EECS 117

Lecture 10: Laplace's Eq, Curl, and Capacitance

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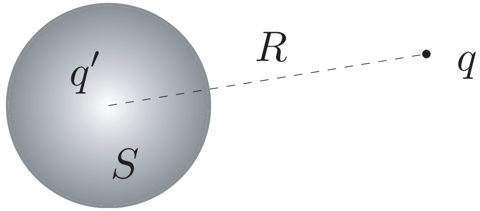
Laplace's Equation

$$\nabla^2 \psi = 0$$

- If ψ satisfies Laplace's eq., then the average value of ψ over a sphere is equal to the value of ψ at the center of the sphere.

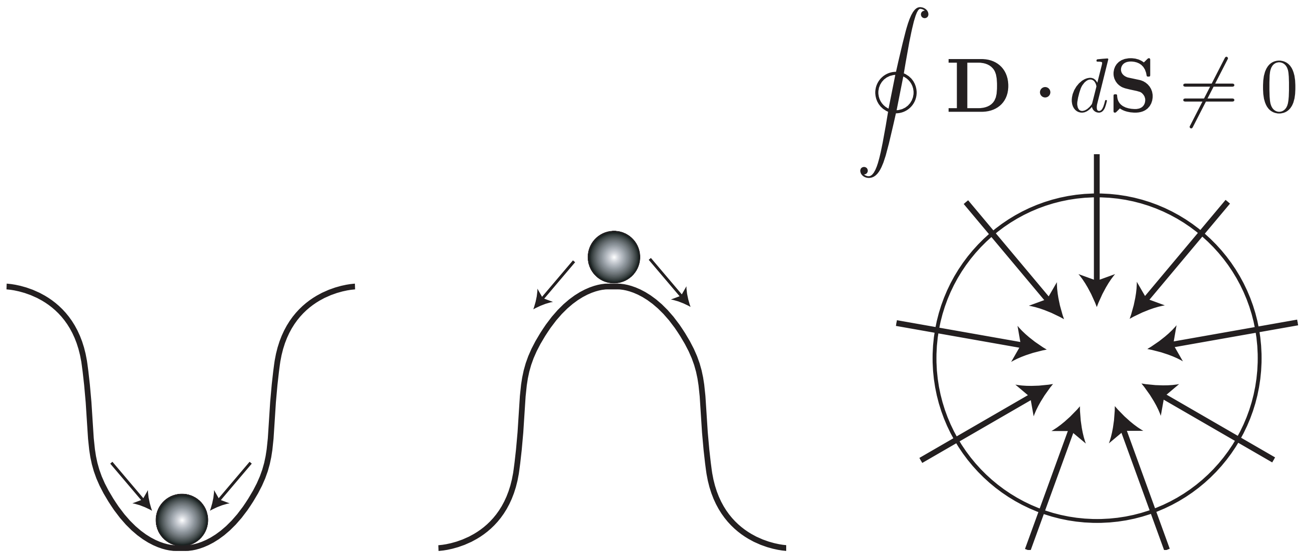


- Proof: We'll use a physical argument to arrive at this result. Consider a point charge q and a spherical surface with uniform charge q' distributed on the surface at a distance R from the point charge. If q is brought in from infinity, the work done is simply $W = \frac{qq'}{4\pi\epsilon R}$

Average Value Property

- On the other hand, if we bring in the sphere from infinity the work required is certainly the same. This work is the average value of the potential due to q computed over the surface S. This work is also $W = \frac{qq'}{4\pi\epsilon R}$ and thus $\bar{\psi} = \frac{q}{4\pi\epsilon R}$. But that's the potential at the center of the sphere due to q.
- This has several interesting consequences. One is that there is no stable electrostatic configuration in empty space. To see this observe that the potential of such a point must be local minima or maxima and a function that has the average value property cannot have a maxima/minima.

