Module 1-2: LTI Systems

Prof. Ali M. Niknejad

LTI Definition

- System is *linear* (studied thoroughly in 16AB):

- System is *time invariant*:
  - There is no “clock” or time reference
  - The transfer function is not a function of time
  - It does not matter when you apply the input. The transfer function is going to be the same …

Linear Systems

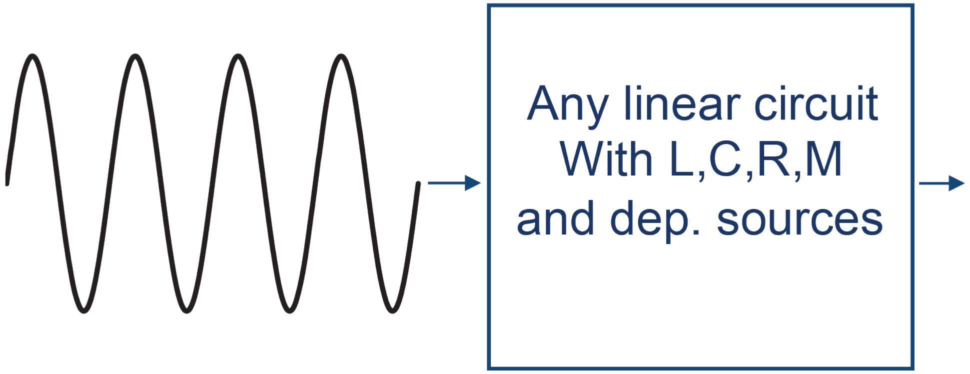
- Continuous time linear systems have a lot in common with finite dimensional linear systems we studied in 16AB:
  - Linearity:
  - Basis Vectors → basis functions:
  - Superposition:
  - Matrix Representation → Integral representation:

Linear Systems (cont)

- Eigenvectors → eigenfunctions
- Orthonormal basis
- Eigenfunction expansion
- Operators acting on eigenfunction expansion

LTI Systems

- Since most periodic (non-periodic) signals can be decomposed into a summation (integration) of sinusoids via Fourier Series (Transform), the response of a LTI system to virtually any input is characterized by the frequency response of the system:

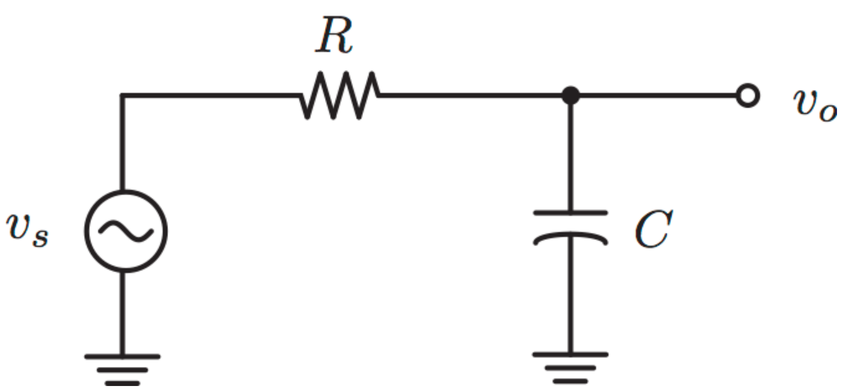


Example: Low Pass Filter (LPF)

- Input signal: $v_s(t) = V_s \cos(\omega t)$
- We know that: $v_o(t) = \underbrace{K \cdot V_s}_{V_0} \cos(\omega t + \phi)$

Phase shift

Amp shift



$$v_0(t) = v_s(t) - i(t)R$$

$$i(t) = C\frac{dv_0}{dt}$$

$$v_0(t) = v_s(t) - RC\frac{dv_0}{dt}$$

$$v_s(t) = v_0(t) + \tau\frac{dv_0}{dt}$$

LPF the "hard way" (cont.)

- Plug the known form of the output into the equation and see if it can satisfy KVL and KCL

$$V_s \cos \omega t = V_0 \cos(\omega t + \phi) - \tau \omega V_0 \sin(\omega t + \phi)$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$V_s \cos \omega t = V_0 \cos \omega t (\cos \phi - \tau \omega \sin \phi) - V_0 \sin \omega t (\sin \phi + \tau \omega \cos \phi)$$

- Since sine and cosine are linearly independent functions:

$$a_1 \sin \omega t + a_2 \cos \omega t = 0$$

IFF $a_1 \equiv a_2 \equiv 0$

LPF: Solving for response…

- Applying linear independence

$$-V_0 \sin\phi - V_0 \tau\omega \cos\phi = 0$$

$$V_0 \cos\phi - V_0 \tau\omega \sin\phi - V_s = 0$$

$$\tan\phi = -\tau\omega$$

Phase Response:

$$\phi = -\tan^{-1}\tau\omega$$

$$V_0(\cos\phi - \tau\omega\sin\phi) = V_s$$

$$V_0 \cos\phi(1 - \tau\omega\tan\phi) = V_s$$

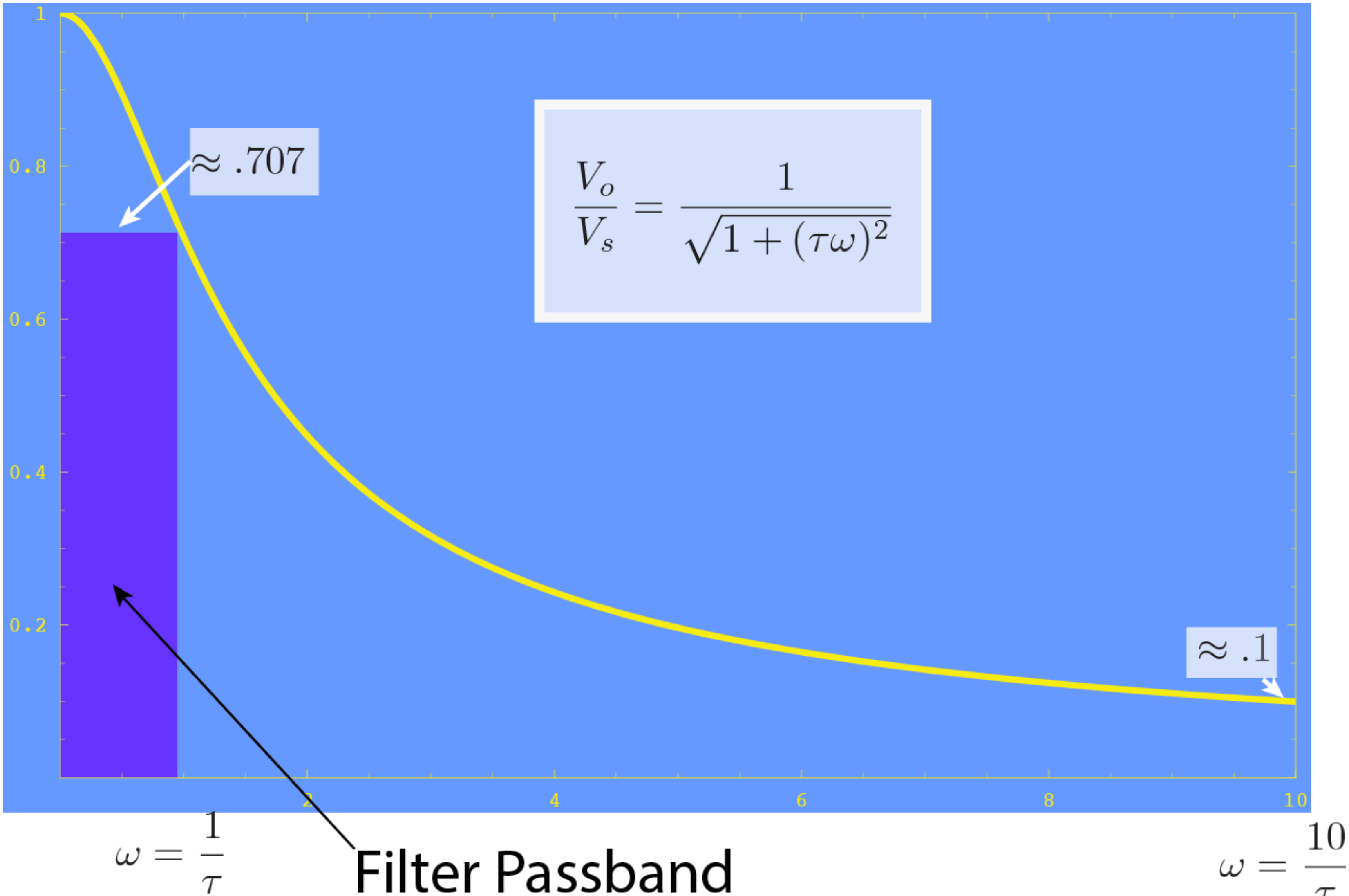
$$V_0 \cos\phi(1 + (\tau\omega)^2) = V_s$$

