

EECS 117

Lecture 12: Dielectric Materials

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Review of Dipoles

- An electric dipole is two equal and opposite charges q separated by a distance d
- We derived that the potential due to an electric dipole at points far removed from the dipole $r \gg d$

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon} \left(\frac{1}{r^+} - \frac{1}{r^-} \right) \approx \frac{-qd \cos \theta}{4\pi\epsilon r^2}$$

- We can define a vector dipole moment $\mathbf{p} = q\mathbf{d}$, where \mathbf{d} is a vector connecting the positive and negative charge
- In terms of \mathbf{p} , we can rewrite the above in a coordinate independent manner

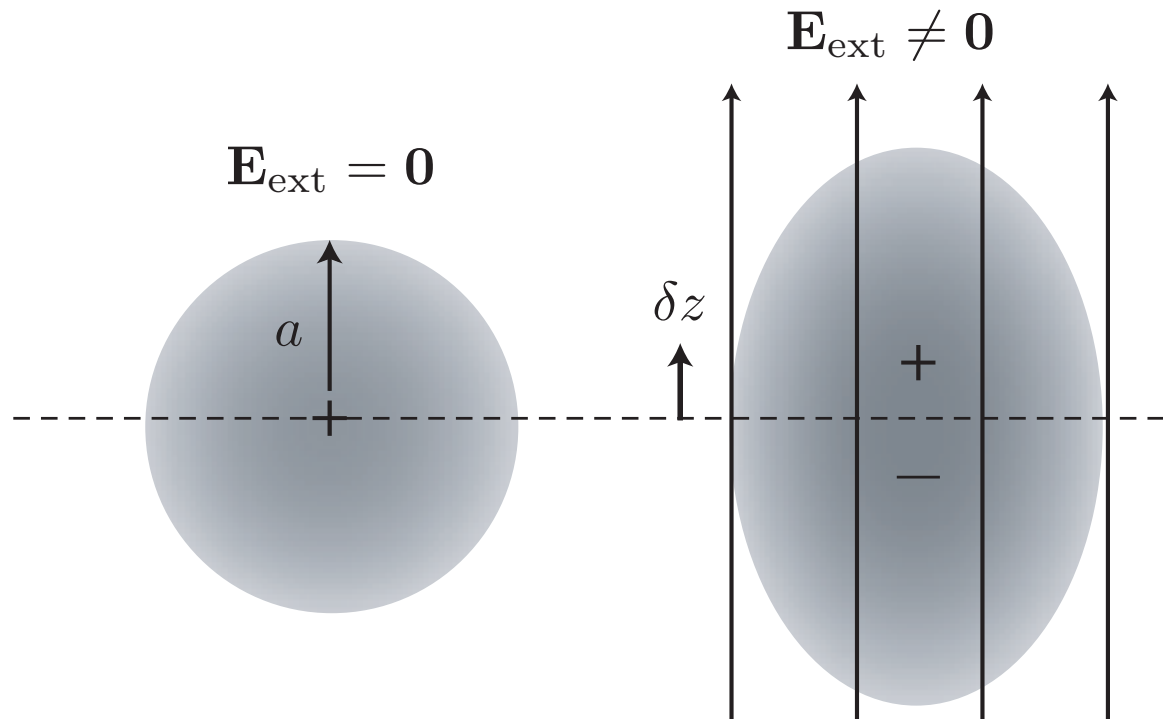
$$\phi(\mathbf{r}) = \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon r^2}$$

Properties of Insulators

- So far we have only considered electrostatics in vacuum and with ideal conductors. We now expand our discussion to include insulators.
- A perfect insulators cannot conduct DC current because there are no mobile charges. All electrons in the material are bound to the nucleus.
- But there is a response to an external field due to the dielectric properties of the material

Electronic Polarization

- For a spherically symmetric atom and electron cloud distribution, for instance, the time average dipole moment is zero. But the application of an external field can distort the electronic charge distribution to produce a net dipole



Electronic Polarization (II)

- It's easy to see why this happens. The electrons are bound to the nucleus through the internal force of the atom. This force is much stronger than our applied field and so the probability that an electron will obtain sufficient energy from the field to leave the atom is negligible.
- But the average electronic distribution is the minimum energy configuration. If the electrons displace slightly from their equilibrium positions, then the new configuration yields lower overall energy in the presence of a field

