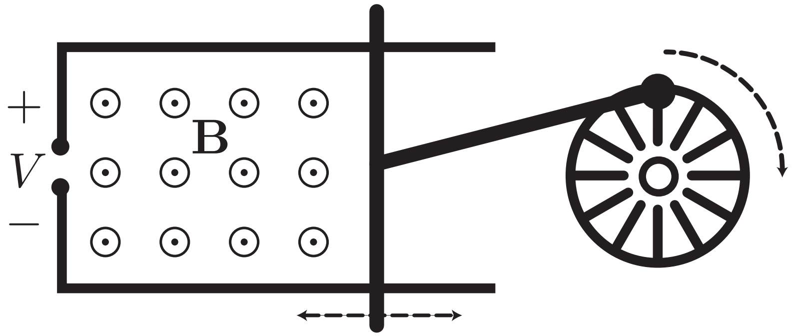
- Note that this problem is just as easy to solve using Faraday's Law
- The flux crossing the loop is increasing at a constant rate

$$\psi(t) = \psi_0 + \ell v_0 t B_0$$

- Where ψ_0 is the initial flux at $t = 0$
- The induced voltage is simply

$$V_{ind} = -\frac{d\psi}{dt} = \ell v_0 B_0$$

An AC "Generator"



- If we connect our metal bar to a piston, in turn connecting to a water-wheel or otherwise rotating wheel, we have a crude generator
- To generate substantial voltage, we need a strong magnetic field
- Say we rotate the wheel at a rate of $\omega = 2\pi \times 10^3 \mathrm{s}^{-1}$, or 1000 RPS (revolutions per second)

An AC "Generator" (cont)

- The flux is now

$$\psi = \psi_0 + \ell \cdot B_0 \cdot A_m \cos \omega t$$

- Where A_m is the amplitude of oscillation. Taking the time derivative

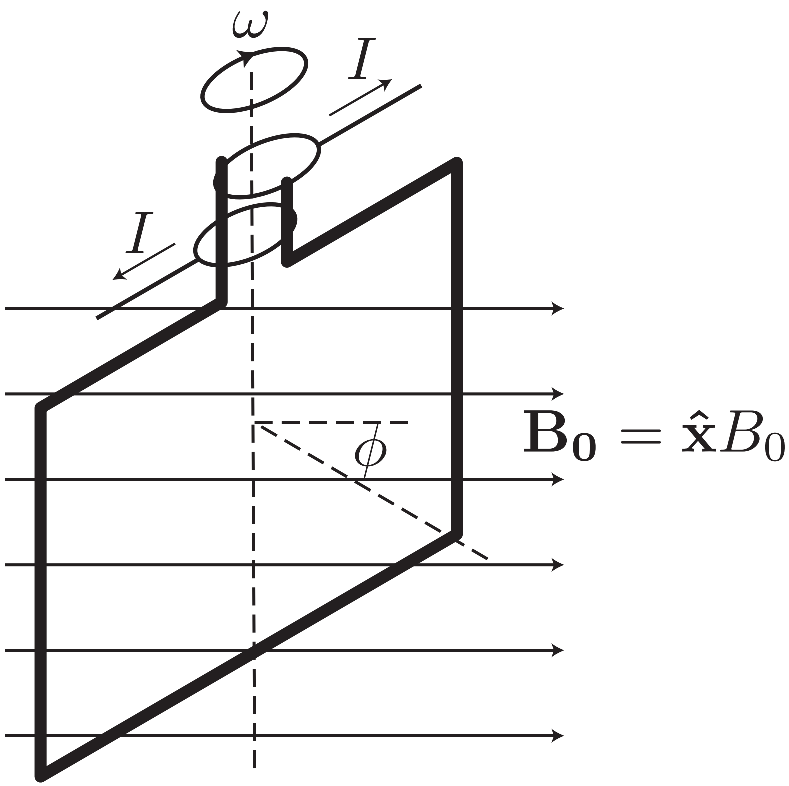
$$\dot{\psi} = -A_m \omega \ell B_0 \sin \omega t = -V_{ind}$$

- Plugging in some numbers, we see that with a relatively strong magnetic field of $1\ \mathrm{T}$, an amplitude $A_m = 1\ \mathrm{m}$, $\ell = 1\ \mathrm{m}$, the voltage generated is "reasonable"

$$V_{ind} = 2\pi \times 10^3 \sin \omega t$$

- The voltage is sinusoidal with frequency equal to the rotation frequency

AC Motor/Generator



- A simple AC motor/generator consists of a rotating loop cutting through a constant magnetic field. The slip rings maintain contact with the loop as it rotates.

AC Motor/Generator

- If AC current is passed through the loop, it rotates at a rate determined by the frequency. If, on the other hand, the loop is rotated mechanically and the circuit is closed with a load, mechanical power is converted to electricity
- The flux in the loop of area A is simply

$$\Psi = AB_0 \cos\phi$$

- The phase $\phi = \omega_0 t$ so

$$V_{ind} = -\dot{\Psi} = AB_0\omega_0 \sin\omega_0 t$$

- This result can also be derived by using $\mathbf{F_m} = q\mathbf{v} \times \mathbf{B}$

Maxwell's Eq (Integral Form)

- We have now studied the complete set of Maxwell's Equations . In Integral form

$$\oint_C \mathbf{E} \cdot d\ell = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}$$

$$\oint_C \mathbf{H} \cdot d\ell = \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS} + \int_S \mathbf{J} \cdot \mathbf{dS}$$

