

# EECS 117

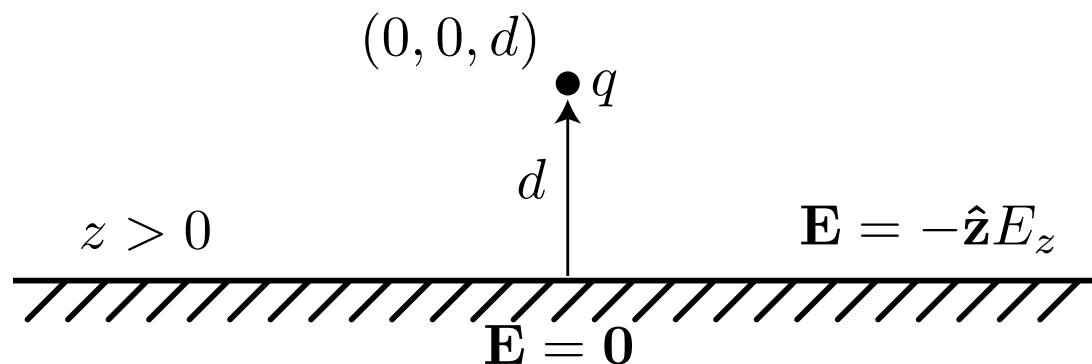
## *Lecture 13: Method of Images / Steady Currents*

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# Point Charge Near Ground Plane

- Consider a point charge  $q$  sitting a distance  $d$  above a conductive ground plane. We'd like to find the electric field  $\mathbf{E}$  everywhere
- Clearly  $\mathbf{E} = 0$  for  $z < 0$  due to the conductor. Also, since  $\rho = 0$  everywhere except for point  $(0, 0, d)$ . Thus we can say that  $\phi$  satisfies Laplace's equation for all points  $z > 0$  except  $(0, 0, d)$  where the point charge lives



# Method of Images

- Consider now a problem we have met before. If we have two charges of equal and opposite magnitude at  $z = \pm d$ , then the solution is trivial. How is this problem related?
- Notice that for this new problem the electric field is normal to the  $xy$  plane at  $z = 0$

$$\mathbf{E}(z = 0) = \frac{q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + d\hat{\mathbf{z}})}{4\pi\epsilon(x^2 + y^2 + d^2)^{3/2}} + \frac{-q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - d\hat{\mathbf{z}})}{4\pi\epsilon(x^2 + y^2 + d^2)^{3/2}} +$$

