EECS 117

Lecture 16: Magnetic Flux and Magnetization

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Magnetic Flux

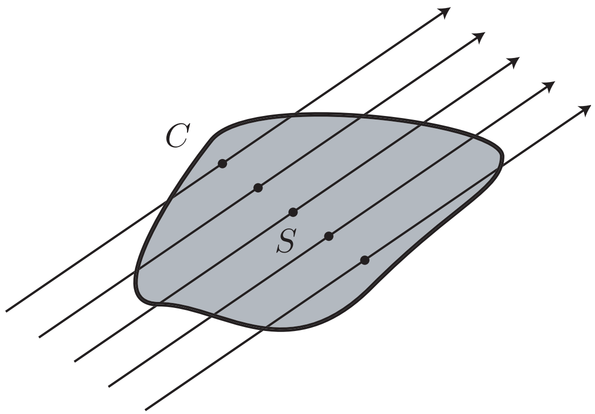
- Magnetic flux plays an important role in many EM problems (in analogy with electric charge)

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

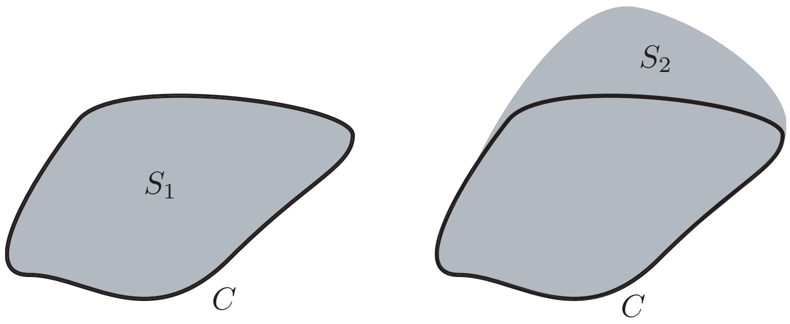
- Due to the absence of magnetic charge

$$\Psi = \oint_S \mathbf{B} \cdot d\mathbf{S} \equiv 0$$

- but net flux can certainly cross an open surface.



Magnetic Flux and Vector Potential

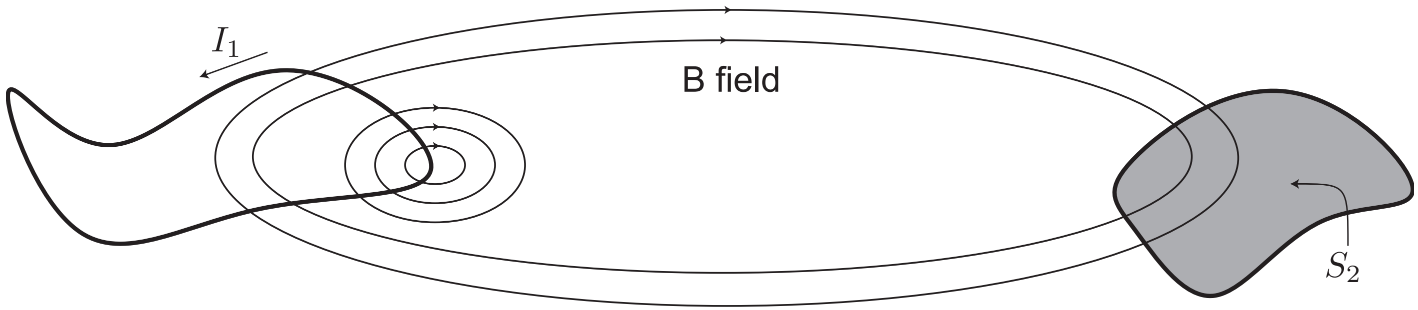


- Magnetic flux is independent of the surface but only depends on the curve bounding the surface. This is easy to show since

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\ell$$

$$\Psi = \int_{S_1} \mathbf{B} \cdot d\mathbf{S} = \int_{S_2} \mathbf{B} \cdot d\mathbf{S}$$