$$V_0\left(1 + (\tau\omega)^2\right)^{1/2} = V_s$$

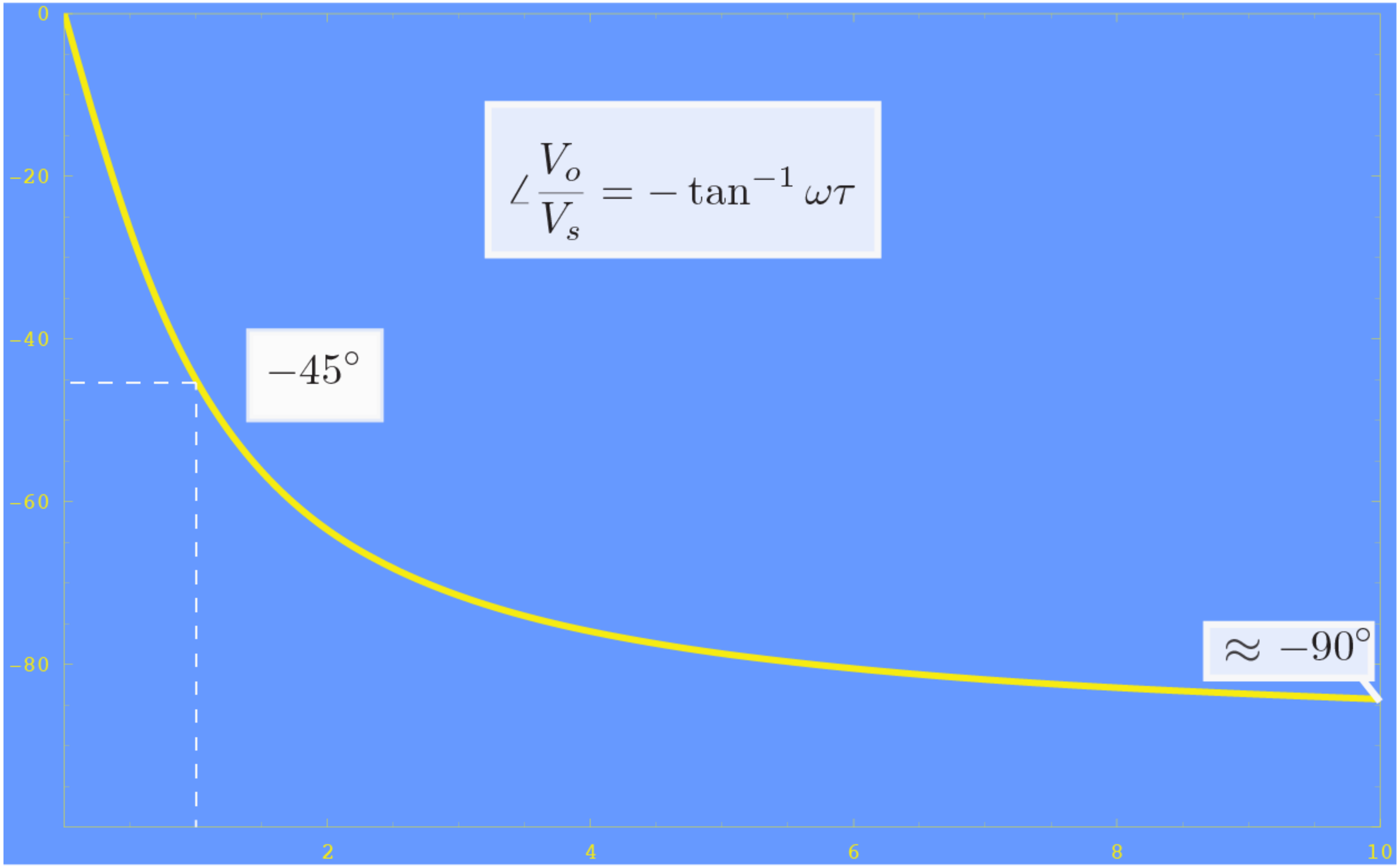
Amplitude Response:

$$\frac{V_0}{V_s} = \frac{1}{\sqrt{1 + (\tau\omega)^2}}$$

LPF Magnitude Response



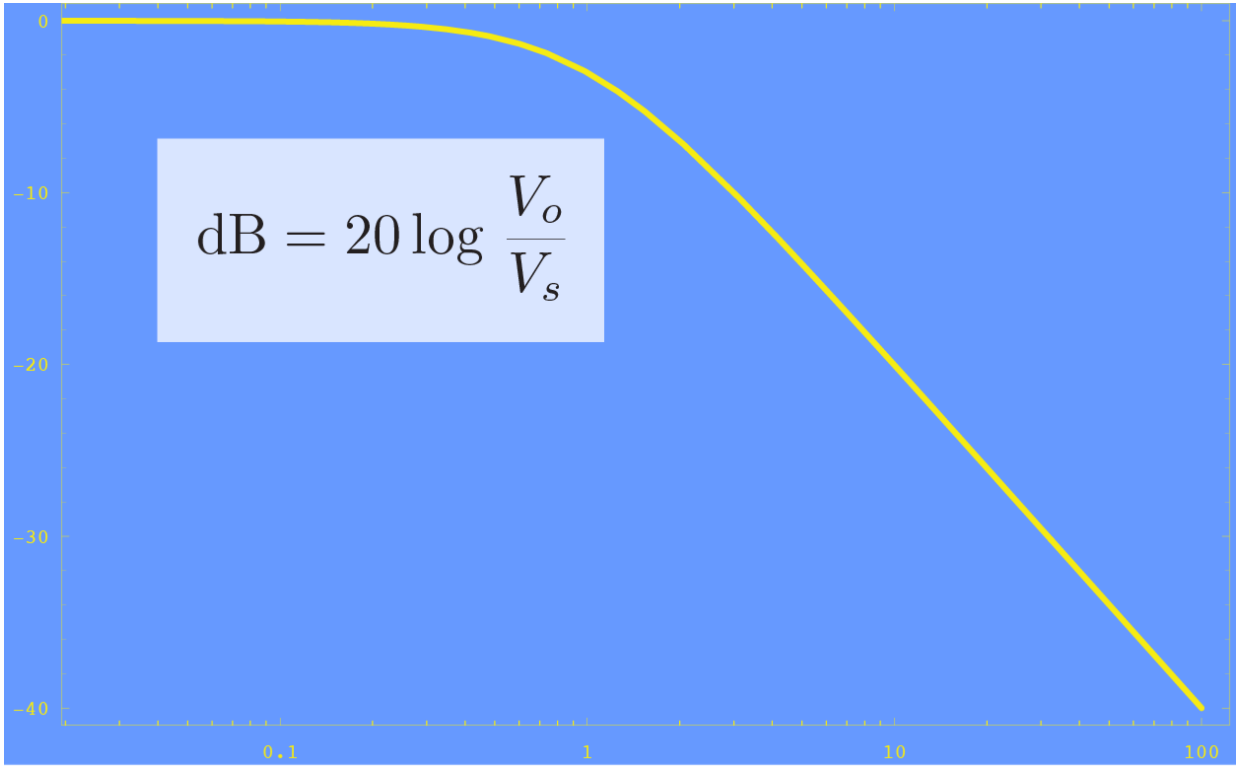
LPF Phase Response



$\omega = \frac{1}{\tau}$ $\omega = \frac{10}{\tau}$

dB: Honor the inventor of the phone…

- The LPF response quickly decays to zero
- We can expand range by taking the log of the magnitude response
  - dB = deciBel (deci = 10)



$$\mathrm{dB} = 20\log\frac{V_o}{V_s}$$

Why 20? Power!