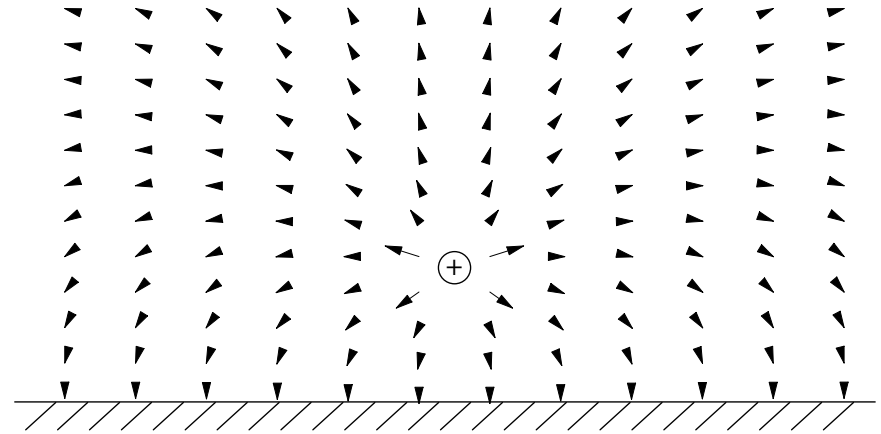
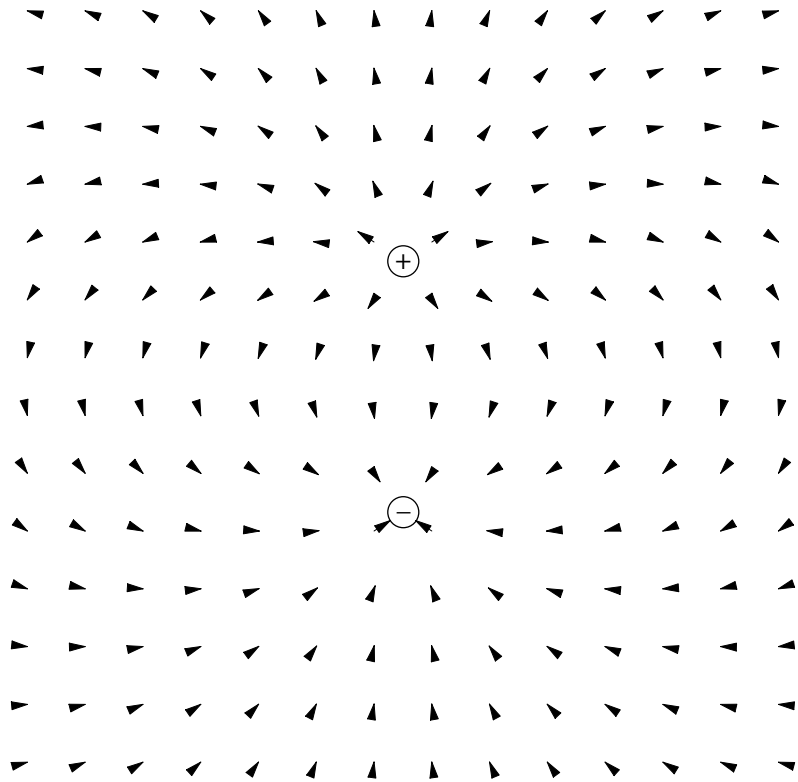
- Combining the terms we have a  $z$ -directed vector

$$\mathbf{E}(z = 0) = \frac{-q2d\hat{\mathbf{z}}}{4\pi\epsilon(x^2 + y^2 + d^2)^{3/2}}$$

# The Image Charge

- Now consider placing a conductive plane at  $z = 0$ . Do the fields need to change? The boundary condition is already satisfied by this field configuration. In other words, this problem satisfies the same boundary conditions as our original problem
- This negative charge is the “image” charge that actually solves our original problem!
- Recall that the solution of Poisson’s equation is uniquely determined by the boundary conditions. Since this image problem satisfies the problem for  $z > 0$  and also the boundary conditions, it’s the solution!
- Of course it’s only the solution for points  $z > 0$  since we know that  $\mathbf{E} = 0$  for  $z < 0$

# E Field Plots



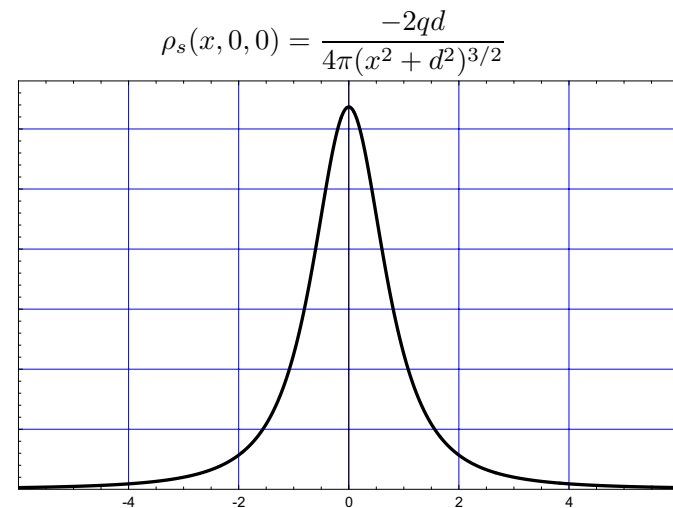
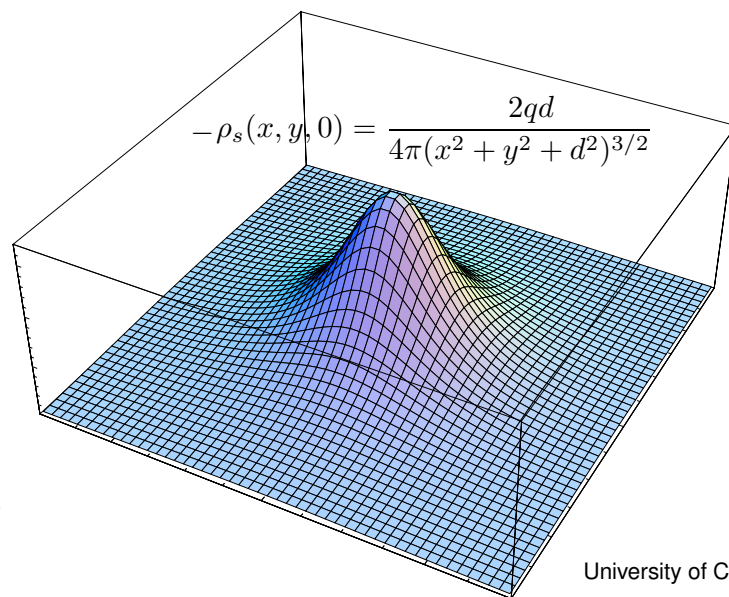
The vector field plot for a dipole distribution is shown on the left. The upper half field plot is also the solution to the monopole charge placed near a ground plane.