Instability of Static Field in Vacuum

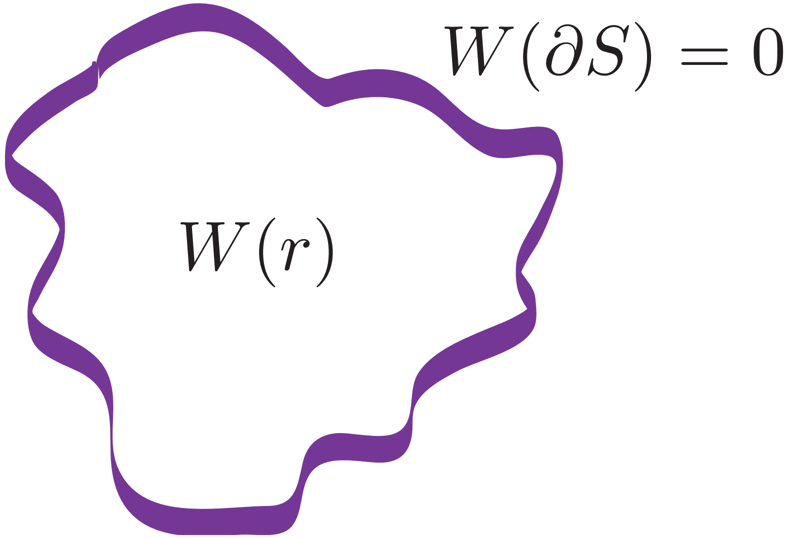


Another argument from Guass' law also confirms this fact. For such a stable point to exists all the field lines would have to either point into (out of) this region. But Gauss' law says that such a region must have charge and so it's not empty space.

Uniqueness Argument

- The solution of Laplace's equation is unique. The proof is simple using the average value property.
- Assume that there is more than one solution to Laplace's equation. Call two such solutions $\psi(r)$ and $\varphi(r)$ and form the difference solution $W(r) = \psi(r) - \varphi(r)$. Due to linearity, W also satisfies the Laplacian equation but with different boundary conditions.

Uniqueness Argument (cont)

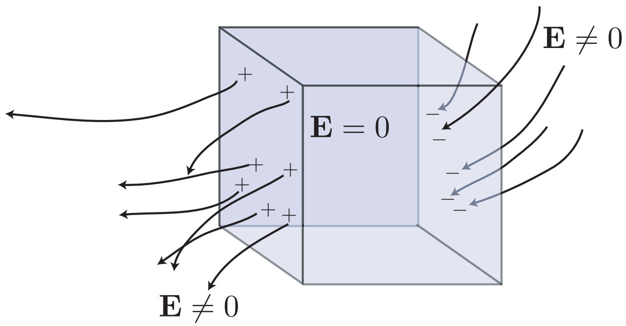


In particular, $W(r)$ is the solution to the problem with zero boundary conditions. But if W is zero on the boundary and non-zero in the interior of the problem space, then clearly W must have maximum or minimum point. But such a function cannot be a solution to Laplace's equation.

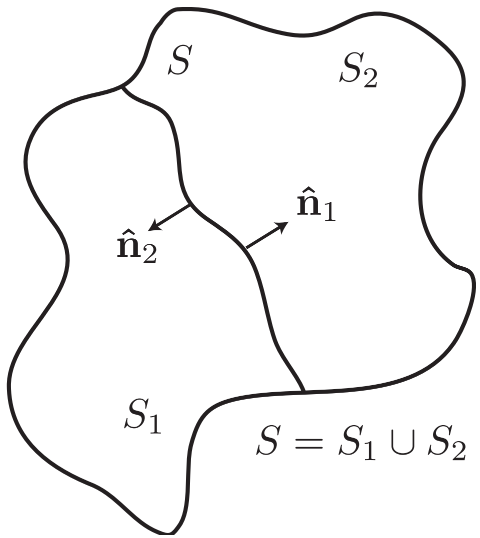
Faraday Cage

Consider the solution of Laplace's equation for a Faraday cage. This is a closed region bounded by a conducting walls. It's easy to now show that $E \equiv 0$ inside this region.

To see this notice that the solution of Laplace's equation in this region must satisfy $\phi = \text{constant}$ on the boundary. But the function $\phi(r) = \text{constant}$ does in fact satisfy Laplace's equation and the boundary conditions. Since the solution is unique, this is indeed the solution we seek. Thus the electric field is zero since $\mathbf{E} = -\nabla\phi$.



