EECS 117

Lecture 17: Magnetic Forces/Torque, Faraday's Law

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Memory Aid

- The following table is a useful way to remember the equations in magnetics. We can draw a very good analogy between the fields [a]

$$
\begin{array}{cccc}
\mathbf{E} & \mathbf{H} & \rho & \mathbf{J} \\
\mathbf{D} & \mathbf{B} & V & A \\
\epsilon & \mu^{-1} & \cdot & \times \\
\mathbf{P} & \mathbf{M} & \times & \cdot
\end{array}
$$

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[a]I personally don't like this choice since to me $\mathbf{E}$ and $\mathbf{B}$ are "real" and so the equations should be arranged to magnify this analogy. Unfortunately the equations are not organized this way (partly due to choice of units) so we'll stick with convention

Boundary Conditions for Mag Field

- We have now established the following equations for a static magnetic field

$$\nabla \times \mathbf{H} = \mathbf{J}$$
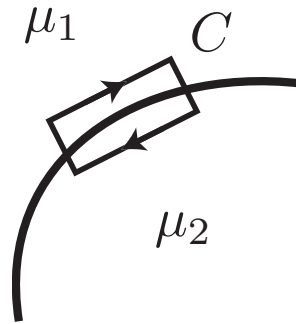
$$\nabla \cdot \mathbf{B} = 0$$

$$\oint_C \mathbf{H} \cdot d\ell = \int_S \mathbf{J} \cdot dS = I$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

- And for linear materials, we find that $\mathbf{H} = \mu^{-1}\mathbf{B}$

Tangential H

- The appropriate boundary conditions follow immediately from our previously established techniques

$$\mu_1 \qquad C$$

$$\mu_2$$

- Take a small loop intersecting with the boundary and take the limit as the loop becomes tiny

$$\int_C \mathbf{H} \cdot d\ell = (H_{t1} - H_{t2})d\ell = 0$$

Tangential H (cont)

- So the tangential component of H is continuous

$$H_{t1} = H_{t2} \quad \mu_1^{-1} B_{t1} = \mu_2^{-1} B_{t2}$$

- Note that B is discontinuous because there is an effective surface current due to the change in permeability. Since B is "real", it reflects this change

- If, in addition, a surface current is flowing in between the regions, then we need to include it in the above calculation

Normal B
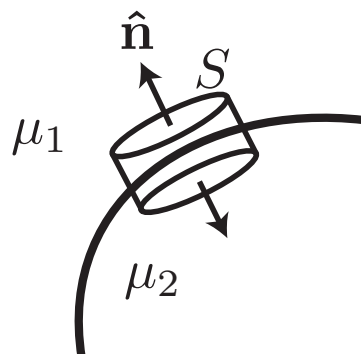
- Consider a pillbox cylinder enclosing the boundary between the layers

- In the limit that the pillbox becomes small, we have

$$\oint \mathbf{B} \cdot d\mathbf{S} = (B_{1n} - B_{2n})dS = 0$$

- And thus the normal component of B is continuous

$$B_{1n} = B_{2n}$$

Boundary Conditions for a Conductor

- If a material is a very good conductor, then we'll show that it can only support current at the surface of the conductor.

- In fact, for an ideal conductor, the current lies entirely on the surface and it's a true surface current

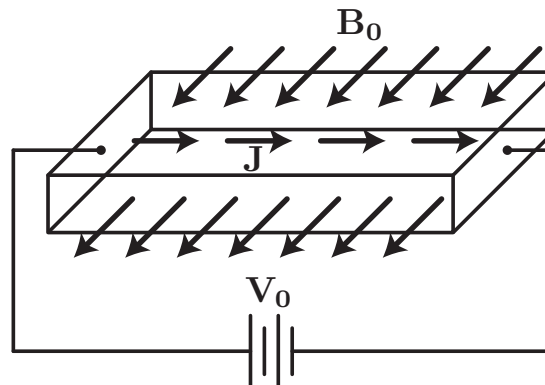- In such a case the current enclosed by even an infinitesimal loop is finite

$$\oint_C \mathbf{H} \cdot d\ell = (H_{t1} - H_{t2})d\ell = \mathbf{J_s}d\ell \quad \mathbf{H_{t1}} - \mathbf{H_{t2}} = \mathbf{J_s}$$

