EECS 117

Lecture 14: Static Magnetic Fields

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Experimental Observations

- Consider a pair of parallel wires carrying steady currents I_1 and I_2.
- Last lecture we found that steady currents imply zero net charge distribution. Therefore, there should be no electrostatic force between these current carrying wires.
- But experimentally we do observe a force which tends to be attractive if the currents are in the same direction and repulsive if the currents are in opposite direction.
- This new force is in fact an electrostatic force if we consider the problem from a relativistic point of view!
- Even though the net charge on each current carrying conductor is zero in a static reference frame, in a moving reference frame there is net charge density and hence force.

Magnetic Force

- Through careful observations, Ampère demonstrated that this force can be computed using the following equation

$$dF_m = \frac{\mu_0}{4\pi} \frac{I_2 d\ell_{\mathbf{2}} \times I_1 d\ell_{\mathbf{1}} \times \hat{\mathbf{R}}}{R^2}$$

- The resemblance to the Coulomb force equation is notable. Both forces fall like $1/R^2$.
- For steady currents, $\nabla \cdot \mathbf{J} = 0$ implies that the currents must flow in loops. Thus we can calculate the force between two loops as follows

$$F_m = \oint_{C_1} \oint_{C_2} \frac{\mu_0}{4\pi} \frac{I_2 d\ell_{\mathbf{2}} \times I_1 d\ell_{\mathbf{1}} \times \hat{\mathbf{R}}}{R^2}$$

Magnetic Field

- Just as in the case of electric forces, the concept of "action at a distance" is disturbing and counterintuitive. Thus we prefer to think of the current in loop C_1 generating a "field" and then we say that this field interacts with the current in loop C_2 to generate a force.
- Just reordering the magnetic force equation gives

$$F_m = \oint_{C_2} I_2 d\ell_{\mathbf{2}} \times \underbrace{\frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 d\ell_{\mathbf{1}} \times \hat{\mathbf{R}}}{R^2}}_{\mathbf{B}}$$

- Here loop 2 is the source and loop one is the field point.
- The unit of B is the tesla (T), where $1\mathrm{T} = 10^4\mathrm{G}$, in terms of the CGS units of gauss (G).

Units of Magnetic Field

- The tesla (T) and gauss (G) are derived units.
- Since $F \propto I^2\mu$, the units of μ are simply $\mathrm{N} \cdot \mathrm{A}^{-2}$. This is more commonly known as $\mathrm{H} \cdot \mathrm{m}^{-1}$.
- The units of the magnetic field is therefore

$$[B] = [\mu]\mathrm{A} \cdot \mathrm{m} \cdot \mathrm{m}^{-2} = \mathrm{H} \cdot \mathrm{A} \cdot \mathrm{m}^{-2}$$

- Not that the units of D are $\mathrm{C} \cdot \mathrm{m}^{-2}$, which can be written as $\mathrm{F} \cdot \mathrm{V} \cdot \mathrm{m}^{-2}$
- From circuit theory we know that voltage is proportional to ωLI, so LI has units of $\frac{\mathrm{V}}{\omega}$. So the unit of $[B]$ is $\mathrm{V} \cdot \mathrm{s} \cdot \mathrm{m}^{-2}$
- For reference, the magnetic field of the earth is only $.5\mathrm{G}$, so $1\mathrm{T}$ is a very large field

Direction of Magnetic Force

- Due to the vector cross product, the direction of the force of the magnetic field is perpendicular to the direction of motion and the magnetic field
- Use the right-hand rule to figure out the direction of $\mathbf{F_m}$ in any given situation.

E and B Duality

- For a point charge dq, the electric force is given by

$$\mathbf{F_e} = q\mathbf{E}$$

- The magnetic force for a point charge in a current loop, we have

$$\mathbf{F_m} = Id\ell \times \mathbf{B} = qNd\ell\mathbf{v} \times \mathbf{B}$$

- The equations for $\mathbf{E}$ and $\mathbf{B}$ are also similar when we consider an arbitrary current density J and charge density ρ

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(r')\mathbf{R}}{R^2}dV'$$

$$\mathbf{B} = \frac{1}{4\pi\mu^{-1}}\int_V \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{R}}}{R^2}dV'$$

Magnetic Charge (I)

- We may now compare the magnetic field to the electric field and look for similarity and differences.
- In this class we shall not discuss the relativistic viewpoint that explains the link between electrostatics and the magnetic field. Instead, we shall assume that the magnetic field is an entity of its own.
- Apparently, the source of magnetic field is moving charges (currents) whereas the source of electric fields is charges. But what about magnetic charges? Is there any reason to believe that nature should be asymmetric and give us electrical charge and not magnetic charge?

Magnetic Charge (II)

- If magnetic charge existed, then the argument for Gauss' law would apply

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = Q_m$$

- Where Q_m is the amount of magnetic charge inside the volume V bounded by surface S.
- But no one has ever observed any magnetic charge!
- So for all practical purposes, we can assume that $Q_m \equiv 0$ and so Gauss' law applied to magnetic fields yields

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Divergence of B

- By the divergence theorem, locally this relation translates into

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{B} dV = 0$$

- Since this is true for any surface S, the integrand must be identically zero

$$\nabla \cdot \mathbf{B} = 0$$

- A vector field with zero divergence is known as a solenoidal field
- We already encountered such a field since $\nabla \cdot \mathbf{J} = 0$. Such a field does not have any sources and thus always curls back onto itself. $\mathbf{B}$ fields are thus always loops.