Divergence Theorem Mini-Proof



- When calculating the flux of an arbitrary function about a surface S, we can always break the sum into a subset of surfaces that make up S

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \sum_i \int_{S_i} \mathbf{D} \cdot dS$$

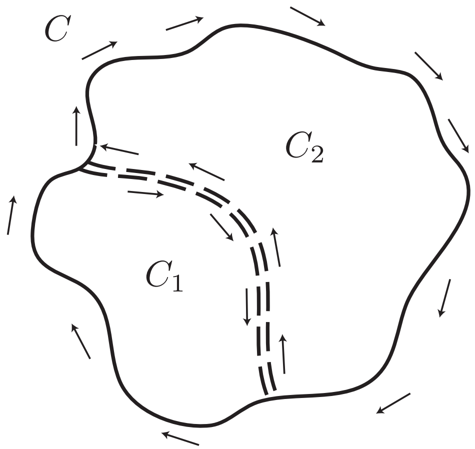
Definition of Curl

- If we simply now divide and multiply the integrand by the volume ΔV of the closed surface S_i and take limits, we have the divergence theorem

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \sum_i \frac{\int_{S_i} \mathbf{D} \cdot dS}{\Delta V_i} \Delta V_i \rightarrow \int_V \nabla \cdot \mathbf{D} dV$$

- We can use a similar argument to arrive at an appropriate definition of curl. Consider the path integral of an arbitrary vector function about a closed loop C. We can always compute this by taking sub-paths C_i since the internal contribution of the integral will cancel out in the integral.

Curl (cont)



$$\oint_C \mathbf{E} \cdot d\ell = \sum_i \oint_{C_i} \mathbf{E} \cdot d\ell$$

- Since $C = C_1 + C_2 + \ldots$. Notice that the surfaces of each sub curve ΔS_i can have *any* shape.
- Let's now divide and multiply by this surface area ΔS_i

$$\oint_C \mathbf{E} \cdot d\ell = \sum_i \frac{\oint_{C_i} \mathbf{E} \cdot d\ell}{\Delta S_i} \Delta S_i$$

Curl as Vector

- This is almost a surface integral! We just need to specify the direction for ΔS_i to make sure that in the limit it's coincident with the normal vector $\hat{\mathbf{n}}$ for surface S.
- In fact, we can define the curl to be this particular integral evaluated about a surface in a particular plane defined by it's normal $\hat{\mathbf{n}}$

$$\hat{\mathbf{n}} \cdot \mathrm{curl}(\mathbf{E}) = \lim_{\Delta S \to 0} \frac{\oint_C \mathbf{E} \cdot d\ell}{\Delta S}$$

- We write this as $\nabla \times \mathbf{E}$ and treat it as a vector.

Curl Vector Components

- To be a true vector, then it must obey vector laws. In particular if we compute

$$\hat{\mathbf{x}} \cdot \nabla \times \mathbf{E} = C_x$$

$$\hat{\mathbf{y}} \cdot \nabla \times \mathbf{E} = C_y$$

$$\hat{\mathbf{z}} \cdot \nabla \times \mathbf{E} = C_z$$

- And now desire to compute the curl about a new direction defined by normal vector $\hat{\mathbf{a}} = \alpha_1 \hat{\mathbf{x}} + \alpha_2 \hat{\mathbf{y}} + \alpha_3 \hat{\mathbf{z}}$, then we must have

$$\hat{\mathbf{a}} \cdot \nabla \times \mathbf{E} = \alpha_1 C_x + \alpha_2 C_y + \alpha_3 C_z$$

Stoke's Theorem

- Returning to our original equation, if we take limits we have

$$\oint_C \mathbf{E} \cdot d\ell = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S}$$

- For a static field we found that for any closed path

$$\oint_C \mathbf{E} \cdot d\ell \equiv 0$$

- That implies that $\nabla \times \mathbf{E} \equiv 0$ for a static field.