- Why multiply log by "20" rather than "10"?
- Power is proportional to voltage squared:

$$dB = 10\log\left(\frac{V_0}{V_s}\right)^2 = 20\log\left(\frac{V_0}{V_s}\right)$$

- At breakpoint: $\omega = 1/\tau \rightarrow \left(\frac{V_0}{V_s}\right)_{\text{dB}} = -3\,\text{dB}$

$$\omega = 100/\tau \rightarrow \left(\frac{V_0}{V_s}\right)_{\text{dB}} = -40\,\text{dB}$$

$$\omega = 1000/\tau \rightarrow \left(\frac{V_0}{V_s}\right)_{\text{dB}} = -60\,\text{dB}$$

- Observe: slope of signal attenuation is 20 dB/decade in frequency

Why introduce complex numbers?

- They actually make things easier
- One insightful derivation of e^{ix}
- Consider a second order homogeneous DE

$$y'' + y = 0$$

$$y = \begin{cases} \sin x \\ \cos x \end{cases}$$

- Since sine and cosine are linearly independent, any solution is a linear combination of the "fundamental" solutions

Insight into Complex Exponential

- But note that e^{ix} is also a solution!
- That means: $e^{ix} = a_1 \sin x + a_2 \cos x$
- To find the constants of prop, take derivative of this equation:

$$i\, e^{ix} = -a_2 \sin x + a_1 \cos x$$

- Now solve for the constants using both equations:

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} e^{ix} \\ i\, e^{ix} \end{pmatrix}$$

$$A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = b \qquad \det A = -1 \neq 0$$

Complex Exponential

- Eulor's Theorem says that

$$e^{jx} = \cos x + j \sin x$$

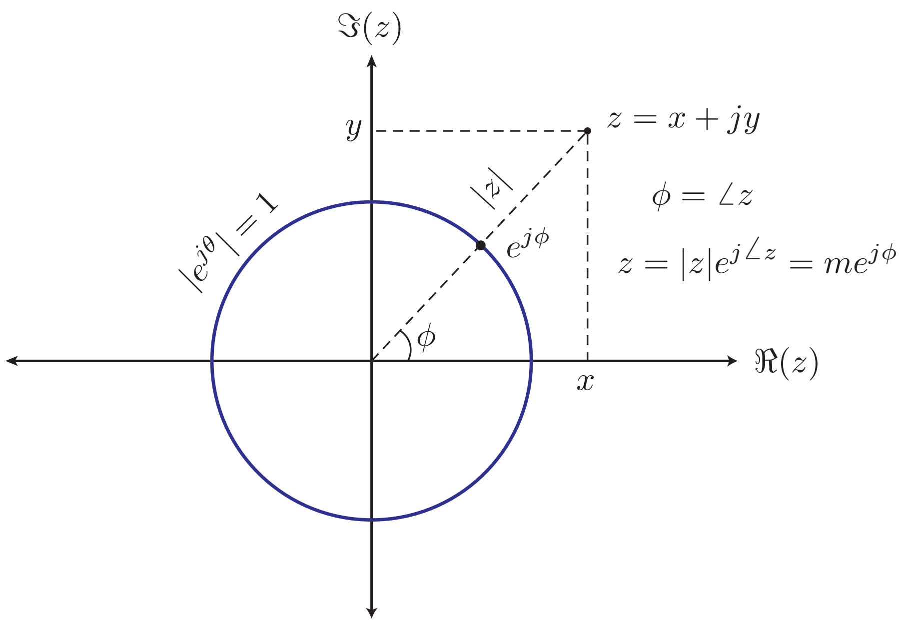
- This can be derived by expanding each term in a power series.
- If take the magnitude of this quantity, it's unity

$$|e^{jx}| = \sqrt{\cos^2 x + \sin^2 x} = 1$$

- That means that $e^{j\phi}$ is a point on the unit circle at an angle of ϕ from the x-axis.

Any complex number z, expressed as have a real and imaginary part $z = x + jy$, can also be interpreted as having a magnitude and a phase. The magnitude $|z| = \sqrt{x^2 + y^2}$ and the phase $\phi = \angle z = \tan^{-1} y/x$ can be combined using the complex exponential

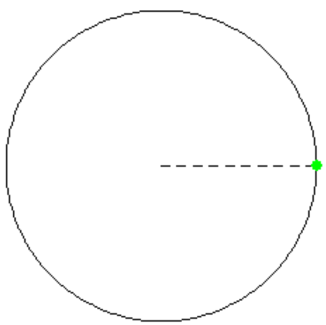
$$x + jy = |z| e^{j\phi}$$



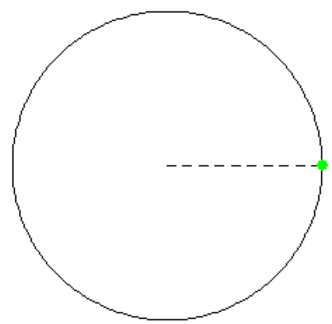
The Rotating Complex Exponential

- So the complex exponential is nothing but a point tracing out a unit circle on the complex plane:

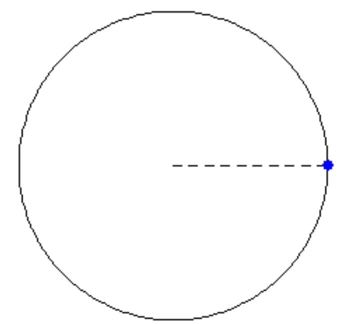
$$e^{ix} = \cos x + i \sin x$$



$$e^{i\omega t}$$



$$e^{-i\omega t}$$



$$\frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

Magic: Turn Diff Eq into Algebraic Eq

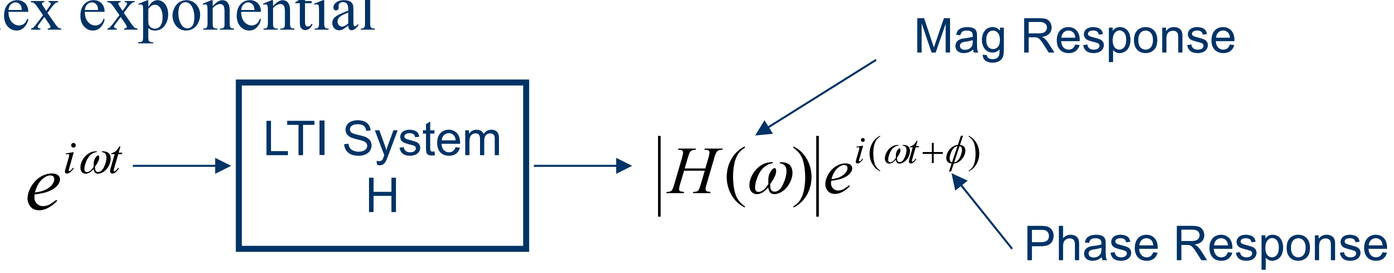
- Integration and differentiation are trivial with complex numbers:

$$\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \qquad \int e^{i\omega\tau}d\tau = \frac{1}{i\omega}e^{i\omega t}$$

- Any ODE is now trivial algebraic manipulations … in fact, we'll show that you don't even need to directly derive the ODE by using phasors
- The key is to observe that the current/voltage relation for any element can be derived for complex exponential excitation

Complex Exponential is Powerful

- To find steady state response we can excite the system with a complex exponential



- At any frequency, the system response is characterized by a single complex number *H*:

$$\left|H(\omega)\right| \qquad\qquad \phi = \measuredangle H(\omega)$$

- This is not surprising since a sinusoid is a sum of complex exponentials (and because of linearity!)