Internal Field of an Atom

- It's important to note that the displacement $\Delta z \ll r_0$ (r_0 is the Bohr radius). This is because the *internal* field of an atom is already very strong

$$E_{\text{int}} \sim \frac{e}{4\pi\epsilon r_0^2} \sim 10^{11} \text{V/m}$$

- This is several orders of magnitude larger than any field we can impart onto the exam externally.
- To first order, then, the response is linear with the field. This is because for a small displacement, any function looks linear

$$f(x + \Delta x) \approx f(x) + \Delta x f'(x)$$

Atomic Dipole Moment

- So based on the fact that $E_{\text{ext}} \ll E_{\text{int}}$, we expect that the induced dipole moment in the atom to be small

$$p = e\Delta z$$

- But since there are a lot of atoms, the net effect can be large
- The constant of proportionality is known as the electronic polarizability coefficient α_e

$$\mathbf{p} = \alpha_e \mathbf{E}$$

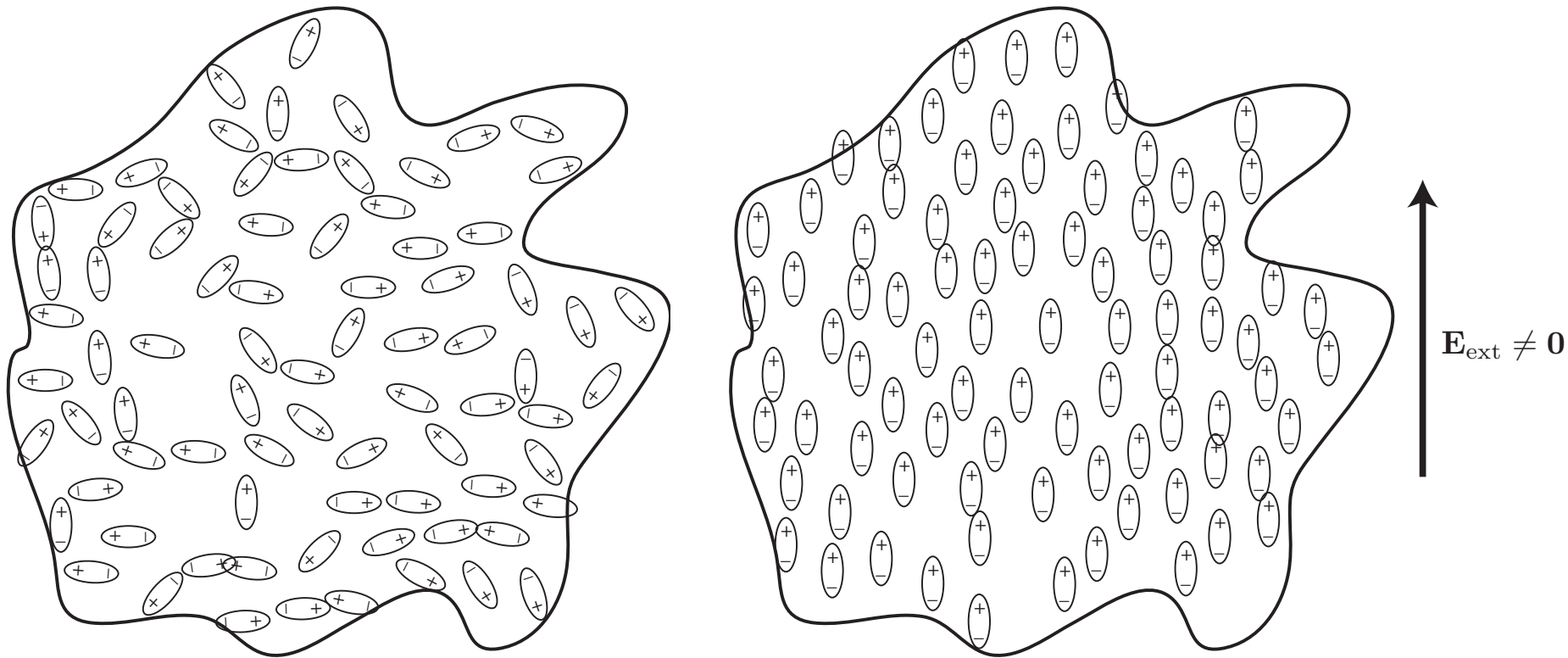
- Typically $\alpha_e \sim 10^{-40} \text{Fm}^2$

Ionic Polarizability

- For more complex molecules, like CH_4 , there can be larger scale polarization. Instead of each individual atom getting polarized, the shared covalent bond electrons can re-arrange and produce a larger polarization
- In other cases, the molecules may have built-in polarization due to ionic or covalent bonds.
- Water is a good example. Normally the dipole moments are pointing in random directions so the net polarization for $E_{\text{ext}} = 0$.

Permanent Dipole Moments

- But the application of an external field can align the dipoles. The response can be huge due to the large number of molecules involved.