# Surface Charge Density

- For a conductive boundary, the normal component of  $D$  is equal to the surface charge density

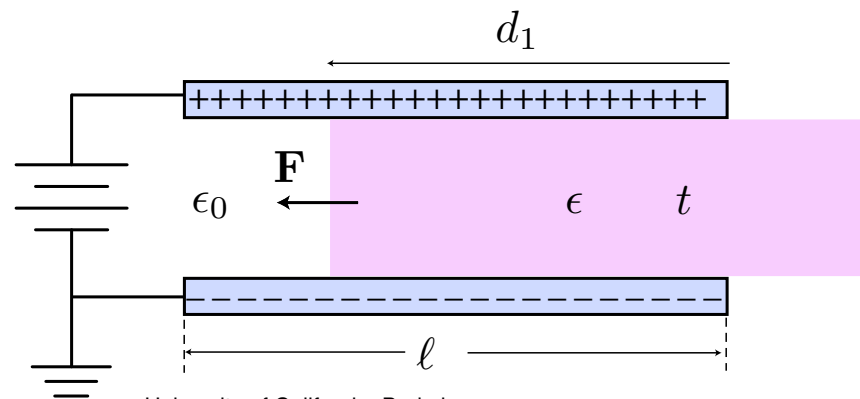
$$\rho_s(x, y, 0) = \hat{\mathbf{z}} \cdot \epsilon \mathbf{E}(z = 0) = \frac{-2qd}{4\pi(x^2 + y^2 + d^2)^{3/2}}$$

- As expected, there is a negative charge density induced on the surface of the plane



# Forces on Dielectrics

- Due to polarization, a dielectric will feel a force when in the presence of a field
- We can use the principle of virtual work to calculate how much work it takes to slightly displace a dielectric in a field. This work must go into the energy of the field
- Example: Consider the force on a dielectric insulator placed part-way into a parallel plate capacitor. Let's assume that the plates of the capacitor are charged to a voltage difference of  $V_0$



# Energy Calculation

- The energy stored in the capacitor volume is (ignoring fringing fields)

$$\begin{aligned} E &= \frac{1}{2}C_1V_0^2 + \frac{1}{2}C_0V_0^2 \\ &= \frac{1}{2}\frac{wd_1\epsilon}{t}V_0^2 + \frac{1}{2}\frac{w(\ell - d_1)\epsilon_0}{t}V_0^2 \end{aligned}$$

- Now displace the dielectric slightly to the right

$$E' = \frac{1}{2}\frac{w(d_1 + \Delta d)\epsilon}{t}V_0^2 + \frac{1}{2}\frac{w(\ell - d_1 - \Delta d)\epsilon_0}{t}V_0^2$$

- The energy of the system has been increased due to the additional amount of material polarization. Does this energy come from us pushing or the battery?



