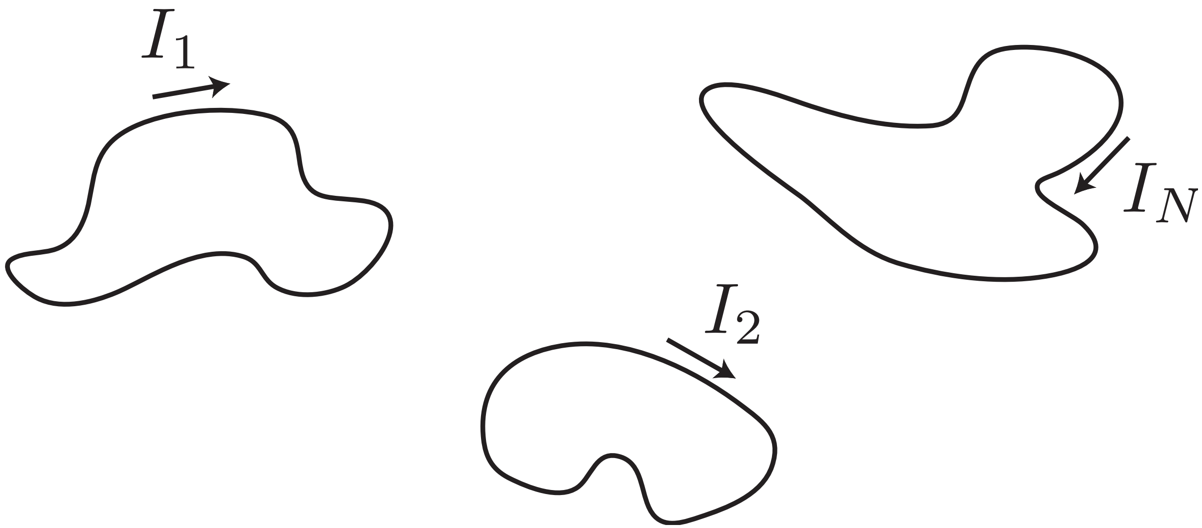
EECS 117

Lecture 18: Magnetic Energy and Inductance

Prof. Niknejad

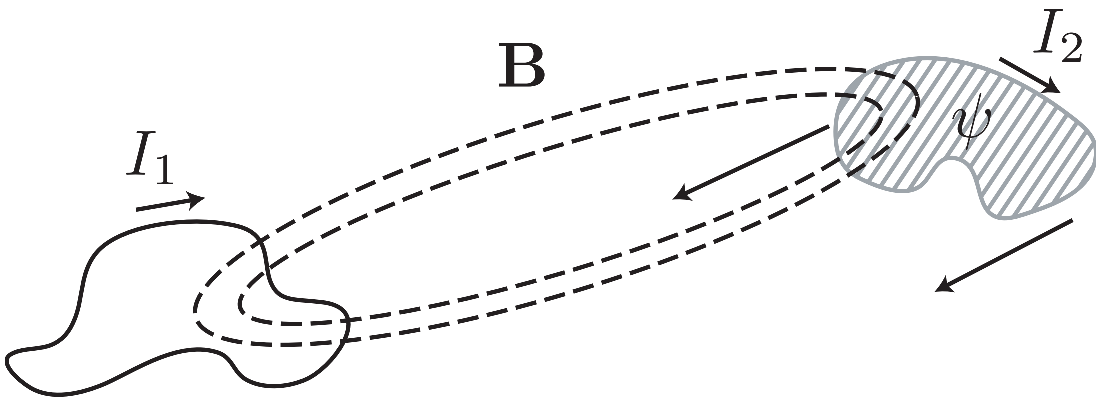
University of California, Berkeley

Energy for a System Of Current Loops



- In the electrostatic case, we assembled our charge distribution one point charge at a time and used electric potential to calculate the energy
- This can be done for the magnetostatic case but there are some complications.

Energy for Two Loops



- As we move in our second loop with current I_2, we'd be cutting across flux from loop 1 and therefore an induced voltage around loop 2 would change the current. When we bring the loop to rest, the induced voltage would drop to zero.
- To maintain a constant current, therefore, we'd have to supply a voltage source in series to cancel the induced voltage. The work done by this voltage source represents the magnetostatic energy in the system.