Dipole Moment Alignment

- Unfortunately the effect of thermal energy tends to re-randomize the orientation so we expect that on average only a small fraction of the dipoles are aligned.
- In fact, one can show that the probability that a molecule to be aligned is related to the energy of a dipole in an external field. The energy is proportional to p^2 and so the fraction of polarized molecules depends on the ratio of p^2 and $3k_B T$

$$\alpha_0 = \frac{p^2}{3k_B T}$$

- Typically $\alpha_0 \sim 10^{-38} \text{Fm}^2$

Total Polarization

- In general, the total polarization arises from different sources (induced and permanent). If the density of polarization is N per volume, the net polarization is

$$\mathbf{P} = N\alpha\mathbf{E} = \epsilon_0\chi_e\mathbf{E}$$

- χ_e is known as the electric susceptibility
- We are now in a position to derive the potential due polarized matter. Let's rewrite the potential due to a single dipole

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = \frac{-1}{4\pi\epsilon_0} \mathbf{p} \cdot \nabla \frac{1}{r}$$

- Notice that $\nabla \frac{1}{r} = \nabla \mathbf{r}^{-1} = \hat{\mathbf{r}} \frac{\partial r^{-1}}{\partial r} = -\frac{\hat{\mathbf{r}}}{r^2}$

Potential due to Polarization

- Integrating over all dipoles we have

$$\phi(\mathbf{r}) = \frac{-1}{4\pi\epsilon_0} \int_V \mathbf{P} \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV = \frac{1}{4\pi\epsilon_0} \int_V \mathbf{P} \cdot \nabla' \frac{1}{|r - r'|} dV$$

- Note that $\nabla R^{-1} = -\nabla' R^{-1}$ where $R = |r - r'|$

- Using the identity $\nabla \cdot (a\mathbf{A}) = a\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla a$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int_V \nabla' \cdot \frac{\mathbf{P}}{|r - r'|} dV - \int_V \frac{\nabla' \cdot \mathbf{P}}{|r - r'|} dV \right)$$

- By divergence theorem we can recast this into

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\oint_S \frac{\mathbf{P} \cdot \hat{\mathbf{n}}}{|r - r'|} dS - \int_V \frac{\nabla' \cdot \mathbf{P}}{|r - r'|} dV \right)$$

Effective Volume and Surface Charge

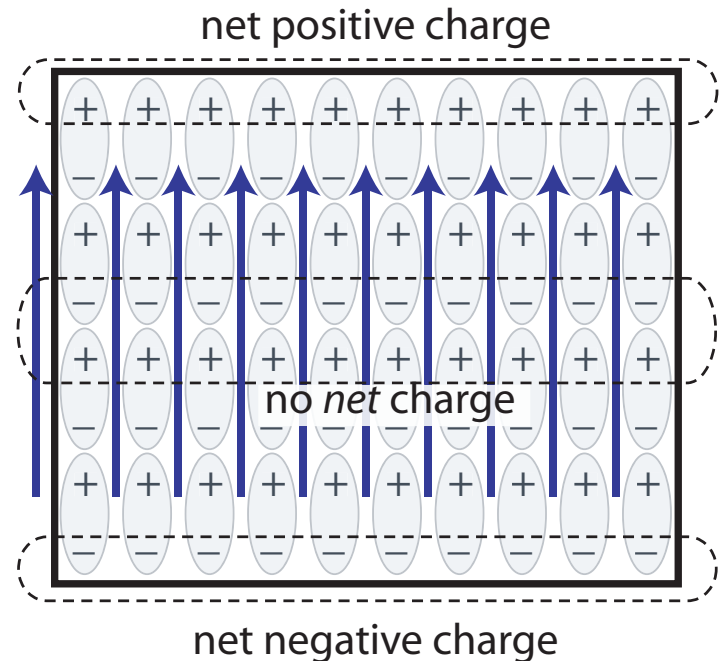
- Notice that these integrals look like an ordinary potential calculation for surface charge and volume charge
- Define a surface charge density $\rho_s = \mathbf{P} \cdot \hat{\mathbf{n}}$ and a volume charge density $\rho_v = -\nabla' \cdot \mathbf{P}$. Then we have

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\oint_S \frac{\rho_s}{|\mathbf{r} - \mathbf{r}'|} dS + \int_V \frac{\rho_v}{|\mathbf{r} - \mathbf{r}'|} dV \right)$$

- There must be a good explanation for this coincidence!