# Energy Change and Force

- The energy difference between the final and initial state is simply

$$\Delta E = E' - E = -\frac{w\Delta dV_0^2}{2t}(\epsilon - \epsilon_0)$$

- If the motion is incremental, we can say that  $\delta E = \delta dF$
- The force is therefore given by

$$F = \frac{2}{2t}V_0^2(\epsilon - \epsilon_0) = \frac{1}{2}wtE_0^2(\epsilon - \epsilon_0)$$

- It's not clear at this point if the force is positive or negative!

# Electrons in Metals

- Electrons are the charge carriers in most metals. They are free to move unimpeded in the crystal at low temperatures. At high temperature the random vibration (phonons) of the positive ion lattice can scatter electrons. Likewise, the presence of impurities can also scatter electrons.
- At thermal equilibrium with no external fields, therefore, the electrons are in motion with random direction and speed. The net velocity of the ensemble of electrons is zero.
- The thermal velocity is quite high ( $\frac{1}{2}mv^2 = \frac{kT}{2}$ ) with  $v_{th} \sim 10^7$  cm/s

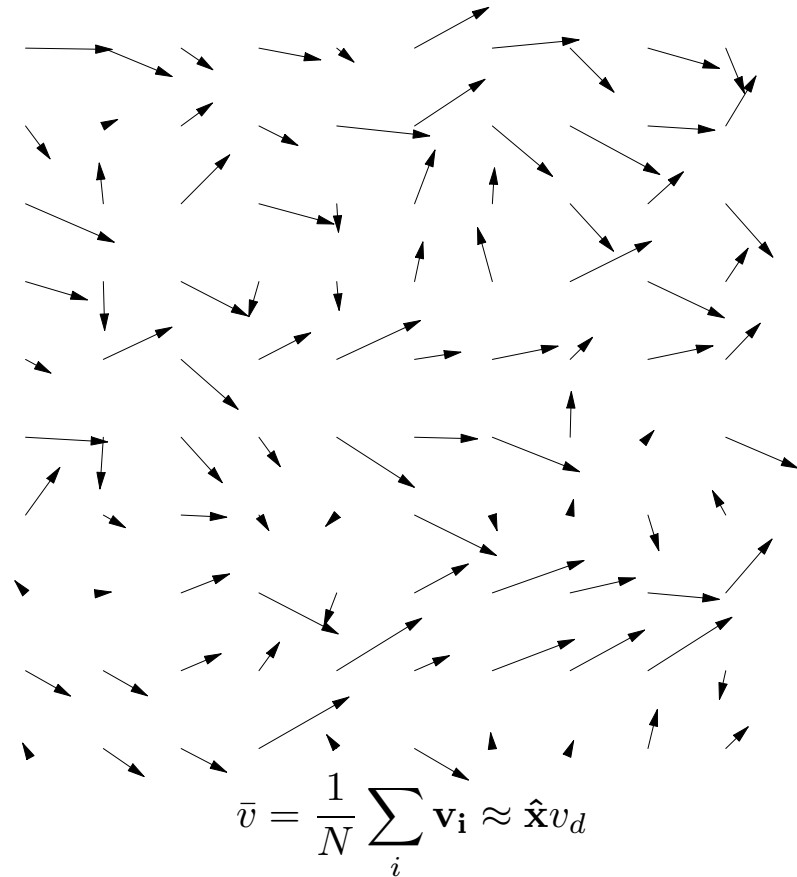
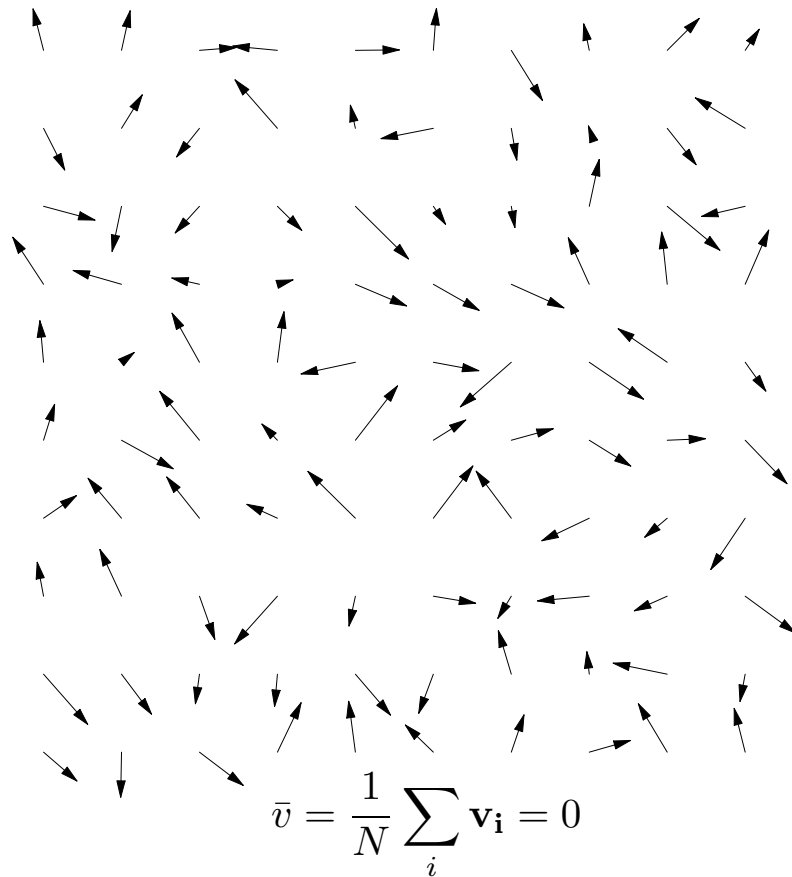
# Drift Velocity