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \qquad\qquad \cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

- From this perspective, the complex exponential is even more fundamental

LPF Example: The "soft way"

- Let's excite the system with a complex exp:

$$v_s(t) = v_0(t) + \tau \frac{dv_0}{dt}$$

$$v_s(t) = V_s e^{j\omega t}$$

$$v_o(t) = |V_0| e^{j(\omega t + \phi)} = V_0 e^{j\omega t}$$

use *j* to avoid confusion

real complex

$$V_s e^{j\omega t} = V_0 e^{j\omega t} + \tau \cdot j\omega \cdot V_0 e^{j\omega t}$$

$$V_s = V_0 (1 + j\omega \cdot \tau)$$

$$\frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)}$$

Easy!!!

Magnitude and Phase Response

- The system is characterized by the complex function

$$H(\omega) = \frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)}$$

- The magnitude and phase response match our previous calculation:

$$|H(\omega)| = \left|\frac{V_0}{V_s}\right| = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \quad \checkmark$$

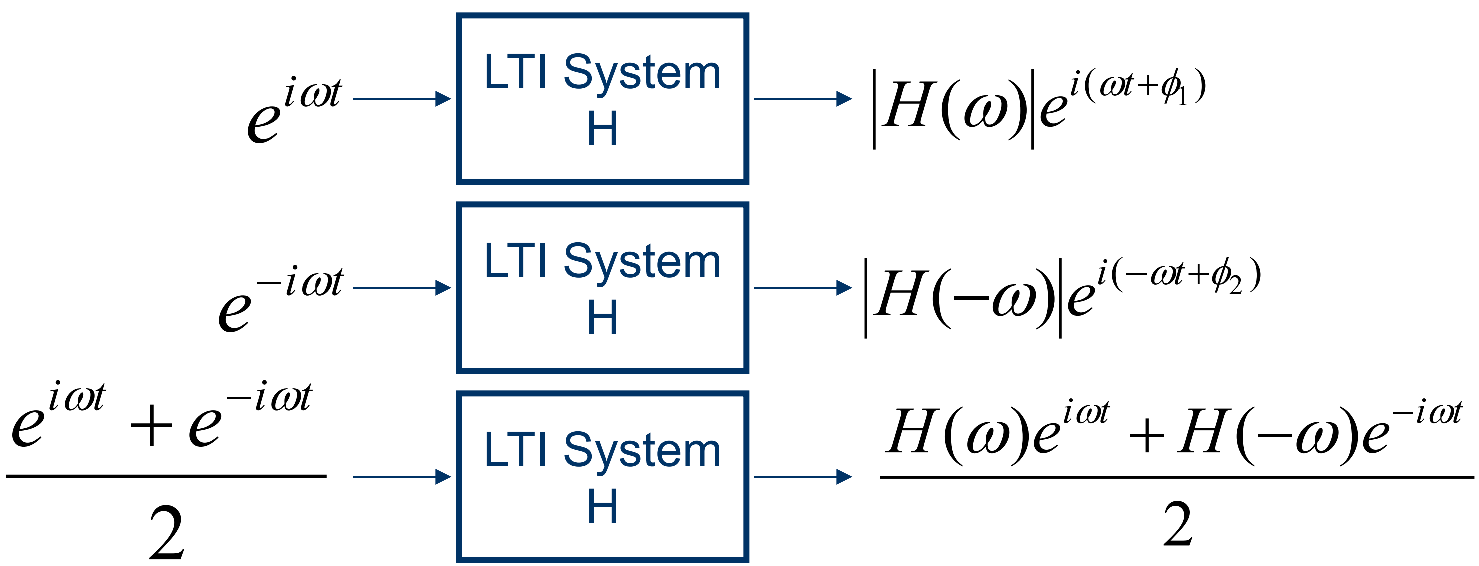
$$\prec H(\omega) = -\tan^{-1}\omega\tau \quad \checkmark$$

Why did it work?

- Again, the system is linear:

$$y = \mathbf{L}(x_1 + x_2) = \mathbf{L}(x_1) + \mathbf{L}(x_2)$$

- To find the response to a sinusoid, we can find the response to $e^{i\omega t}$ and $e^{-i\omega t}$ and sum the results:



(cont.)

- Since the input is real, the output has to be real:

$$y(t) = \frac{H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}}{2}$$

- That means the second term is the conjugate of the first:

$$|H(-\omega)| = |H(\omega)| \qquad \text{(even function)}$$

$$\prec H(-\omega) = -\prec H(\omega) = -\phi \qquad \text{(odd function)}$$

- Therefore the output is:

$$y(t) = \frac{|H(\omega)|}{2}\left(e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right)$$

$$= |H(\omega)|\cos(\omega t + \phi)$$

✓

"Proof" for Linear Systems

- For an arbitrary linear circuit (L,C,R,M, and dependent sources), decompose it into linear sub-operators, like multiplication by constants, time derivatives, or integrals:

$$y = \mathbf{L}(x) = ax + b_1 \frac{d}{dt} x + b_2 \frac{d^2}{dt^2} x + \cdots + \int x + \iint x + \iiint x + \cdots$$

- For a complex exponential input x this simplifies to:

$$y = \mathbf{L}(e^{j\omega t}) = ae^{j\omega t} + b_1 \frac{d}{dt} e^{j\omega t} + b_2 \frac{d^2}{dt^2} e^{j\omega t} + \cdots + c_1 \int e^{j\omega t} + c_2 \iint e^{j\omega t} + \cdots$$

$$y = ae^{j\omega t} + b_1 j\omega e^{j\omega t} + b_2 (j\omega)^2 e^{j\omega t} + \cdots + c_1 \frac{e^{j\omega t}}{j\omega} + c_2 \frac{e^{j\omega t}}{(j\omega)^2} + \cdots$$

$$y = Hx = e^{j\omega t}\left(a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots \right)$$

"Proof" (cont.)

- Notice that the output is also a complex exp times a complex number:

$$y = Hx = e^{j\omega t}\left(a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots \right)$$

- The amplitude of the output is the magnitude of the complex number and the phase of the output is the phase of the complex number

$$y = Hx = e^{j\omega t}\left(a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots \right)$$

$$y = e^{j\omega t} \left| H(\omega) \right| e^{j \prec H(\omega)}$$

$$\mathrm{Re}[y] = \left| H(\omega) \right| \cos(\omega t + \prec H(\omega))$$

Phasors

- With our new confidence in complex numbers, we go full steam ahead and work directly with them … we can even drop the time factor $e^{i\omega t}$ since it will cancel out of the equations.
- Excite system with a phasor: $\tilde{V}_1 = V_1 e^{j\phi_1}$
- Response will also be phasor: $\tilde{V}_2 = V_2 e^{j\phi_2}$
- For those with a Linear System background, we're going to work in the frequency domain
  – This is the Laplace domain with $s = j\omega$