- This can be expressed compactly as

$$\hat{\mathbf{n}} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$$

- But for a perfect conductor, we'll see that $H_2 = 0$, so $H_{1t} = J_s$
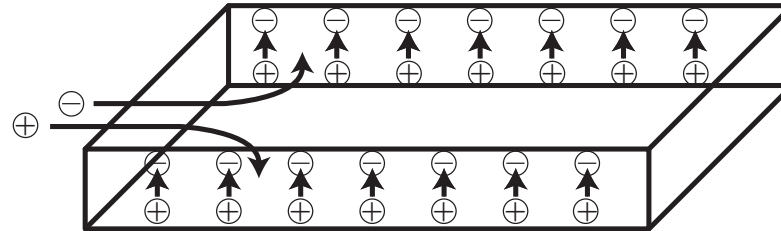
Hall Effect

- When current is traveling through a conductor, at any instant it experiences a force given by the Lorentz equation

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

- The force $q\mathbf{E}$ leads to conduction along the length of the bar (due to momentum relaxation) with average speed v_d but the magnetic field causes a downward deflection

$$\mathbf{F} = q\hat{\mathbf{x}}E_0 - q\hat{\mathbf{y}}v_d B_0$$
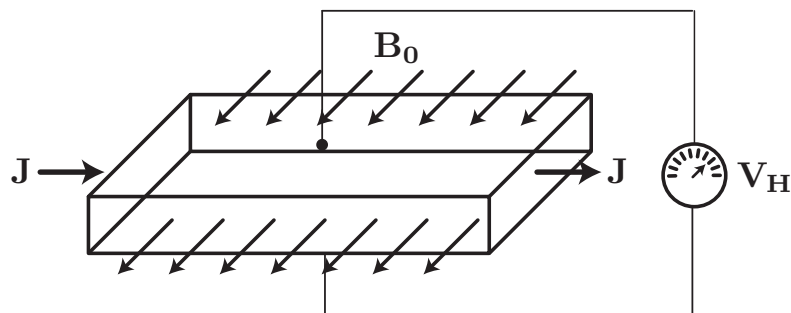
Hall Effect: Vertical Internal Field

- In steady-state, the movement of charge down (or electrons up) creates an internal electric field which must balance the downward pull

- Thus we expect a "Hall" voltage to develop across the top and bottom faces of the conducting bar

$$V_H = E_y d = v_d B_0 d$$

Hall Effect: Density of Carriers (I)

- Since $v_d = \mu E_x$, and $J_x = \sigma E_x$, we can write $v_d = \mu J_x / \sigma$

$$V_H = \frac{\mu J_x}{\sigma} B_0 d$$

- Recall that the conductivity of a material is given by $\sigma = qN\mu$, where q is the unit charge

- Since $v_d = \mu E_x$, and $J_x = \sigma E_x$, we can write $v_d = \mu J_x / \sigma$

$$V_H = \frac{\mu J_x}{\sigma} B_0 d$$

Hall Effect: Density of Carriers (II)

- Recall that the conductivity of a material is given by $\sigma = qN\mu$, where q is the unit charge, N is the density of mobile charge carriers, and μ is the mobility of the carriers