Divergence of Curl

- Let's calculate the divergence of the curl of an arbitrary vector field $\mathbf{A}$, $\nabla \cdot \nabla \times \mathbf{A}$
- Let's compute the volume of the above quantity and apply the divergence theorem

$$\int_V \nabla \cdot \nabla \times \mathbf{A} dV = \oint_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

- To compute the surface integral, consider a new surface S' with a hole in it. The surface integral of $\nabla \times \mathbf{A}$ can be written as the line integral using Stoke's Theorem

$$\int_{S'} \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\ell$$

Divergence of Curl

- Where C is the perimeter of the hole. As we shrink this hole to a point, the right hand side goes to zero and the surface integral turns into the closed surface integral. Thus

$$\int_V \nabla \cdot \nabla \times \mathbf{A} dV = \oint_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = 0$$

- Since this is true for any volume V, it must be that

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

- Thus a solenoidal vector can always be written as the curl of another vector. Thus the magnetic field $\mathbf{B}$ can be written as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Ampère's Law

- One of the fundamental relations for the magnetic field is Amère's law. It is analogous to Gauss' law.
- We can derive it by taking the curl of the magnetic field

$$\nabla \times \mathbf{B} = \nabla \times \frac{\mathbf{1}}{\mathbf{4}\pi\mu^{-\mathbf{1}}} \int_{\mathbf{V}} \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{R}}}{\mathbf{R^2}} \mathbf{dV}'$$

- After some *painful* manipulations (see Appendix B), this can be simplified to Ampère's famous law

$$\nabla \times \mathbf{B} = \mu \mathbf{J}$$

- Now apply Stoke's Theorem

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\ell = \int_S \mu \mathbf{J} \cdot d\mathbf{S} = \mu I$$

Application of Ampère's Law

- Ampère's Law is very handy in situations involving cylindrical symmetry. This is analogous to applying Gauss' law to problems with spherical symmetry
- For example, consider the magnetic field due to a long wire. The field should have no r or z dependence (by symmetry) so the integral of B over a circle enclosing the wire is simply a constant times the perimeter

$$\oint_C \mathbf{B} \cdot d\ell = 2\pi r B_r = \mu I$$

- So the magnetic field drops like $1/r$

$$B(r) = \frac{\mu I}{2\pi r}$$

Magnetic Vector Potential

- Earlier we showed that we can also define a vector $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. Since $\nabla \times \mathbf{B} = \mu \mathbf{J}$

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}$$

- We can apply the vector identity for $\nabla \times \nabla \times \mathbf{A}$

$$\nabla \times \nabla \times \mathbf{A} = \nabla \left(\nabla \cdot \mathbf{A} \right) - \nabla^2 \mathbf{A}$$

- Since the divergence of $\mathbf{A}$ is arbitrary, let's choose the most convenient value. In magnetostatics that is $\nabla \cdot \mathbf{A} = 0$
- Then we have

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

- We have met this equation before!

Equations for Potential

- The vector Laplacian can be written as three scalar Laplacian equations (using rectangular coordinates). For instance, the x-component is given by

$$\nabla^2 A_x = -\mu J_x$$

- By analogy with the scalar potential, therefore, the solution is given by

$$A_x = \frac{\mu}{4\pi} \int_V \frac{J_x(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|}$$

- And in general, the total vector potential is given by

$$\mathbf{A} = \frac{\mu}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|}$$

Why use Vector Potential?

- The vector potential $\mathbf{A}$ is easier to calculate than $\mathbf{B}$ since each component is a simple scalar calculation.
- Also, the direction of $\mathbf{A}$ is easy to determine since it follows $\mathbf{J}$
- At this point the vector potential seems like a mathematical creation. It does not seem to have physical relevance.
- This is compounded by the fact that $\nabla \cdot \mathbf{A}$ is arbitrary!
- Later on, we'll see that $\mathbf{A}$ has a lot of physical relevance and in some ways it's more fundamental than the vector $\mathbf{B}$

From Vector A to B

- Now that we have an equation for $\mathbf{A}$, we can verify that it is indeed consistent with the experimentally observed equation for $\mathbf{B}$

$$\nabla \times \mathbf{A} = \nabla \times \frac{\mu}{4\pi} \int_{\mathbf{V}} \frac{\mathbf{J}(\mathbf{r}')\mathbf{dV}'}{|\mathbf{r} - \mathbf{r}'|}$$

- Interchanging the order of integration and differentiation

$$\nabla \times \mathbf{A} = \frac{\mu}{4\pi} \int_V \nabla \times \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

- We now need some fancy footwork to go further.

Yet Another Vector Identify

- It's relatively easy to show that for a scalar function ϕ times a vector field $\mathbf{F}$

$$\nabla \times \phi \mathbf{F} = \nabla \phi \times \mathbf{F} + \phi \nabla \times \mathbf{F}$$

- Applying this to our case note that ∇ operates on the coordinates $\mathbf{r}$ whereas $\mathbf{J}(\mathbf{r}')$ is a function of the primed coordinates, and hence a constant

$$\nabla \times \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \times \mathbf{J}(\mathbf{r}') = -\frac{\hat{\mathbf{R}}}{|\mathbf{r} - \mathbf{r}'|^2} \times \mathbf{J}(\mathbf{r}')$$

- where we have used

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{\hat{\mathbf{R}}}{|\mathbf{r} - \mathbf{r}'|^2}$$

Back to B

- The expression for B matches up with the experimentally observed equation

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{R}}}{R^2} dV'$$