- When a carrier in a conductor is accelerated by an external field, it initially travels in the direction of the field. But after scattering with an impurity or phonon, the electron's new trajectory is random and independent of the field
- Let's suppose that the average time between collisions is  $t_c$ . Then in this period the momentum of electrons is increased by  $qE$  due to the external field,  $mv_d/t_c = qE$ .
- The average velocity of carriers increases proportional to the field  $E$

$$v_d = \frac{qt_c}{m} E = \mu E$$

- The constant  $\mu$  is the mobility and the “drift” velocity  $v_d$  is so named because most of the motion is still random as  $v_d \ll v_{th}$

# Electron Drift



In the left figure, electrons are moving with random velocity (net motion is zero). On the right figure, electrons are moving with random velocity plus a drift component to the right (drift has been exaggerated).

# Definition of Current

- Current is defined as the amount of charge crossing a particular cross sectional area per unit time. Consider a density of  $N$  carriers carrying a charge  $q$  and moving with drift velocity  $v_d$ . How many cross a perpendicular surface  $A$ ?
- In a time  $\Delta t$ , carriers move a distance of  $v_d \Delta t$ . So any carriers in the volume  $A v_d \Delta t$  will cross the surface if they are all moving towards it. The current is therefore

$$I = \frac{A v_d \Delta t N q}{\Delta t} = v_d N q A$$

- The vector current density  $\mathbf{J}$  is therefore

$$\mathbf{J} = N q \mathbf{v}_d$$

# Conductivity

- Since we found that the velocity is proportional to the field, it follows that  $J$  is also proportional

$$\mathbf{J} = Nq\mathbf{v}_d = Nq\mu\mathbf{E}$$

- The constant  $\sigma = \mu qN$  is known as the conductivity of the material. It varies over several orders of magnitude for conductors and dielectrics
- This is in fact a statement of Ohm's law  $V = RI$  in microscopic form. Often we work with the resistivity of the material  $\rho = \sigma^{-1}$
- Since  $\sigma$  is *very* easy to measure, we can indirectly calculate  $t_c$  for a material. For a good conductor  $\sigma \sim 10^7 \text{S/m}$ .