$$V_H = \frac{J_x B_0 d}{qN}$$

$$N = \frac{J_x B_0 d}{qV_H} = \frac{I B_0 d}{AqV_H}$$

- Notice that all the quantities on the RHS are either known or easily measured. Thus the density of carriers can be measured indirectly through measuring the Hall Voltage
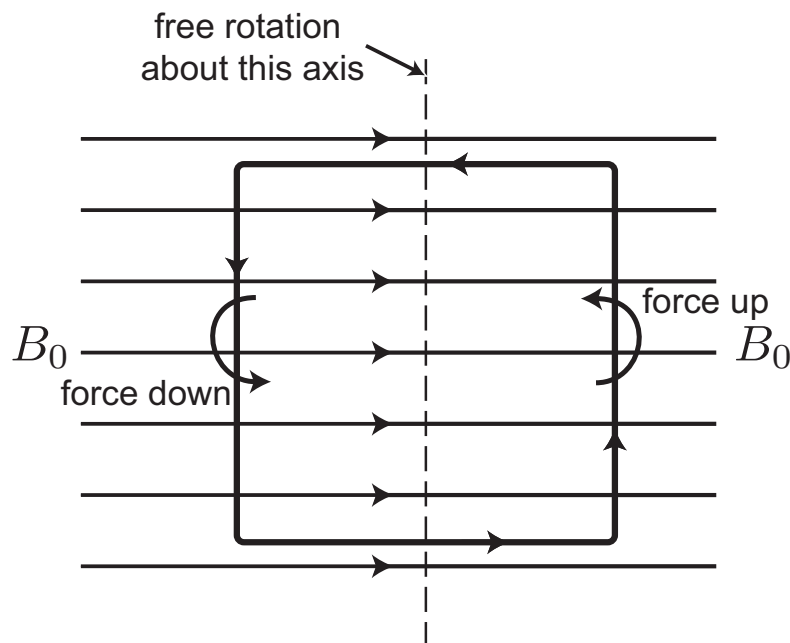
Forces on Current Loops
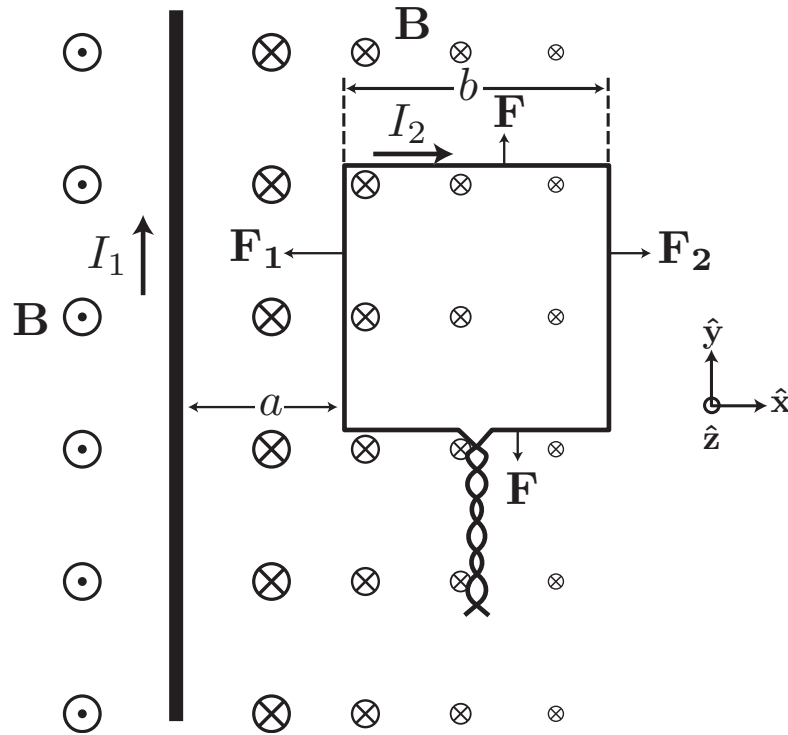
- Since the field is not uniform, the net force is not zero. Note the force on the $\perp$ sides cancel out

$$\mathbf{F_1} = -\hat{\mathbf{y}}\frac{\mu I_1 I_2}{2\pi a}d \qquad\qquad \mathbf{F_2} = +\hat{\mathbf{y}}\frac{\mu I_1 I_2}{2\pi(a+b)}d$$

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} = -\hat{\mathbf{y}}\frac{\mu I_1 I_2 d}{2\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Torques on Current Loops (I)

free rotation
about this axis

B_0

force down

force up

B_0

- In a uniform field, the net force on the current loop is zero. But the net torque is not zero. Thus the loop will tend to rotate.
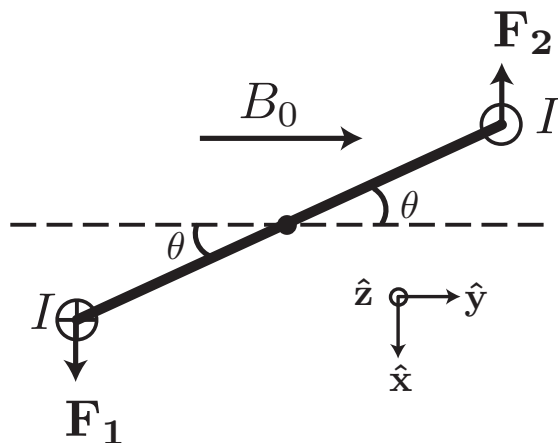
$$\mathbf{T} = \mathbf{r} \times \mathbf{F}$$