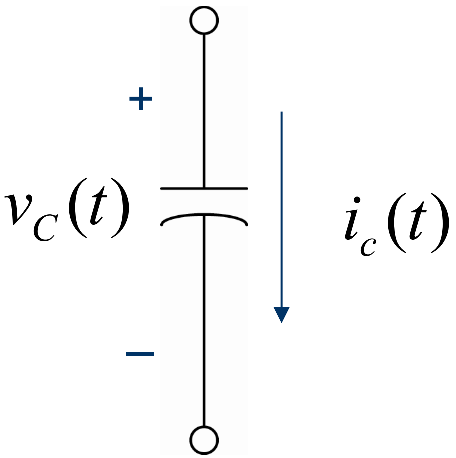
Capacitor I-V Phasor Relation

- Find the Phasor relation for current and voltage in a cap:

$$i_c(t) = C\frac{dv_C(t)}{dt}$$

$$i_c(t) = I_c e^{j\omega t}$$

$$v_c(t) = V_c e^{j\omega t}$$



$$I_c e^{j\omega t} = C\frac{d}{dt}[V_c e^{j\omega t}]$$

$$CV_c\frac{d}{dt}e^{j\omega t} = j\omega CV_c e^{j\omega t}$$

$$I_c e^{j\omega t} = j\omega CV_c e^{j\omega t}$$

$$I_c = j\omega CV_c$$

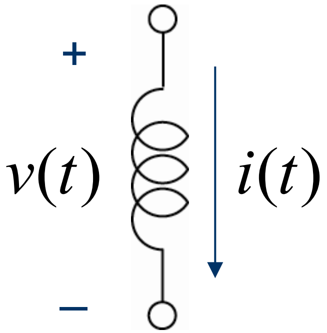
Inductor I-V Phasor Relation

- Find the Phasor relation for current and voltage in an inductor:

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = Ie^{j\omega t}$$

$$v(t) = Ve^{j\omega t}$$



$$Ve^{j\omega t} = L\frac{d}{dt}[Ie^{j\omega t}]$$

$$LI\frac{d}{dt}e^{j\omega t} = j\omega LIe^{j\omega t}$$

$$Ve^{j\omega t} = j\omega LIe^{j\omega t}$$

$$V = j\omega L I$$

Complex Transfer Function

- Excite a system with an input voltage (current) x
- Define the output voltage y (current) to be any node voltage (branch current)
- For a complex exponential input, the "transfer function" from input to output:

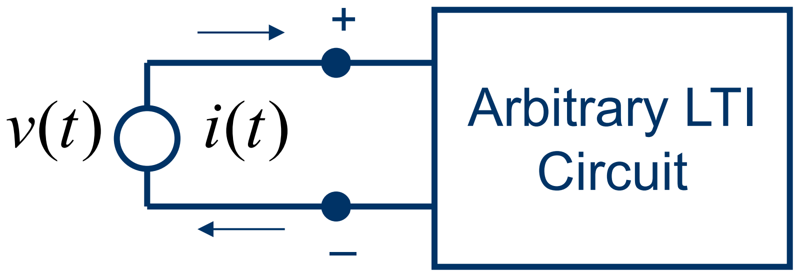
$$H \equiv \frac{y}{x} = \left(a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots \right)$$

- We can write this in canonical form as a rational function:

$$H(\omega) = \frac{n_1 + n_2 j\omega + n_3 (j\omega)^2 + \cdots}{d_1 + d_2 j\omega + d_3 (j\omega)^2 + \cdots}$$

Impede the Currents !

- Suppose that the "input" is defined as the current of a terminal pair (*port*) and the "output" is defined as the voltage into the port:



$$v(t) = Ve^{j\omega t} = |V|e^{j(\omega t + \phi_v)}$$

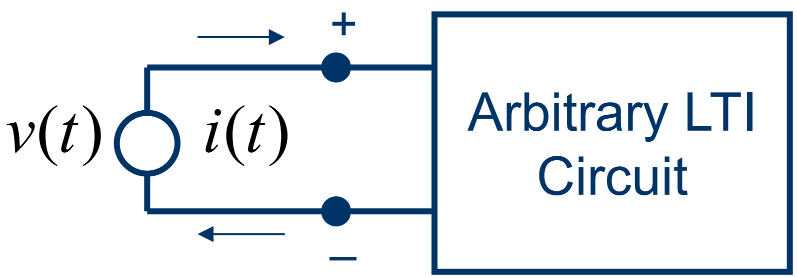
$$i(t) = Ie^{j\omega t} = |I|e^{j(\omega t + \phi_i)}$$

- The impedance Z is defined as the ratio of the phasor voltage to phasor current ("self" transfer function)

$$Z(\omega) = H(\omega) = \frac{V}{I} = \left|\frac{V}{I}\right| e^{j(\phi_v - \phi_i)}$$

Admit the Currents!

- Suppose that the "input" is defined as the current of a terminal pair (*port*) and the "output" is defined as the voltage into the port:



$$v(t) = Ve^{j\omega t} = |V|e^{j(\omega t+\phi_v)}$$

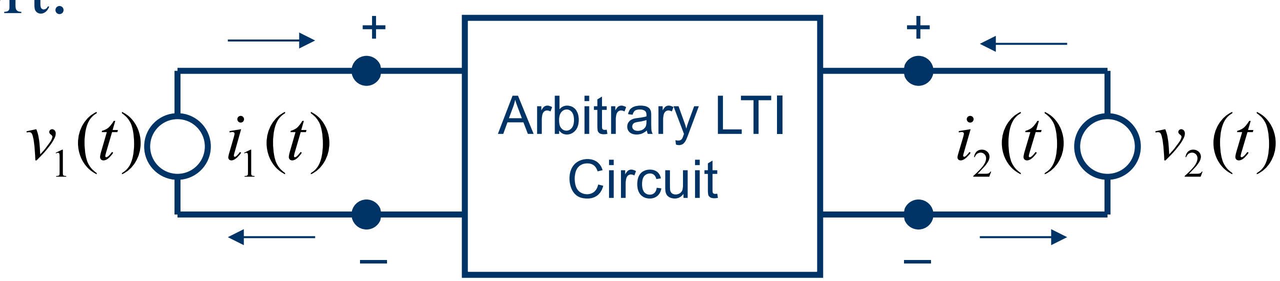
$$i(t) = Ie^{j\omega t} = |I|e^{j(\omega t+\phi_i)}$$

- The <u>admmittance</u> Z is defined as the ratio of the phasor current to phasor voltage ("self" transfer function)

$$Y(\omega) = H(\omega) = \frac{I}{V} = \left|\frac{I}{V}\right| e^{j(\phi_i - \phi_v)}$$

Voltage and Current Gain

- The voltage (current) gain is just the voltage (current) transfer function from one port to another port:



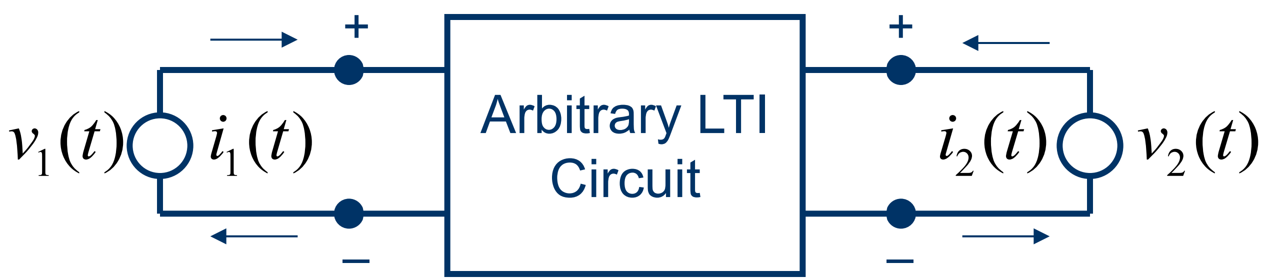
$$G_v(\omega) = \frac{V_2}{V_1} = \left|\frac{V_2}{V_1}\right| e^{j(\phi_2 - \phi_1)}$$

$$G_i(\omega) = \frac{I_2}{I_1} = \left|\frac{I_2}{I_1}\right| e^{j(\phi_2 - \phi_1)}$$

- If $G > 1$, the circuit has voltage (current) gain
- If $G < 1$, the circuit has loss or attenuation