Flux Linkage



- Consider the flux crossing surface S_2 due to a current flowing in loop I_1

$$\Psi_{21} = \int_{S_2} \mathbf{B_1} \cdot d\mathbf{S}$$

- Likewise, the "self"-flux of a loop is defined by the flux crossing the surface of a path when a current is flowing in the path

$$\Psi_{11} = \int_{S_1} \mathbf{B_1} \cdot d\mathbf{S}$$

The Geometry of Flux Calculations

- The flux is linearly proportional to the current and otherwise only a function of the geometry of the path
- To see this, let's calculate Ψ_{21} for filamental loops

$$\Psi_{21} = \oint_{C_2} \mathbf{A_1} \cdot d\ell_2$$

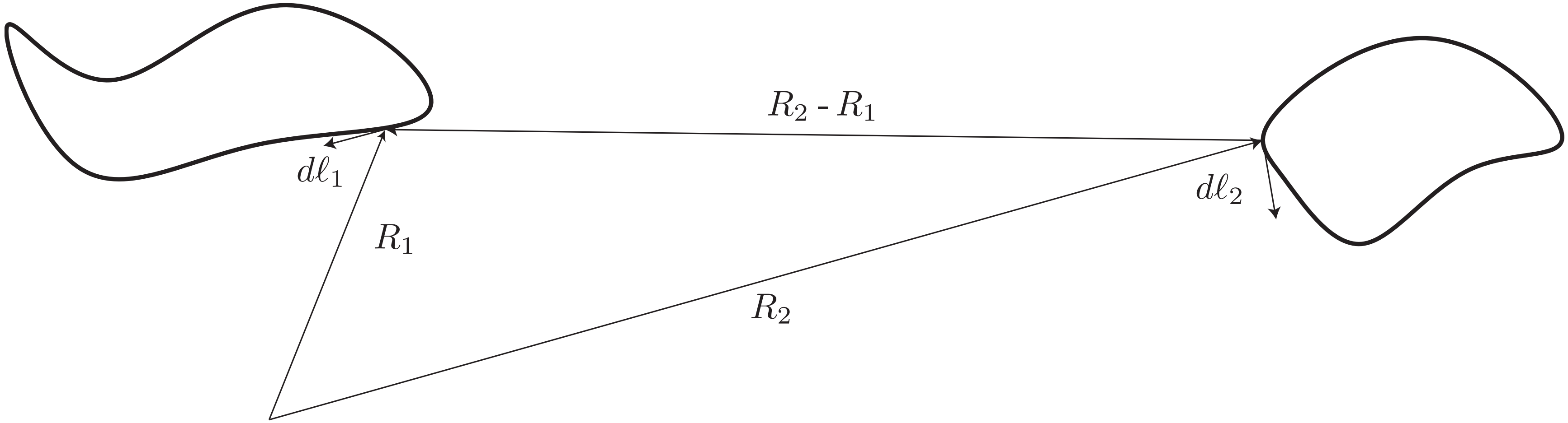
but

$$\mathbf{A_1} = \frac{1}{4\pi\mu_0^{-1}} \oint_{C_1} \frac{I_1 d\ell_1}{R - R_1}$$

substituting, we have a double integral

$$\Psi_{21} = \frac{I_1}{4\pi\mu_0^{-1}} \oint_{C_2} \left(\oint_{C_1} \frac{d\ell_1}{R - R_1} \right) \cdot d\ell_2$$

Geometry of Flux (cont)



- We thus have a simple formula that only involves the magnitude of the current and the average distance between every two points on the loops

$$\Psi_{21} = \frac{I_1}{4\pi\mu_0^{-1}} \oint_{C_2} \oint_{C_1} \frac{d\ell_1 \cdot d\ell_2}{R_2 - R_1}$$