$$\mathbf{F_1} = -I_1 B_0 d\hat{\mathbf{x}}$$

$$\mathbf{F_2} = +I_1 B_0 d\hat{\mathbf{x}}$$

$$\mathbf{F_1} + \mathbf{F_2} = 0$$

Torques on Current Loops (II)

$$\mathbf{T_1} = -(b/2)I_1 B_0 d \sin\theta \hat{\mathbf{z}}$$

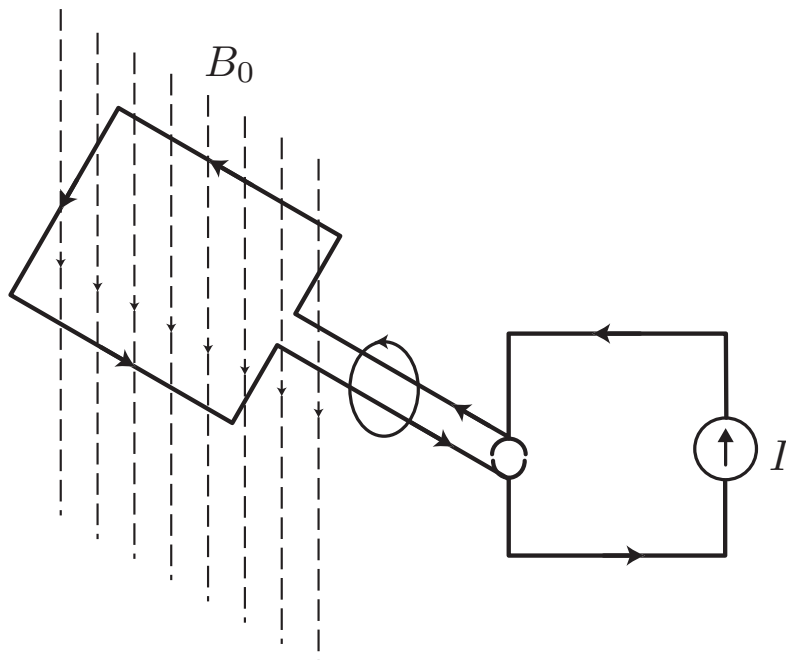$$\mathbf{T_2} = -(b/2)I_1 B_0 d \sin\theta \hat{\mathbf{z}}$$

$$\mathbf{T} = \mathbf{T_1} + \mathbf{T_2} = -\hat{\mathbf{z}}B_0 \overbrace{I \times \underbrace{b \times d}_{\text{Area of loop}}}^{\text{moment}} \sin\theta$$

- In general the torque can be expressed as

$$\mathbf{T} = \mathbf{m} \times \mathbf{B}$$

- where the moment is defined as $m = I \times \text{Area}$

Electric Motors

- A DC electric motor operates on this principle. A uniform strong magnetic field cuts across a current loop causing it to rotate.

- When the loop is ∥ to the field, the torque drops to zero but the rotational inertia of the loop keeps it rotating. Simultaneously, the direction of the current is reversed as the loop flips around and cuts into the field. This generates a new torque that favors the continuous rotation.

Faraday's Big Discovery

- In electrostatics we learned that $\oint \mathbf{E} \cdot d\ell = 0$

- Let's use the analogy between B and D (and E and H). Since $q = Cv$ and $\psi = Li$, and $i = \dot{q} = C\dot{v}$, should we not expect that $\dot{\psi} = L\dot{i} = v$?

energy of field
converted to heat!