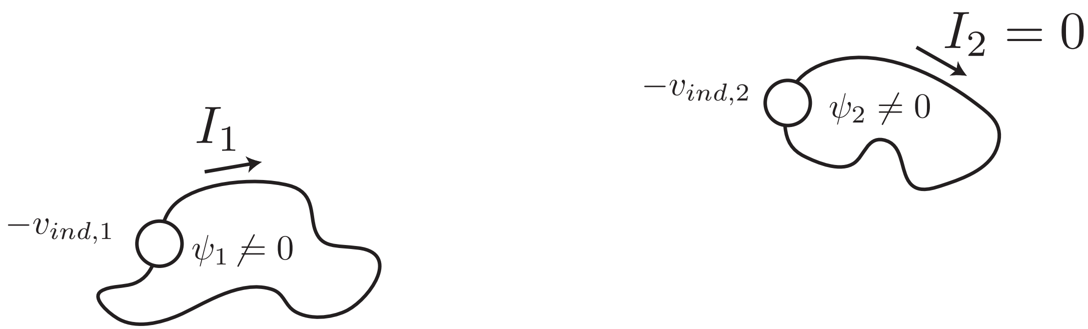
Energy for Two Loops (another way)

- A simpler approach is to bring in the two loops with zero current and then increase the current in each loop one at a time
- First, let's increase the current in loop 1 from zero to I_1 in some time t_1. Note that at any instant of time, a voltage is induced around loop number 1 due to it's changing flux

$$v_{ind,1} = -\frac{d\psi}{dt} = -L_1 \frac{di_1}{dt}$$

- where i_1 represents the instantaneous current.

Current in Loop 1 (I)



- Note that this induced voltage will tend to decrease the current in loop 1. This is a statement of Lenz's law. In other words, the induced voltage in loop 1 tends to create a magnetic field to oppose the field of the original current!
- To keep the current constant in loop 1, we must connect a voltage source to cancel the induced voltage

Work Done by Source 1

- The work done by this voltage source is given by

$$w_1 = \int_0^{t_1} p(\tau) d\tau$$

- where $p(t) = -v_{ind,1} i_1(t) = L_1 i_1 \frac{di_1}{dt}$
- The net work done by the source is simply

$$w_1 = L_1 \int_0^{t_1} i_1 \frac{di_1}{d\tau} d\tau = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$

Current in Loop 1 (II)

- Note that to keep the current in loop 2 equal to zero, we must also provide a voltage source to counter the induced voltage

$$v_{ind,2}(t) = -M_{21}\frac{di_1}{dt}$$

- This voltage source does not do any work since $i_2(t) = 0$ during this time

Current in Loop 2 (I)

- By the same argument, if we increase the current in loop 2 from 0 to I_2 in time t_2, we need to do work equal to $\frac{1}{2}L_2 I_2^2$.
- But is that all? No, since to keep the current in loop 1 constant at I_1 we must connect a voltage source to cancel the induced voltage

$$-v_{ind,1} = \frac{d\psi_1}{dt} = M_{12}\frac{di_2}{dt}$$

- The additional work done is therefore

$$w_1' = \int_0^{t_2} M_{12}\frac{di_2}{d\tau} I_1 d\tau = M_{12} I_1 I_2$$

Energy for Two Loops

- The total work to bring the current in loop 1 and loop 2 to I_1 and I_2 is therefore

$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$

- But the energy should not depend on the order we turn on each current. Thus we can immediately conclude that $M_{12} = M_{21}$
- We already saw this when we derived an expression for M_{12} using the Neumann equation

Generalize to N Loops

- We can now pretty easily generalize our argument for 2 loops to N loops

$$W = \frac{1}{2} \sum_i L_i I_i^2 + \sum \sum_{i>j} M_{ij} I_i I_j$$

- The first term represents the "self" energy for each loop and the second term represents the interaction terms. Let's rewrite this equation and combine terms

$$W = \frac{1}{2} \sum \sum_{i=j} M_{ij} I_i I_j + \frac{1}{2} \sum \sum_{i \neq j} M_{ij} I_i I_j$$

$$W = \frac{1}{2} \sum_i \sum_j M_{ij} I_i I_j$$

Neumann's Equation

- We derived the mutual inductance between two filamentary loops as Neumann's equation

$$M_{ij} = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\ell_i \cdot d\ell_j}{R_{ij}}$$