Transimpedance/admittance

- Current/voltage gain are unitless quantities
- Sometimes we are interested in the transfer of voltage to current or vice versa



$$J(\omega) = \frac{V_2}{I_1} = \left|\frac{V_2}{I_1}\right| e^{j(\phi_2 - \phi_1)} \qquad [\Omega]$$

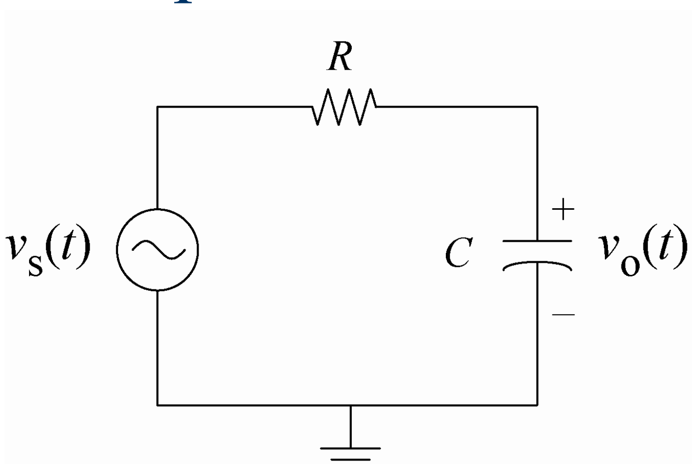
$$K(\omega) = \frac{I_2}{V_1} = \left|\frac{I_2}{V_1}\right| e^{j(\phi_2 - \phi_1)} \qquad [S]$$

Direct Calculation of *H* (no DEs)

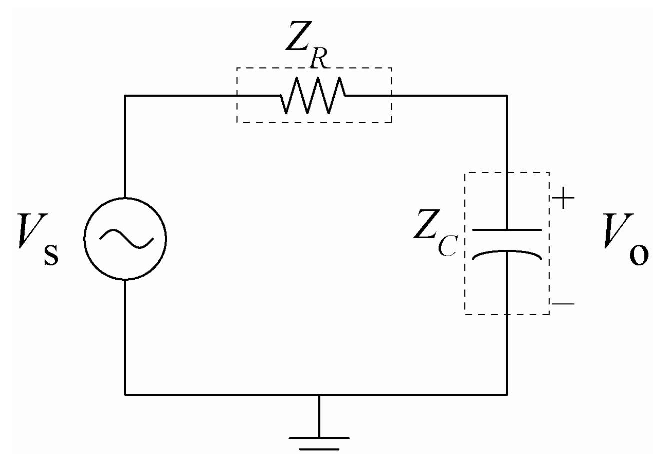
- To directly calculate the transfer function (impedance, trans-impedance, etc) we can generalize the circuit analysis concept from the "real" domain to the "phasor" domain
- With the concept of impedance (admittance), we can now directly analyze a circuit without explicitly writing down any differential equations
- Use KVL, KCL, mesh analysis, loop analysis, or node analysis where inductors and capacitors are treated as complex resistors

LPF Example: Again!

- Instead of setting up the DE in the time-domain, let's do it directly in the frequency domain
- Treat the capacitor as an imaginary "resistance" or impedance:



time domain "real" circuit

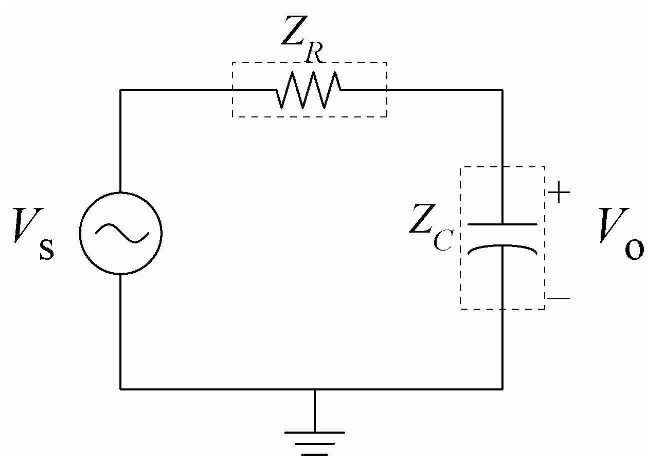


frequency domain "phasor" circuit

- We know the impedances:

$$Z_R = R \qquad Z_C = \frac{1}{j\omega C}$$

LPF … Voltage Divider

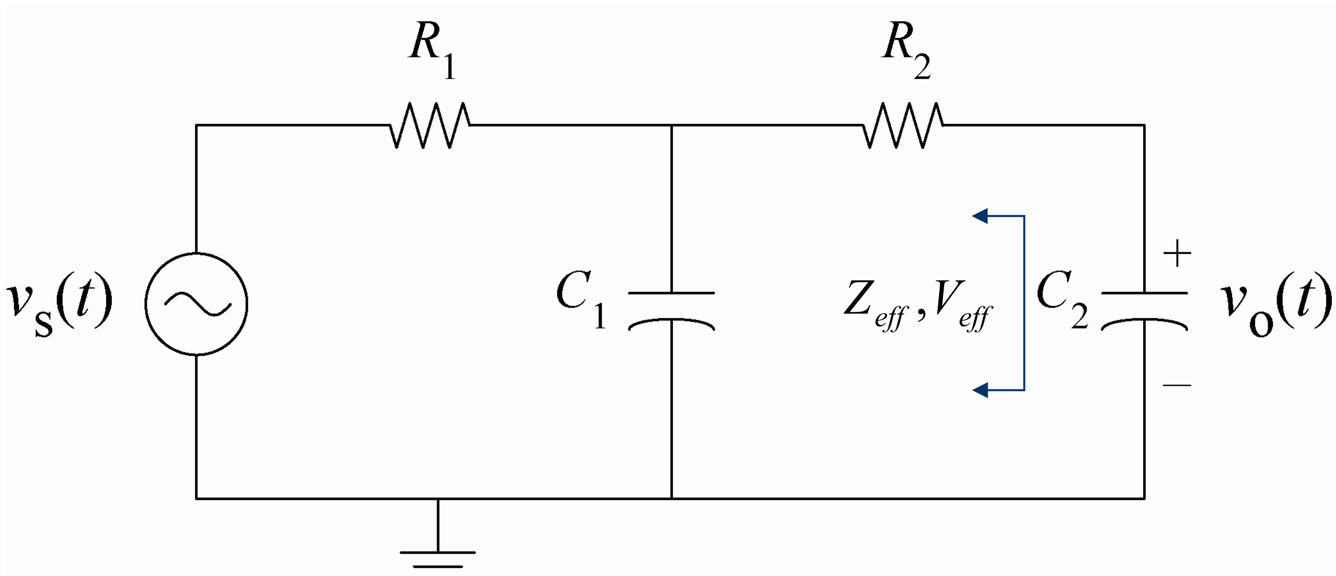


- Fast way to solve problem is to say that the LPF is really a voltage divider

$$H(\omega) = \frac{V_o}{V_s} = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \quad \checkmark$$

Bigger Example (no problem!)

- Consider a more complicated example:



$$H(\omega) = \frac{V_o}{V_s} = \frac{Z_{C2}}{Z_{eff} + Z_{C2}} \frac{V_{eff}}{V_s} \qquad Z_{eff} = R_2 + R_1 \parallel Z_{C1}$$

$$\frac{V_{eff}}{V_s} = \frac{Z_{C1}}{R_1 + Z_{C1}} \qquad H(\omega) = \frac{Z_{C2}}{R_2 + R_1 \parallel Z_{C1} + Z_{C2}} \cdot \frac{Z_{C1}}{R_1 + Z_{C1}}$$

Second Order Transfer Function

- Series RLC circuit

Poles/Zeros of Shunt RLC Circuit

Does it sound better?