Mutual and Self Inductance

- Since the flux is proportional to the current by a geometric factor, we may write

$$\Psi_{21} = M_{21} I_1$$

- We call the factor M_{21} the mutual inductance

$$M_{21} = \frac{\Psi_{21}}{I_1} = \frac{1}{4\pi\mu_0^{-1}} \oint_{C_2} \oint_{C_1} \frac{d\ell_1 \cdot d\ell_2}{R_2 - R_1}$$

- The units of M are H since $[\mu] = \mathrm{H/m}$.
- It's clear that mutual inductance is reciprocal, $M_{21} = M_{12}$
- The "self-flux" mutual inductance is simply called the self-inductance and donated by $L_1 = M_{11}$

System of Mutual Inductance Equations

- If we generalize to a system of current loops we have a system of equations

$$\begin{aligned}\Psi_1 &= L_1 I_1 + M_{12} I_2 + \ldots M_{1N} I_N \\ &\vdots \\ \Psi_N &= M_{N1} I_1 + M_{N2} I_2 + \ldots L_N I_N\end{aligned}$$

- Or in matrix form $\psi = M\mathbf{i}$, where M is the inductance matrix.
- This equation resembles $\mathbf{q} = C\mathbf{v}$, where C is the capacitance matrix.

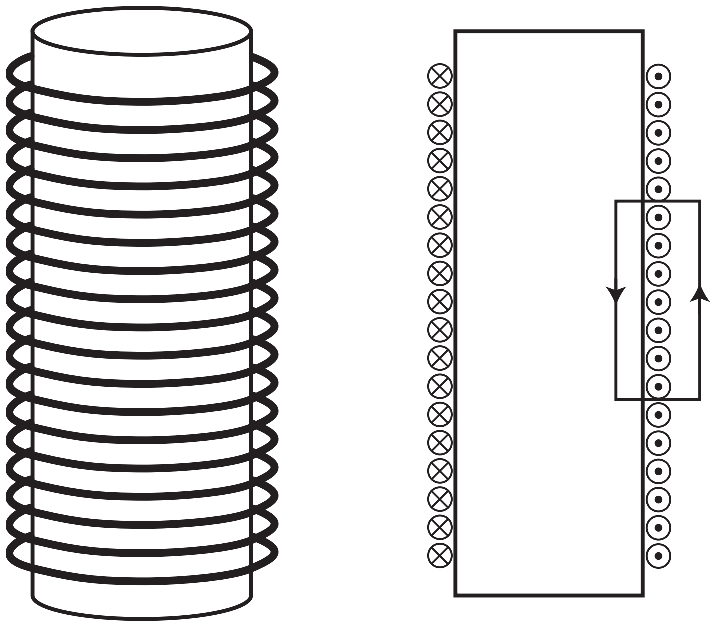
Solenoid Magnetic Field

- We have seen that a tightly wound long long solenoid has $B = 0$ outside and $B_x = 0$ inside, so that by Ampère's law

$$B_y \ell = N I \mu_0$$

- where N is the number of current loops crossing the surface of the path.
- The vertical magnetic field is therefore constant

$$B_y = \frac{N I \mu_0}{\ell} = \mu_0 I n$$



Solenoid Inductance

- The flux per turn is therefore simply given by

$$\Psi_{\text{turn}} = \pi a^2 B_y$$

- The total flux through N turns is thus

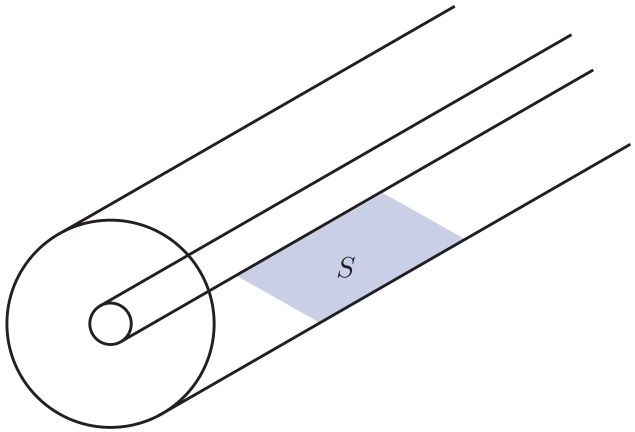
$$\Psi = N\Psi_{\text{turn}} = N\pi a^2 B_y$$