Intuitive Picture

- If the material is uniform and the external field is uniform, then the polarization is also uniform and $\nabla' \cdot \mathbf{P} = 0$. The interpretation is that all the internally induced polarizations cancel out
- The dipoles on the boundary, though, remain unbalanced and thus produce a net charge
- If the polarization is non-uniform, though, then $\nabla' \cdot \mathbf{P} \neq 0$ and a volume polarization arises



Dielectric Boundary



- By the same token, if there is a discontinuity in the material, one would expect surface charge to occur at the boundary.
- Imagine the interface of two materials with $\epsilon_1 > \epsilon_2$. Since material one has a stronger polarization, it has more bound charge which is imbalanced at the interface. This means there is a discontinuity in the electric field.

Back to Gauss' Law

- Since the bound charges can create net charge, we must include their contribution in Gauss Law

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_{\text{free}} + \rho_{\text{bound}}$$

- Where $\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$. Substituting in the above equation

$$\nabla \cdot \epsilon_0 \mathbf{E} + \nabla \cdot \mathbf{P} = \rho_{\text{free}}$$

- Grouping terms we have (where ρ is now understood to be free charge)

$$\nabla \cdot \epsilon_0 \mathbf{E} + \mathbf{P} = \rho_{\text{free}} = \rho$$

Electric Flux Density Vector

- We can re-define the electric flux density

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

- so that it depends only on free charge

$$\nabla \cdot \mathbf{D} = \rho$$

- Since the polarization is proportional to \mathbf{E}

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

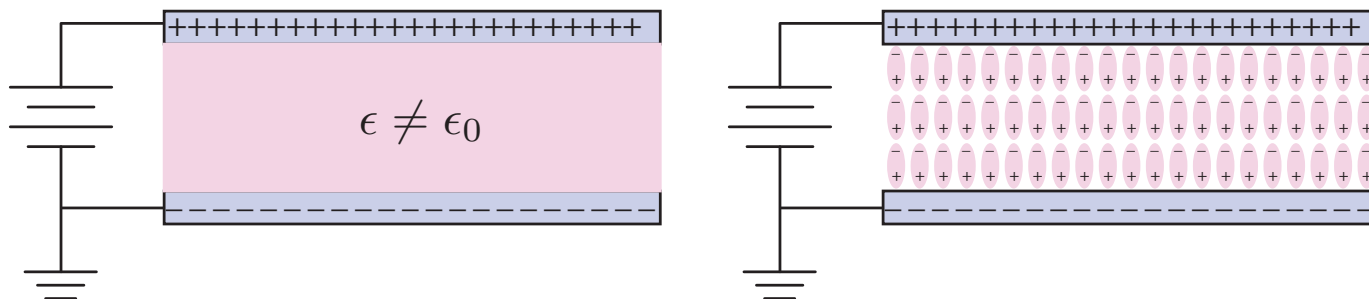
- The dielectric constant captures the polarization effects in a macroscopic coefficient $\epsilon = \epsilon_r \epsilon_0 = \epsilon_0 (1 + \chi_e)$

Dielectric Materials

- One important application for a dielectric is to insulate conductors from one another (coaxial transmission line) while providing mechanical support
- In a capacitor there is the additional benefit that the capacitance increases. The capacitance increases because the effect of the polarization is to produce a net charge density at the top and bottom of the plates that subtracts from the net charge on the conductors

Parallel Plate Capacitor with Dielectric

- So for a fixed applied field (voltage), the parallel plate capacitor can support more charge
- Another perspective is if we fix the amount of charge on the plates, then the field in the dielectric is reduced and hence the applied electric field is smaller



Boundary Conditions: Normal D

- Applying Gauss' law to the interface of two materials

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = (\hat{\mathbf{n}} \cdot \mathbf{D}_1)\Delta S - (\hat{\mathbf{n}} \cdot \mathbf{D}_2)\Delta S = \rho_s \Delta S$$

- Thus the discontinuity in the normal field must be due to surface charge

$$\mathbf{D}_{1n} - \mathbf{D}_{2n} = \rho_s$$

- For dielectrics, there is no “free” surface charge and so $\rho_s = 0$

$$\hat{\mathbf{n}} \cdot \mathbf{D}_1 = \hat{\mathbf{n}} \cdot \mathbf{D}_2$$

- Or $D_{1n} = D_{2n}$. The normal electric field is discontinuous
 $\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$

Boundary Conditions: Tangential E

- Take the line integral around the interface. Since the field is static, the line integral along any closed path is identically zero

$$\int_C \mathbf{E} \cdot d\ell = (E_{1t} - E_{2t})\Delta w = 0$$

- This means that the tangential field is always continuous, $E_{1t} = E_{2t}$. Then the tangential component of D is *not* continuous

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$