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho dV$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$$

Maxwell's Eq (Differential Form)

- The fields are related by the following material parameters

$$\mathbf{B} = \mu \mathbf{H} \qquad \mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{J} = \sigma \mathbf{E}$$

- For most materials we assume that these are scalar relations.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = -\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 (\mathbf{J} + \frac{\partial P}{\partial t} + \nabla \times \mathbf{M})$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} (\rho - \nabla \cdot \mathbf{P})$$

$$\nabla \cdot \mathbf{B} = 0$$

Source Free Regions

- In source free regions $\rho = 0$ and $J = 0$. Assume the material is uniform (no bound charges or currents)

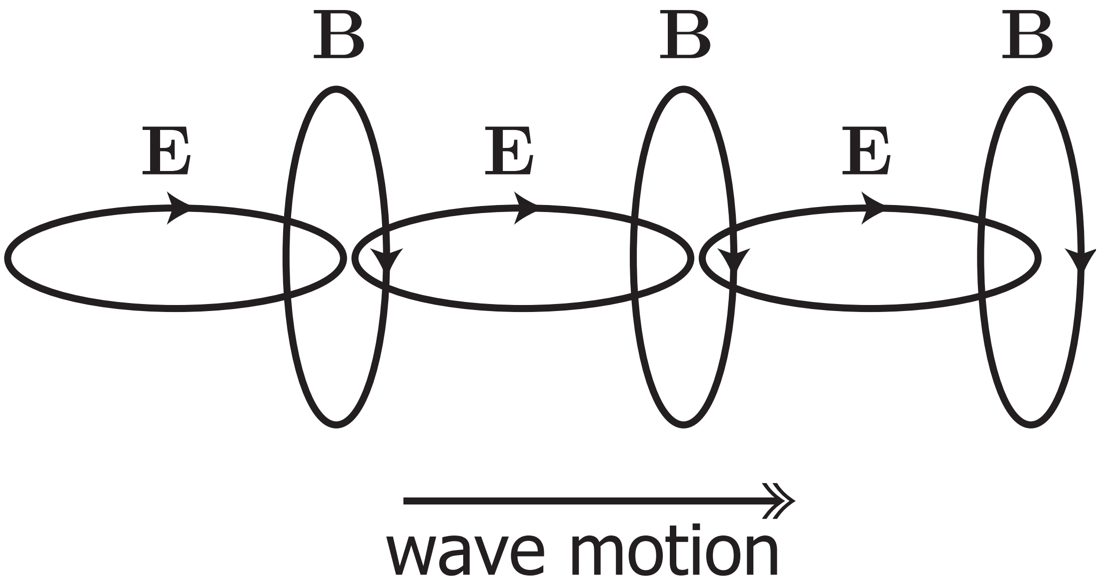
$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$$

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Wave Motion



- We can see intuitively that $\frac{\partial E}{\partial t} \rightarrow \frac{\partial B}{\partial t} \rightarrow \frac{\partial E}{\partial t} \rightarrow \ldots$, that wave motion is possible

Time Harmonic Maxwell's Eq.

- Under time-harmonic conditions (many important practical cases are time harmonic, or nearly so, or else Fourier analysis can handle non-harmonic cases)

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D}$$

$$\nabla \cdot \mathbf{E} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

- These equations are not all independent. Take the divergence of the curl, for instance

$$\nabla \cdot (\nabla \times \mathbf{E}) \equiv 0 = -j\omega \nabla \cdot \mathbf{B}$$

Harmonic Eq. (cont)

- In other words, the non-existence of magnetic charge is built-in to our curl equation. If magnetic charge is ever observed, we'd have to modify our equations
- This is analogous to the displacement current that Maxwell introduced to make the curl of H equation self-consistent

$$\nabla \cdot (\nabla \times \mathbf{H}) \equiv 0 = \nabla \cdot \mathbf{J} + j\omega \nabla \cdot \mathbf{D}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -j\omega\rho$$

- This implies that $\nabla \cdot \mathbf{D} = \rho$, so Gauss' law is built-in to our curl equations as well.

Tangential Boundary Conditions

- The boundary conditions on the E-field at the interface of two media is

$$\hat{\mathbf{n}} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$$

- Or equivalently, $E_{1t} = E_{2t}$. If magnetic charges are ever found, then this condition will have to include the possibility of a surface magnetic current
- The boundary conditions on $\mathbf{H}$ are similar

$$\hat{\mathbf{n}} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$$

- For the interface of a perfect conductor, for example, a surface current flows so that ($H_2 = 0$)

$$\hat{\mathbf{n}} \times \mathbf{H_1} = \mathbf{J_s}$$

Boundary Conditions for Current

- Applying the "pillbox" argument to the divergence of current

$$\int_V (\nabla \cdot \mathbf{J}) dV = \oint_S \mathbf{J} \cdot \mathbf{dS} = -\int_V \frac{\partial \rho}{\partial t} dV$$

  in the limit

$$J_{1n} - J_{2n} = -\frac{\partial \rho_s}{\partial t}$$

- where ρ_s is the surface current. In the static case

$$J_{1n} = J_{2n}$$

- implies that $\sigma_1 E_1 = \sigma_2 E_2$. This implies that $\rho_s \neq 0$ since $\epsilon_1 E_1 \neq \epsilon_2 E_2$ (unless the ratios of σ match the ratio of ϵ perfectly!)