R

$\dfrac{\partial B}{\partial t}$
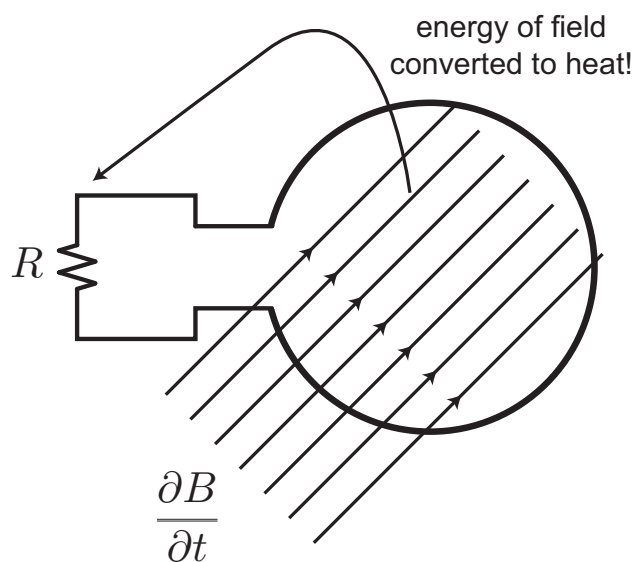
- In fact, this is true! Faraday was able to show this experimentally

$$\oint_C \mathbf{E} \cdot d\ell = -\frac{d\psi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

- The force is no longer conservative, $\mathbf{E} \neq -\nabla\phi$

Faraday's Law in Differential Form
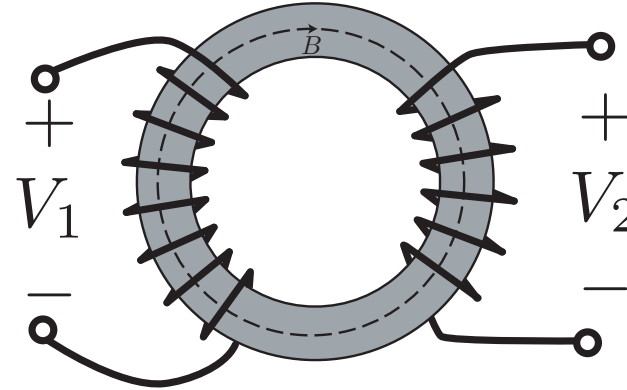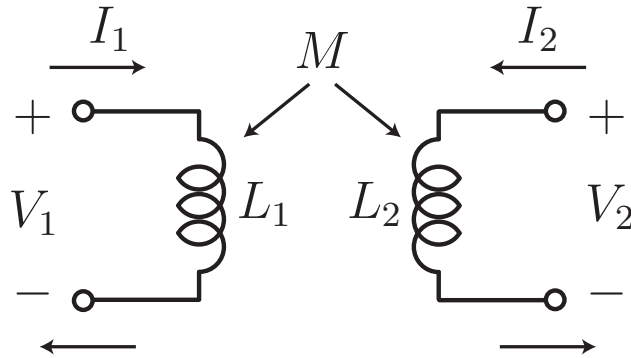
- Using Stoke's Theorem

$$\int_C \mathbf{E} \cdot d\ell = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

- Since this is true for any arbitrary curve C, this implies that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- Faraday's law is true for any region of space, including free space.