- Let's substitute the above relation into the expression for energy

$$W = \frac{1}{2} \sum_i I_i \left[\sum_j M_{ij} I_j \right]$$

$$W = \frac{1}{2} \sum_i I_i \left[\sum_j I_j \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\ell_i \cdot d\ell_j}{R_{ij}} \right]$$

Rework Expression for Energy

- Let's change the order of integration and summation

$$W = \frac{1}{2} \sum_i I_i \oint_{C_i} \left[\sum_j I_j \frac{\mu_0}{4\pi} \oint_{C_j} \frac{\cdot d\ell_j}{R_{ij}} \right] \cdot d\ell_i$$

- Each term of the bracketed expression represents the vector potential due to loop j evaluated at a position on loop i. By superposition, the sum represents the total voltage potential due to all loops

$$W = \frac{1}{2} \oint_{C_i} I_i A \cdot d\ell_i$$

Energy in terms of Vector Potential

- We derived this for filamental loops. Generalize to an arbitrary current distribution and we have

$$W_m = \frac{1}{2}\int_V \mathbf{J}\cdot\mathbf{A}dV$$

- Compare this to the expression for electrostatic energy

$$W_e = \frac{1}{2}\int_V \rho\phi dV$$

- Thus the vector potential $\mathbf{A}$ really does represent the magnetic potential due to a current distribution in an analogous fashion as ϕ represents the electric potential

Energy in terms of the Fields

- Let's replace $\mathbf{J}$ by Ampère's law $\nabla \times \mathbf{H} = \mathbf{J}$

$$W_m = \frac{1}{2}\int_V (\nabla \times \mathbf{H}) \cdot \mathbf{A} dV$$

- Using the identity

$$\nabla \cdot (\mathbf{H} \times \mathbf{A}) = (\nabla \times \mathbf{H}) \cdot \mathbf{A} + \mathbf{H} \cdot (\nabla \times \mathbf{A})$$

$$W_m = \frac{1}{2}\int_V \nabla \cdot (\mathbf{H} \times \mathbf{A}) dV + \frac{1}{2}\int_V (\nabla \times \mathbf{A}) \cdot \mathbf{H} dV$$

- Apply the Divergence Theorem to the first term to give

$$W_m = \frac{1}{2}\int_S \mathbf{H} \times \mathbf{A} \cdot dS + \frac{1}{2}\int_V (\nabla \times \mathbf{A}) \cdot \mathbf{H} dV$$

Vanishing Surface Term

- We'd like to show that the first term is zero. To do this. Consider the energy in all of space $V \to \infty$. To do this, consider a large sphere of radius r and take the radius to infinity
- We know that if we are sufficiently far from the current loops, the potential and field behave like $A \sim r^{-1}$ and $H \sim r^{-2}$. The surface area of the sphere goes like r^2
- The surface integral, therefore, gets smaller and smaller as the sphere approaches infinity

Energy in Terms of B and H

- The remaining volume integral represents the total magnetic energy of a system of currents

$$W_m = \frac{1}{2}\int_V (\nabla \times \mathbf{A}) \cdot \mathbf{H} dV$$

- But $\nabla \times \mathbf{A} = \mathbf{B}$

$$W_m = \frac{1}{2}\int_V \mathbf{B} \cdot \mathbf{H} dV$$

- And the energy density of the field is seen to be

$$w_m = \mathbf{B} \cdot \mathbf{H}$$

- Recall that $w_e = \mathbf{D} \cdot \mathbf{E}$

Another Formula for Inductance

- The self-inductance of a loop is given by

$$L = \frac{1}{I}\int_S \mathbf{B}\cdot dS$$

- Since the total magnetic energy for a loop is $\frac{1}{2}LI^2$, we have an alternate expression for the inductance

$$\frac{1}{2}LI^2 = \frac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}dV$$