- Application of LPF: Noise Filter
- Listen to the following sound file (corrupted with noise)
- Since the noise has a flat frequency spectrum, if we LPF the signal we should get rid of the high-frequency components of noise
- The filter cutoff frequency should be above the highest frequency produced by the human voice (~ 5 kHz).
- A high-pass filter (HPF) has the opposite effect, it amplifies the noise and attenuates the signal.

Tones

Tones+ Noise

LPF

BPF on 1st tone

BPF (both tones)

Building Tents: Poles and Zeros

- For most circuits that we'll deal with, the transfer function can be shown to be a rational function

$$H(\omega)=\frac{n_1+n_2 j\omega+n_3(j\omega)^2+\cdots}{d_1+d_2 j\omega+d_3(j\omega)^2+\cdots}$$

- The behavior of the circuit can be extracted by finding the roots of the numerator and denominator

$$H(\omega)=\frac{(z_1-j\omega)(z_2-j\omega)\cdots}{(p_1-j\omega)(p_2-j\omega)\cdots}=\frac{\prod(z_i-j\omega)}{\prod(p_i-j\omega)}$$

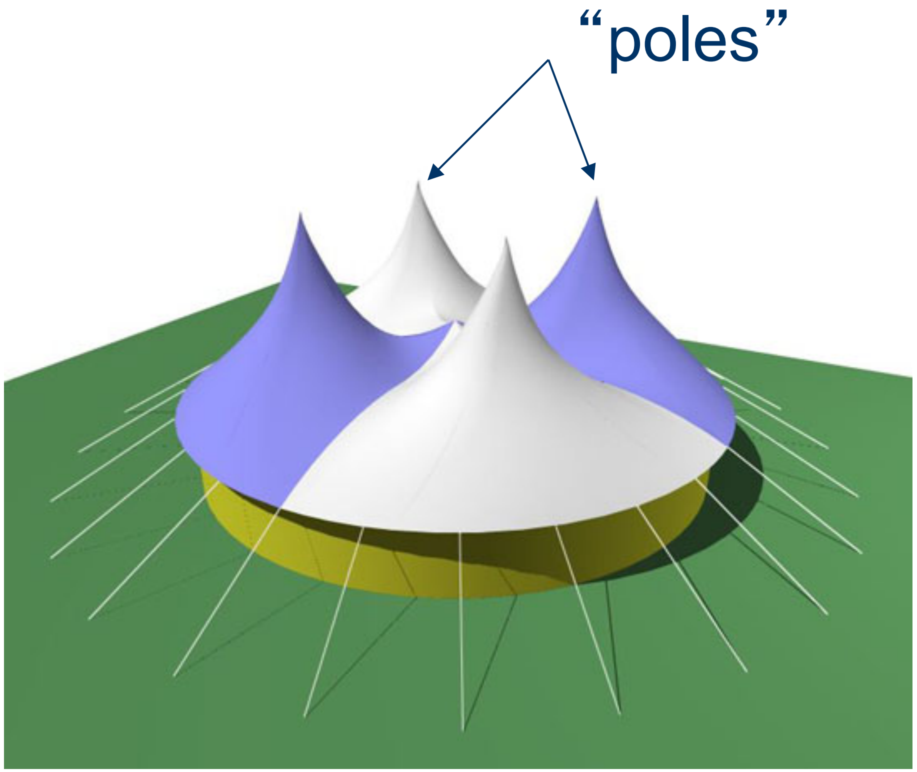
- Or another form (DC gain explicit)

$$H(\omega)=G_0(j\omega)^K\frac{(1-j\omega\tau_{z1})(1-j\omega\tau_{z2})\cdots}{(1-j\omega\tau_{p2})(1-j\omega\tau_{p2})\cdots}=G_0(j\omega)^K\frac{\prod(1-j\omega\tau_{z,i})}{\prod(1-j\omega\tau_{p,i})}$$

Poles and Zeros (cont)

- The roots of the numerator are called the "zeros" since at these frequencies, the transfer function is zero

- The roots of the denominator are called the "poles", since at these frequencies the transfer function peaks (like a pole in a tent)



$$H(\omega) = \frac{(z_1 - j\omega)(z_2 - j\omega)\cdots}{(p_1 - j\omega)(p_2 - j\omega)\cdots}$$

Finding the Magnitude (quickly)

- The magnitude of the response can be calculated quickly by using the property of the mag operator:

$$|H(\omega)| = \left| G_0 (j\omega)^K \frac{(1 - j\omega\tau_{z1})(1 - j\omega\tau_{z2})\cdots}{(1 - j\omega\tau_{p2})(1 - j\omega\tau_{p2})\cdots} \right|$$

$$= |G_0| \omega^K \frac{|1 - j\omega\tau_{z1}||1 - j\omega\tau_{z2}|\cdots}{|1 - j\omega\tau_{p2}||1 - j\omega\tau_{p2}|\cdots}$$

- The magnitude at DC depends on G_0 and the number of poles/zeros at DC. If $K > 0$, gain is zero. If $K < 0$, DC gain is infinite. Otherwise if $K=0$, then gain is simply G_0

Finding the Phase (quickly)

- The phase can be computed quickly with the following formula:

$$\angle H(\omega) = \angle G_0 (j\omega)^K \frac{(1 - j\omega\tau_{z1})(1 - j\omega\tau_{z2})\cdots}{(1 - j\omega\tau_{p2})(1 - j\omega\tau_{p2})\cdots}$$

$$= \angle G_0 + \angle (j\omega)^K + \angle (1 - j\omega\tau_{z1}) + \angle (1 - j\omega\tau_{z2}) + \cdots$$

$$- \angle (1 - j\omega\tau_{p1}) - \angle (1 - j\omega\tau_{p2}) - \cdots$$

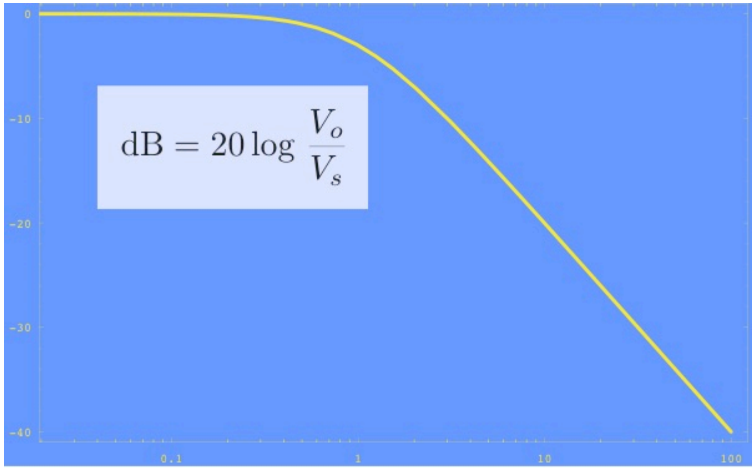
- No the second term is simple to calculate for positive frequencies:

$$\angle (j\omega)^K = K\frac{\pi}{2}$$

- Interpret this as saying that multiplication by j is equivalent to rotation by 90 degrees