- In particular, if C is bounded by an actual loop of wire, then the flux cutting this loop will induce a voltage around the loop.
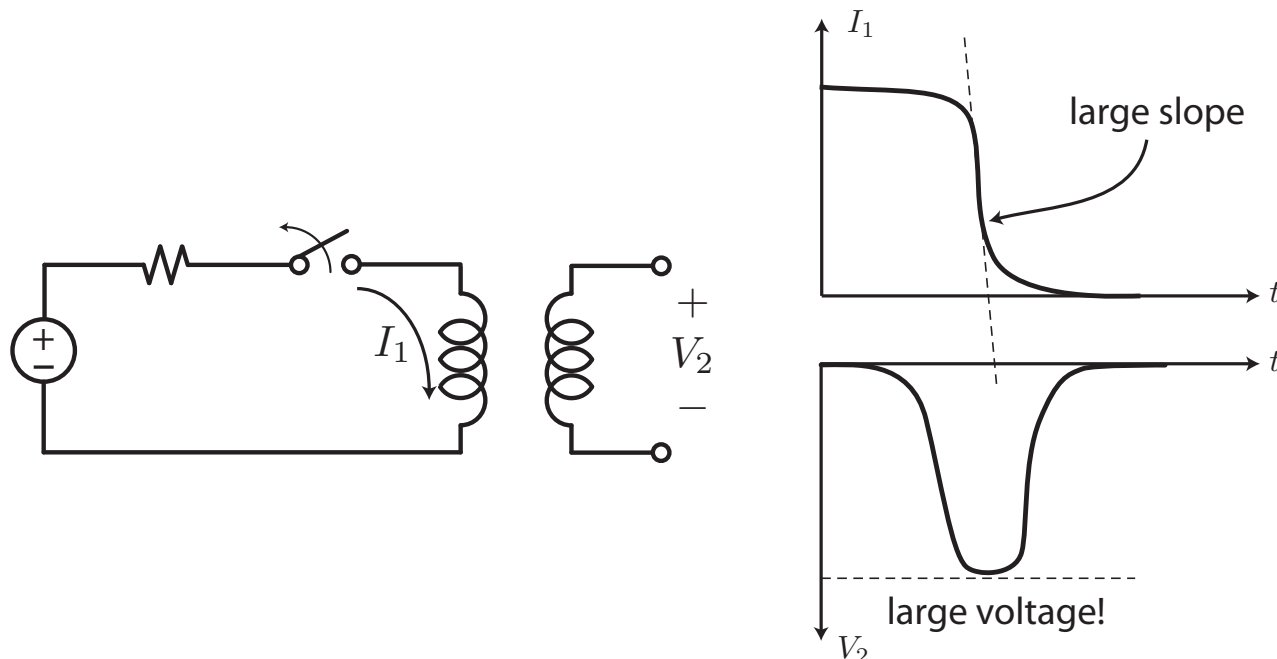
Example: Transformers

- In a transformer, by definition the flux in the "primary" side is given by $\psi_1 = L_1 I_1$

- Likewise, the flux crossing the "secondary" is given by $\psi_2 = M_{21} I_1 = M_{12} I_1 = M I_1$ (assuming $I_2 = 0$)

- Thus if the current in the primary changes, a voltage is induced in the secondary

$$V_2 = \dot{\psi}_2 = M \dot{I}_1$$

Generating Sparks!

Since the voltage at the secondary is proportional to the rate of change of current in loop 1, we can generate very large voltages at the secondary by interrupting the current with a switch

Vector Potential

- Since $\nabla \cdot \mathbf{B} \equiv 0$, we can write $\mathbf{B} = \nabla \times \mathbf{A}$. Thus

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial \mathbf{t}}$$

- If we group terms we have

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial \mathbf{t}} \right) = 0$$

- So, as we saw in electrostatics, we can likewise write

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi$$

More on Vector Potential

- We choose a negative sign for ϕ to be consistent with electrostatics. Since if $\frac{\partial}{\partial t} = 0$, this equation breaks down to the electrostatic case and then we identify ϕ as the scalar potential.

- This gives us some insight into the electromagnetic response as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{E} = \underbrace{-\nabla\phi}_{\text{electric response}} \underbrace{-\frac{\partial \mathbf{A}}{\partial t}}_{\text{magnetic response}}$$

- In reality the EM fields are linked so this viewpoint is not entirely correct.

Is the vector potential real?

- We can now re-derive Faradays' law as follows

$$V = \oint_C \mathbf{E} \cdot d\ell = - \oint_C \nabla\phi \cdot d\ell - \frac{\partial}{\partial t} \oint_C A \cdot d\ell$$

- The line integral involving $\nabla\phi$ is zero by definition so we have the induced emf equal to the line integral of $\mathbf{A}$ around the loop in question

$$V = -\frac{\partial}{\partial t} \oint_C \mathbf{A} \cdot d\ell$$

- We also found that equivalently

$$V = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$$

The Reality of the Vector Potential

$$V = -\frac{\partial}{\partial t} \oint_C \mathbf{A} \cdot d\ell$$

- This equation is somewhat more satisfying that Faraday's law in terms of the flux. Although it's mathematically equivalent, it explicitly shows us the shape of the loop's role in determining the induced flux.