# Ohm's Law

- We assert that the ratio between the voltage and current in a conductor is always constant. We see this stems from  $J = \sigma E$

$$I = \int_A \mathbf{J} \cdot d\mathbf{S} = \int_A \sigma \mathbf{E} \cdot d\mathbf{S}$$

$$V = \int_C \mathbf{E} \cdot d\ell$$

- The ratio is called the resistance

$$R = \frac{V}{I} = \frac{\int_C \mathbf{E} \cdot d\ell}{\int_A \sigma \mathbf{E} \cdot d\mathbf{S}}$$

# Resistance

- The integrals over  $E$  depend only on the geometry of the problem and scale with voltage
- Equivalently, if we increase  $V$ , the flux lines for  $E$  remain the same but of a higher magnitude. Thus the current  $J$  is the same and only  $I$  increases by the same factor that  $V$  increases.
- Example: Block of conductor. The electric field will be shown to be everywhere constant and parallel to the conductor

$$V = E_0 \ell$$

$$I = A \sigma E_0$$

$$J = \sigma E_0 = \frac{I}{A}$$

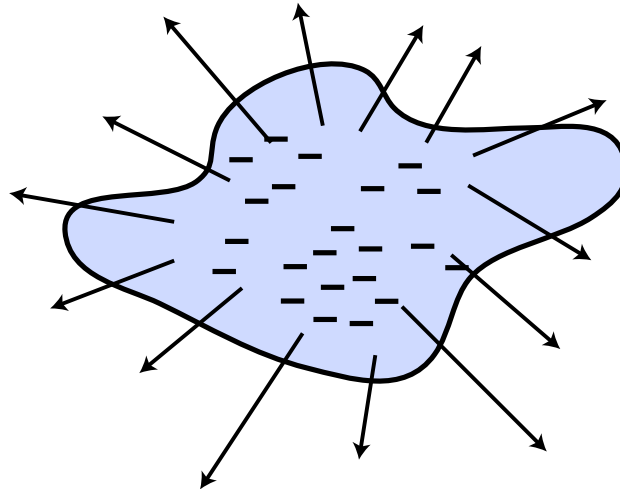
$$R = \frac{V}{I} = \frac{\ell}{\sigma A}$$



# Charge Conservation

- One of the fundamental facts of nature is charge conservation. This has been experimentally observed and verified in countless experiments.
- Therefore if we look at a volume bounded by surface  $S$ , if there is net current leaving this volume, it must be accompanied by a charge pile-up in the region of opposite polarity

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_V \rho dV$$



# Current Continuity

- We can transform this simple observation by employing the Divergence Theorem

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{J} dV$$

- For any volume we have found that

$$\int_V \nabla \cdot \mathbf{J} dV - \frac{\partial}{\partial t} \int_V \rho dV = \int_V \left( \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dV = 0$$

- If this is true for any volume  $V$ , it must be true that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

# Excess Charge

- Since  $\mathbf{J} = \sigma \mathbf{E}$ , we have

$$\nabla \cdot \sigma \mathbf{E} + \frac{\partial \rho}{\partial t} = 0$$

- If the conductivity is constant, we have

$$\sigma \nabla \cdot \mathbf{E} + \frac{\partial \rho}{\partial t} = 0$$

- It is experimentally observed that Gauss' law is satisfied in all reference frames of constant velocity so  $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$

$$\frac{\sigma}{\epsilon} \rho + \frac{\partial \rho}{\partial t} = 0$$

# Excess Charge (cont)

- If we assume that  $\rho$  is a function of time only, the equation

$$\frac{\sigma}{\epsilon}\rho + \frac{d\rho}{dt} = 0$$

- Let  $\tau = \sigma/\epsilon$  and solve the above equation

$$\rho = \rho_0 e^{-\frac{t}{\tau}}$$

- Thus the charge density in a conductor decays exponentially to zero with a rate determined by the time constant  $\tau = \sigma/\epsilon$ , or the relaxation time of the material
- Earlier we deduced that conductors have zero volume charge density under electrostatic conditions. Now we can estimate the time it takes to reach this equilibrium

# Relaxation Time for Good Conductors

- In fact, a good conductor like copper has  $\sigma \sim 10^8 S/m$
- It's hard to measure the dielectric constant of a good conductor since the effect is masked by the conductive response. For  $\epsilon \sim \epsilon_0$ , we can estimate the relaxation time

$$\tau \sim \frac{10 \times 10^{12}}{10^8} s \sim 10^{-19} s$$

- This is an *extremely* short period of time!
- How do electrons move so fast? Note this result is independent of the size of the conductor.
- Answer: They don't, only a slight displacement in the charge distribution can balance the net charge and thus transfer it to the border region.