$$\Psi = \mu_0 \frac{N^2 \pi a^2}{\ell} I$$

- The solenoid inductance is thus

$$L = \frac{\Psi}{I} = \frac{\mu_0 N^2 \pi a^2}{\ell}$$

Coaxial Conductor



- In transmission line problems, we need to compute inductance/unit length. Consider the shaded area from $r = a$ to $r = b$
- The magnetic field in the region between conductors if easily computed

$$\oint \mathbf{B} \cdot d\ell = B_\phi 2\pi r = \mu_0 I$$

- The external flux (excluding the volume of the ideal conductors) is given by

$$\psi' = \int_a^b B_\phi dr = \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{b}{a}\right)$$

Coaxial Transmission Line (cont)

- The inductance per unit length is therefore

$$L' = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \;\; [\mathrm{H/m}]$$

- Recall that the product of inductance and capacitance per unit length is a constant

$$L'C' = \frac{1}{c^2}$$

- where c is the speed of light in the medium. Thus we can also calculate the capacitance per unit length without any extra work.

Magnetization Vector

- We'd like to study magnetic fields in magnetic materials. Let's define the magnetization vector as

$$\mathbf{M} = \lim_{\Delta V \to 0} \frac{\sum_k \mathbf{m_k}}{\Delta V}$$

- where $\mathbf{m_k}$ is the magnetic dipole of an atom or molecule
- The vector potential due to these magnetic dipoles is given by in a differential volume dv' is given by

$$d\mathbf{A} = \mu_0 \frac{\mathbf{M} \times \hat{\mathbf{r}}}{4\pi R^2} dv'$$

so

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{M} \times \hat{\mathbf{r}}}{R^2} dv'$$

Vector Potential

- Using

$$\nabla' \left(\frac{1}{R}\right) = \frac{\hat{\mathbf{r}}}{R^2}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \mathbf{M} \times \nabla' \left(\frac{1}{R}\right) dv'$$

- Consider the vector identity

$$\nabla' \times \left(\frac{\mathbf{M}}{R}\right) = \frac{1}{R}\nabla' \times \mathbf{M} + \nabla' \left(\frac{1}{R}\right) \times \mathbf{M}$$

- We can thus break the vector potential into two terms

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R}\right) dv'$$

Another Divergence Theorem

- Consider the vector $\mathbf{u} = \mathbf{a} \times \mathbf{v}$, where $\mathbf{a}$ is an arbitrary constant. Then

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\mathbf{a} \times \mathbf{v}) = (\nabla \times \mathbf{a}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{a} = -(\nabla \times \mathbf{v}) \cdot \mathbf{a}$$

- Now apply the Divergence Theorem to $\nabla \cdot \mathbf{u}$

$$\int_V -(\nabla \times \mathbf{v}) \cdot \mathbf{a} dV = \oint_S ((\mathbf{a} \times \mathbf{v}) \cdot \mathbf{u}) \cdot d\mathbf{S}$$

- Re-ordering the vector triple product

$$-\oint_S (\mathbf{a} \cdot \mathbf{v} \times \mathbf{n}) \cdot d\mathbf{S}$$

Another Divergence Thm (cont)

- Since the vector $\mathbf{a}$ is constant, we can pull it out of the integrals

$$\mathbf{a} \cdot \int_V (-\nabla \times \mathbf{v})\, dV = \mathbf{a} \cdot \oint_S \mathbf{r} \times \mathbf{n} \cdot d\mathbf{S}$$