- The flux equation, though, depends on a surface bounding the loop, in fact *any* surface. Sometimes it's even difficult to imagine the shape of such a surface (e.g. a coil)

Solenoid Transformer
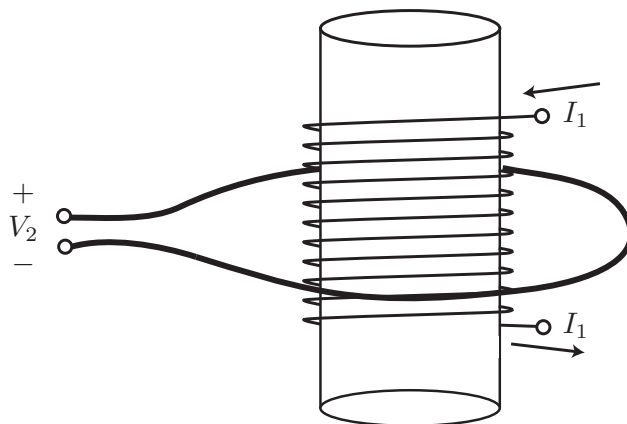
- Consider the magnetic coupling between a solenoid and a large loop surrounding the solenoid.

- We found that for an ideal solenoid, $\mathbf{B} = 0$ outside of the cylinder. Certainly we can assume that $\mathbf{B} \approx 0$ outside of this region.

Vector Potential Outside of Solenoid

- Then the voltage induced into the outer loop only depends on the constant flux generated within the center section coincident with the solenoid

- What's disturbing is that even though $\mathbf{B} = 0$ along the loop, there is a force pushing electrons inside the outer metal.

- The force is therefore *not* magnetic since $\mathbf{B} = 0$.

- The viewpoint with vector potential, though, does not pose any problems since $\mathbf{A} \neq 0$ outside of the loop. Therefore when we integrate $\mathbf{A}$ outside of the loop, there is a nonzero result.

Circuit Application of Transformers
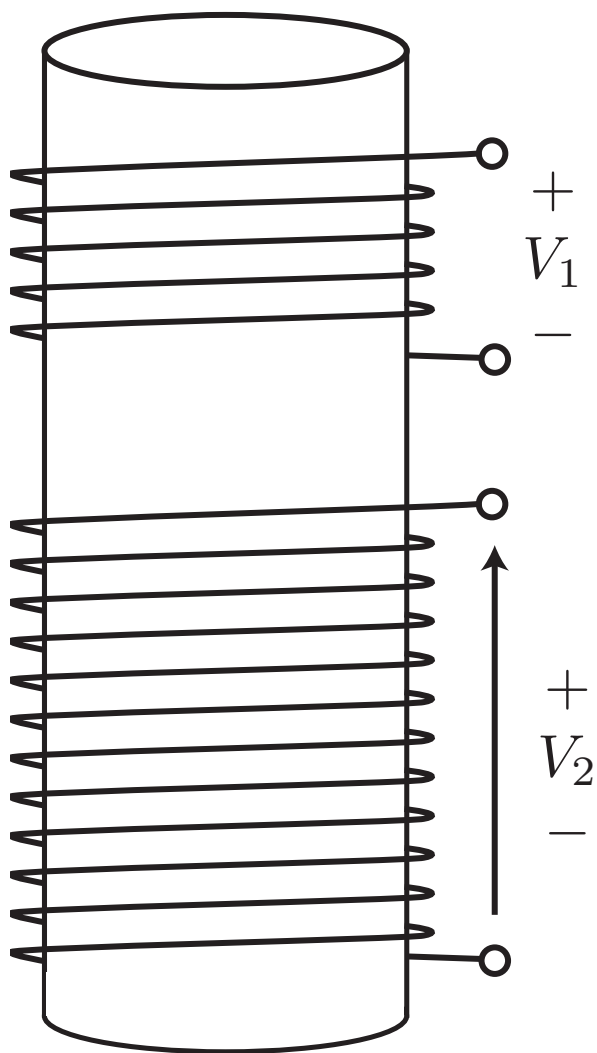
- The transformer is a very important circuit element

- Before switching power supplies, transformers were ubiquitous in voltage/current transformation applications (taking wall voltage of say 120V and converting it to say 3V).