$$L = \frac{1}{I^2}\int_V \mathbf{B}\cdot\mathbf{H}dV$$

- This alternative expression is sometimes easier to calculate

Self-Inductance of Filamentary Loops

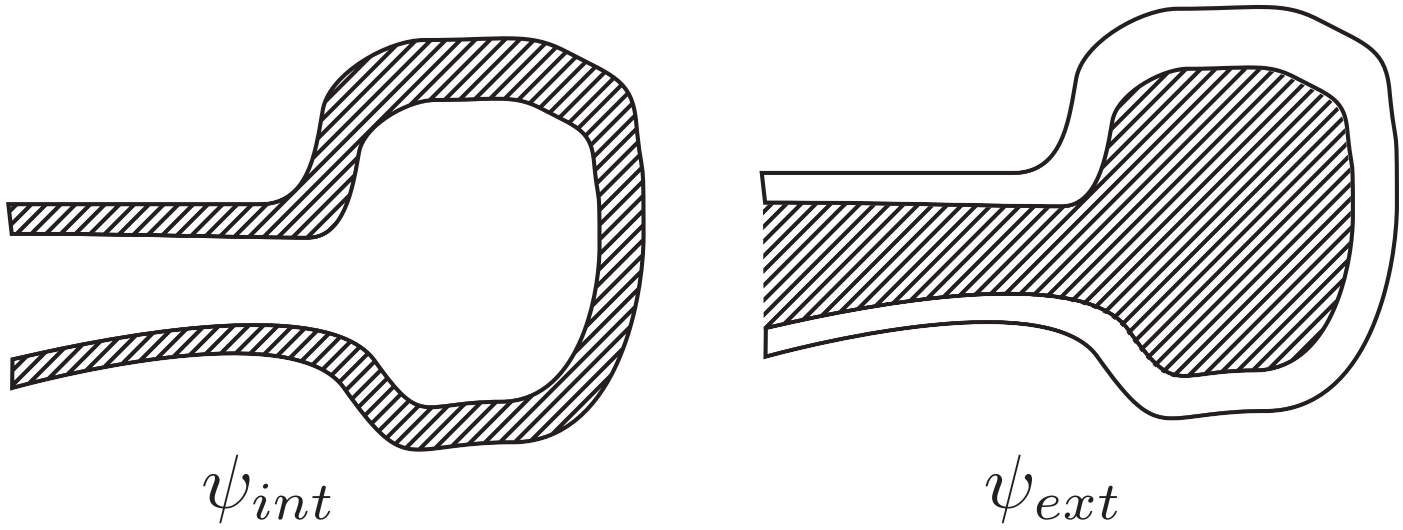
- We have tacitly assumed that the inductance of a loop is a well-defined quantity. But for a filamentary loop, we can expect trouble.
- By definition

$$L = \frac{1}{I}\int_S \mathbf{B}\cdot dS = \frac{1}{I}\oint_C \mathbf{A}\cdot d\ell$$

$$L = \frac{\mu}{4\pi}\oint_C\oint_C \frac{d\ell'\cdot d\ell}{R}$$

- This is just Neumann's equation with $C_1 = C_2$. But for a filamental loop, $R = 0$ when both loops traverse the same point. The integral is thus not defined for a filamental loop!

Internal and External Inductance



- It's common to split the flux in a loop into two components. One component is defined as the flux crossing the internal portions of the conductor volume. The other, is external to the conductors

$$L = \frac{\psi}{I} = \frac{\psi_{int}}{I} + \frac{\psi_{ext}}{I} = L_{int} + L_{ext}$$

Internal Inductance of a Round Wire

- Usually if the wire radius is small relative to the loop area, $L_{int} \ll L_{ext}$
- We shall see that at high frequencies, the magnetic field decays rapidly in the volume of conductors and thus the $\psi_{int} \to 0$ and $L(f \to \infty) = L_{ext}$
- Consider a round wire carrying uniform current. We can easily derive the magnetic field through Amére's law

$$B_{inside} = \frac{\mu_0 I r}{2\pi a^2}$$

Round Wire (cont)

- Using this expression, we can find the internal inductance

$$W_m = \frac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}dV = \frac{1}{2}\int_{V_{inside}} \mathbf{B}\cdot\mathbf{H}dV + \frac{1}{2}\int_{V_{outside}} \mathbf{B}\cdot\mathbf{H}dV +$$

$$W_m = \frac{1}{2}L_{int}I^2 + \frac{1}{2}L_{ext}I^2$$

- The "inside" term is easily evaluated

$$W_{m,int} = \frac{1}{2}\frac{\mu_0 I^2}{(2\pi a^2)^2}\int_0^a r^2 2\pi r dr = \frac{1}{2}\frac{\mu_0 I^2}{8\pi}$$

Internal Inductance Calculation

- The internal inductance per unit length is thus

$$L_{int} = \frac{\mu_0}{8\pi}$$

- Numerically, this is $50\mathrm{pH}/\mathrm{mm}$, a pretty small inductance. Recall that this is only the inductance due to energy stored inside of the wires. The external inductance is likely to be much larger.