- The vector $\mathbf{a}$ is arbitrary, so we have

$$\int_V (\nabla \times \mathbf{v})\, dV = -\oint_S \mathbf{r} \times \mathbf{n} \cdot d\mathbf{S}$$

- Applying this to the second term of the vector potential

$$\int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R}\right) dv' = -\oint_S \frac{\mathbf{M} \times \hat{\mathbf{n}}}{R} \cdot d\mathbf{S}$$

Vector Potential due to Magnetization

- The vector potential due to magnetization has a volume component and a surface component

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{M} \times \hat{\mathbf{n}}}{R} \cdot d\mathbf{S} dv'$$

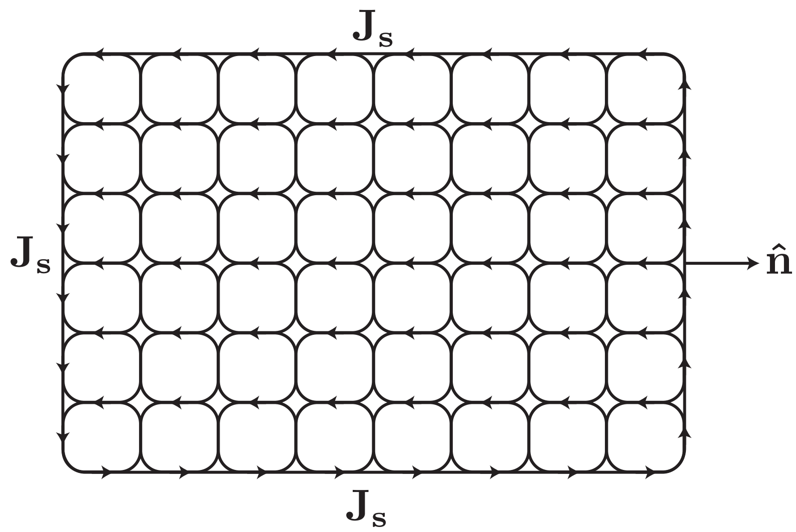
- We can thus define an equivalent magnetic volume current density

$$\mathbf{J_m} = \nabla \times \mathbf{M}$$

- and an equivalent magnetic surface current density

$$\mathbf{J_s} = \mathbf{M} \times \hat{\mathbf{n}}$$

Volume and Surface Currents



- In the figure above, we can see that for uniform magnetization, all the internal currents cancel and only the magnetization vector on the boundary (surface) contributes to the integral

Relative Permeability

- We can include the effects of materials on the macroscopic magnetic field by including a volume current $\nabla \times \mathbf{M}$ in Ampère's eq

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0(\mathbf{J} + \nabla \times \mathbf{M})$$

or

$$\nabla \times \frac{\mathbf{B}}{\mu_\mathbf{0}} - \mathbf{M} = \mathbf{J}$$

- We thus have defined a new quantity $\mathbf{H}$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

- The units of $\mathbf{H}$, the *magnetic field*, are $\mathrm{A/m}$

Ampere's Equation for Media

- We can thus state that for any medium under static conditions

$$\nabla \times \mathbf{H} = \mathbf{J}$$

equivalently

$$\oint_C \mathbf{H} \cdot d\ell = I$$

- Linear materials respond to the external field in a linear fashion, so

$$\mathbf{M} = \chi_m \mathbf{H}$$

so

$$\mathbf{B} = \mu(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$$

or

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

Magnetic Materials

- Magnetic materials are classified as follows
- Diamagnetic: $\mu_r \leq 1$, usually χ_m is a small negative number
- Paramagnetic: $\mu_r \geq 1$, usually χ_m is a small positive number
- Ferromagnetic: $\mu_r \gg 1$, thus χ_m is a large positive number
- Most materials in nature are diamagnetic. To fully understand the magnetic behavior of materials requires a detailed study (and quantum mechanics)
- In this class we mostly assume $\mu \approx \mu_0$