- In fact, the name "trans-former" comes from this very application

Voltage Transformer

$$V_1 = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$V_2 = M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt}$$

- If $I_2 \approx 0$, or for a light load on the secondary, we have

$$V_1 = L_1 \frac{dI_1}{dt}$$

$$V_2 = M \frac{dI_1}{dt}$$

$$\frac{V_2}{V_1} = \frac{M}{L_1} = k \frac{\sqrt{L_1 L_2}}{L_1} = k \sqrt{\frac{L_2}{L_1}} = n$$

Power Transmission

- Transformers are also used to boost the voltage for long range power transmission

- This follows since power loss proportional to $I^2 R$, so to transmit a given power P, it's best to use the largest voltage to minimize the current $I = P/V$.

- This is the reason we use AC power versus DC, since transformers don't work with DC!

Summary So Far...

● Let's summarize what we've learned thus far. There are no magnetic charges, so

$$\nabla \cdot \mathbf{B} = 0$$

and electric fields diverge on physical charge

$$\nabla \cdot \mathbf{D} = \rho$$

Faraday's laws tell us that

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$$

and Ampère's law relate magnetic fields to currents by

$$\nabla \times \mathbf{H} = J$$

Are These Equations Complete?

- Are these equations complete and self-consistent? In other words, do they over-specify the problem or are some equations still missing? Furthermore, are they self-consistent?

- Mathematics tells us that $\nabla \cdot (\nabla \times \mathbf{H}) = 0$, which implies that

$$\nabla \cdot \mathbf{J} = 0$$

- But this can only hold for steady fields. In general, by conservation of charge we know that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Maxwell's Displacement Current

- In other words, we have to add something to the RHS of Ampére's eq. to make it self-consistent!

- Maxwell was the first to make this observation. Since $\nabla \cdot \mathbf{D} = \rho$, it's natural to add a *displacement current* to the Ampére's eq.
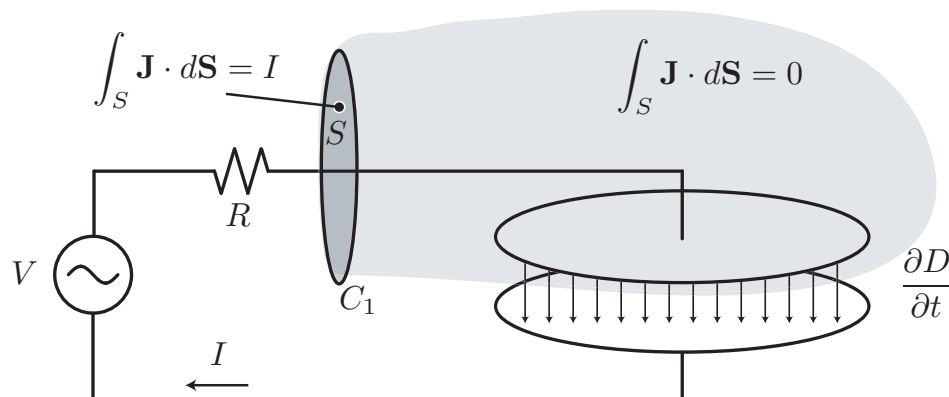
$$\nabla \times \mathbf{H} = J + \frac{\partial \mathbf{D}}{\partial t}$$

- This now makes our eq. self-consistent since

$$\nabla \cdot \nabla \times \mathbf{H} = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial \mathbf{t}}$$

$$\nabla \cdot \nabla \times \mathbf{H} = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}$$

Magnetic Field of a Capacitor

- Now we can resolve a contradiction in Ampére's eq. If we consider the magnetic field of the following circuit, we know that there is a magnetic field around loop C_1 since current cuts through surface S_1

- But Ampére's law says that *any* surface bounded by C_1 can be used to calculate the magnetic field. If we use surface S_2, then the current cutting through this surface is zero, which would yield a zero magnetic field!

Displacement Current of a Capacitor

- The answer to this contradiction is displacement current. If current is flowing in this circuit, then the electric field between the capacitor plates must be changing. Thus $\frac{\partial \mathbf{D}}{\partial t} \neq 0$

- So the displacement current cutting surface S_2 must be the same as the conductive current cutting through surface S_1

$$\int_{S_1} \mathbf{J_c} \cdot d\mathbf{S} = \int_{S_2} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$