Capacitance

- Capacitance of a conductor is defined as

$$C \triangleq \frac{q}{\phi}$$

- where a larger C means that an object can store more charge at a fixed potential. Hence it's a measure of capacity to store charge.
- As an example consider an isolated spherical conductor of radius a. By Gauss' law we know the radial field is given by ($r \geq a$)

$$D_r = \frac{q}{4\pi a^2}$$

Capacitance of a Sphere

- If we integrate this field to arrive at the potential, we have

$$\phi = -\int_{\infty}^{a} E_r dr = -\frac{q}{4\pi\epsilon}\int_{\infty}^{a}\frac{dr}{r^2} = \frac{q}{4\pi\epsilon}\left(\frac{1}{r}\right|_{\infty}^{a} = \frac{q}{4\pi\epsilon a}$$

- The capacitance is therefore

$$C = 4\pi\epsilon a \propto a$$

- The larger sphere can hold more charge at a fixed potential. Equivalently, dumping a given charge onto a larger sphere will increase it's potential less than a smaller sphere.

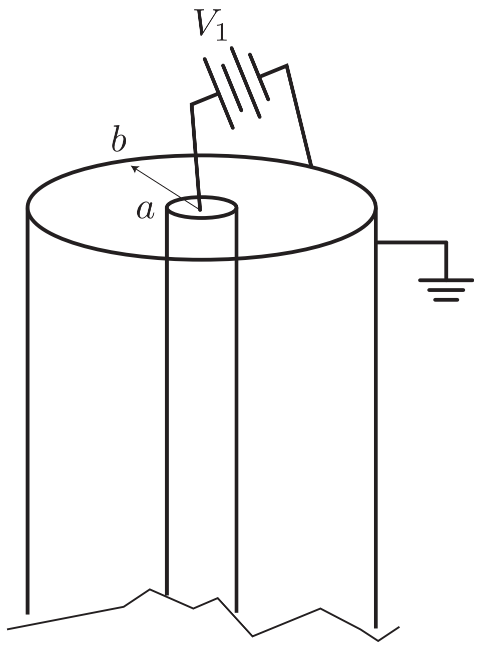
Capacitance Between Objects

- We can also define the capacitance between two objects as

$$C_{12} = \frac{q_1}{\phi_{12}}$$

- Where we fix the potential between the two objects at ϕ_{12} and measure the amount of charge transfer between the objects. In the above equation a charge of q_1 has been transfered from object 2 to 1 and therefore a charge of $q_2 = -q_1$ will reside on object 2.
- We can re-interpret the capacitance of a single object as the capacitance relative to a reference at infinity.

Capacitance of a Coaxial Cylinder



- Let’s practice and use Poisson’s equation to find the potential in the region between the conductors

$$\nabla^2 \phi = 0$$

Coaxial Cylinder (II)

- By symmetry ϕ is only a function of r and not a function of θ or z

$$\nabla \cdot \nabla \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = 0$$

$$r\frac{\partial \phi}{\partial r} = C_1$$

$$\frac{\partial \phi}{\partial r} = \frac{C_1}{r}$$

- The general solution is therefore $\phi(r) = C_1 \ln r + C_2$.

Coaxial Cylinder (III)

- This solution must satisfy the boundary conditions that $\phi(a) = V_1$ and $\phi(b) = 0$.

$$\phi(r = a) = V_1 = C_1 \ln a + C_2$$

$$\phi(r = b) = 0 = C_1 \ln b + C_2$$

- Solving these equations we have

$$\phi(r) = \frac{V_1 \ln r/b}{\ln a/b}$$

- The field is given by $\mathbf{E} = \hat{\mathbf{r}} E_r = -\hat{\mathbf{r}} \frac{\partial \phi}{\partial r}$

$$\mathbf{E} = \frac{\hat{\mathbf{r}} V_1}{r \ln b/a}$$

Charge Density

- Recall that $\mathbf{D} \cdot \hat{\mathbf{n}} = \rho_s$ on the surface of conductors. Therefore for a length of ℓ of the coaxial conductors

$$\frac{\epsilon V_1}{a \ln(b/a)} = \frac{q}{2\pi a \ell}$$

- Solve for the charge to find the capacitance

$$q = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \ell V_1$$

- The capacitance per unit length is therefore

$$C' = \frac{C}{\ell} = \frac{2\pi\epsilon}{\ln \frac{b}{a}}$$

References

- “Electricity and Magnetism,” by Edward Purcell (second edition) published by McGraw-Hill Book Company (1985).