Bode Plots

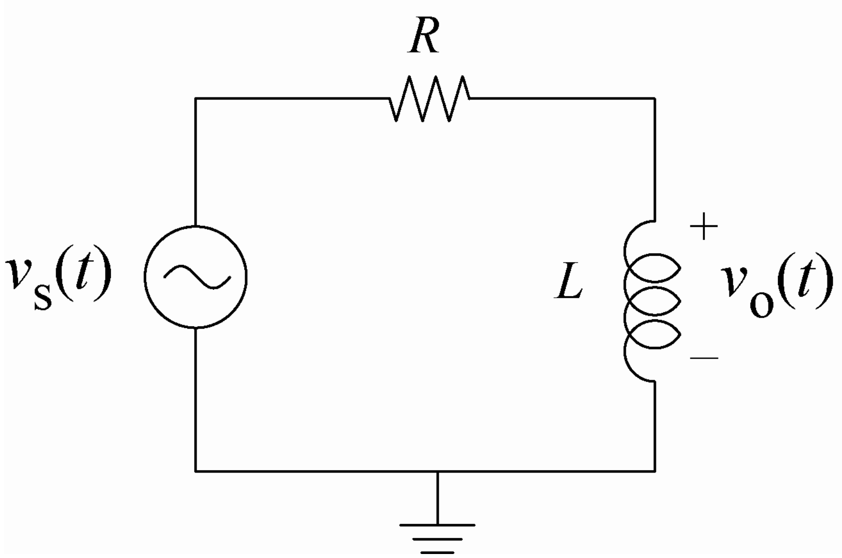
- Simply the log-log plot of the magnitude and phase response of a circuit (impedance, transimpedance, gain, …)
- Gives insight into the behavior of a circuit as a function of frequency
- The "log" expands the scale so that breakpoints in the transfer function are clearly delineated
- In EECS 140, Bode plots are used to "compensate" circuits in feedback loops



Example: High-Pass Filter

- Using the voltage divider rule:

$$H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}}$$



$$H(\omega) = \frac{j\omega\tau}{1 + j\omega\tau}$$

$$\omega \to \infty \qquad |H| \to \left| \frac{j\omega\tau}{j\omega\tau} \right| = 1$$

$$\omega \to 0 \quad |H| \to \frac{0}{1+0} = 0$$

$$\omega = \frac{1}{\tau} \quad |H| = \left| \frac{j}{1+j} \right| = \frac{1}{\sqrt{2}}$$

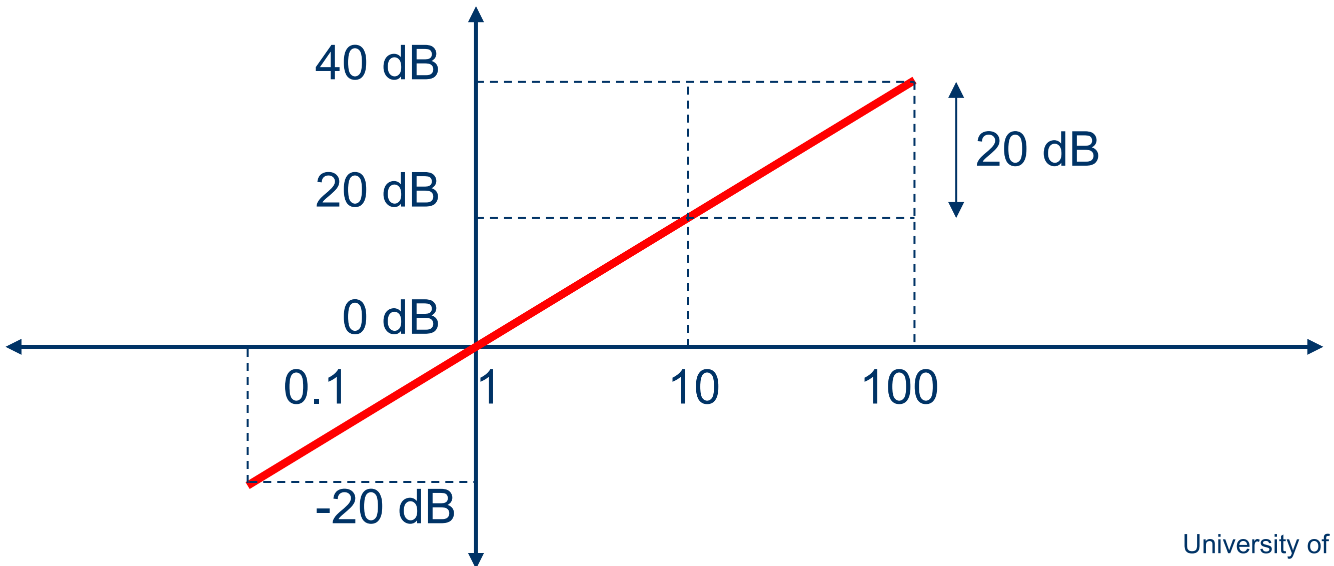
HPF Magnitude Bode Plot

- Recall that log of product is the sum of log

$$\left|H(\omega)\right|_{dB} = \left|\frac{j\omega\tau}{1+j\omega\tau}\right|_{dB} = \left|j\omega\tau\right|_{dB} + \left|\frac{1}{1+j\omega\tau}\right|_{dB}$$

$\left|j\omega\tau\right|_{dB}$ ← Increase by 20 dB/decade

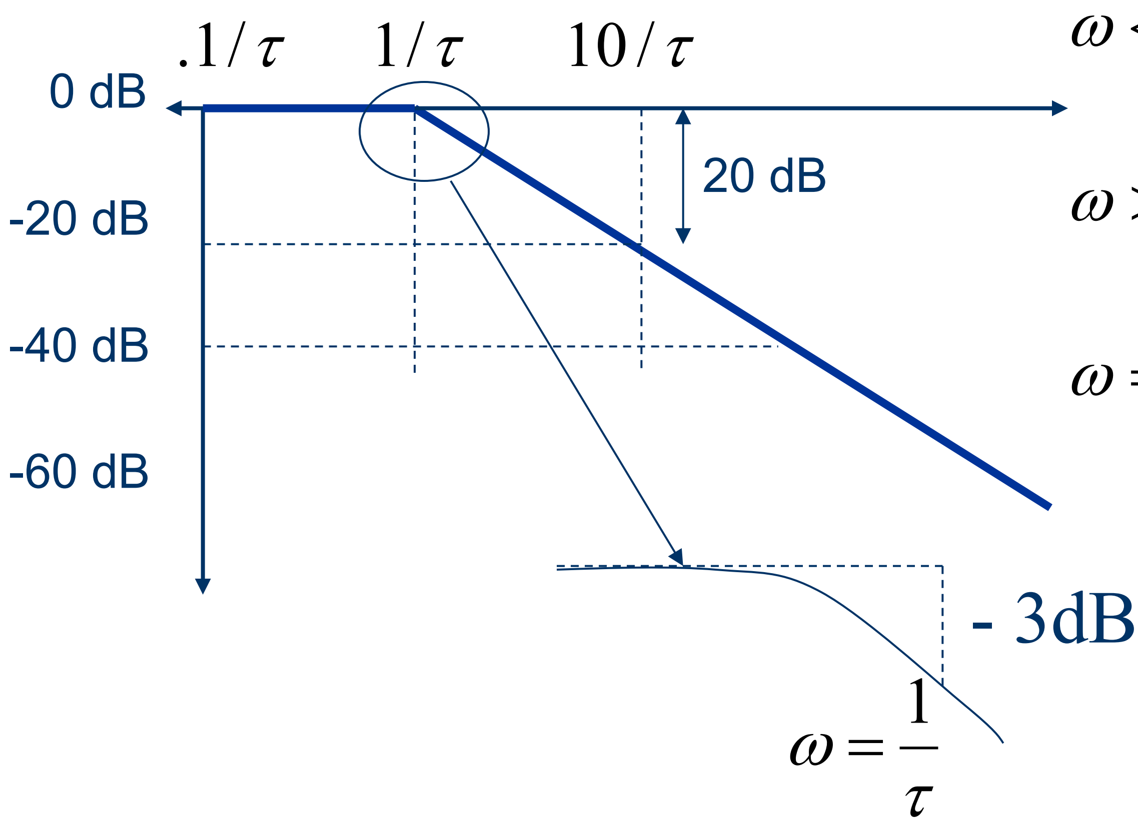
$\omega\tau = 1 \Rightarrow \left|j\omega\tau\right|_{dB} = 0\,\text{dB}$ Equals unity at breakpoint



HPF Bode Plot (dissection)

- The second term can be further dissected:

$$\left|\frac{1}{1+j\omega\tau}\right|_{dB} = 0\,\text{dB} - \left|1+j\omega\tau\right|_{dB}$$



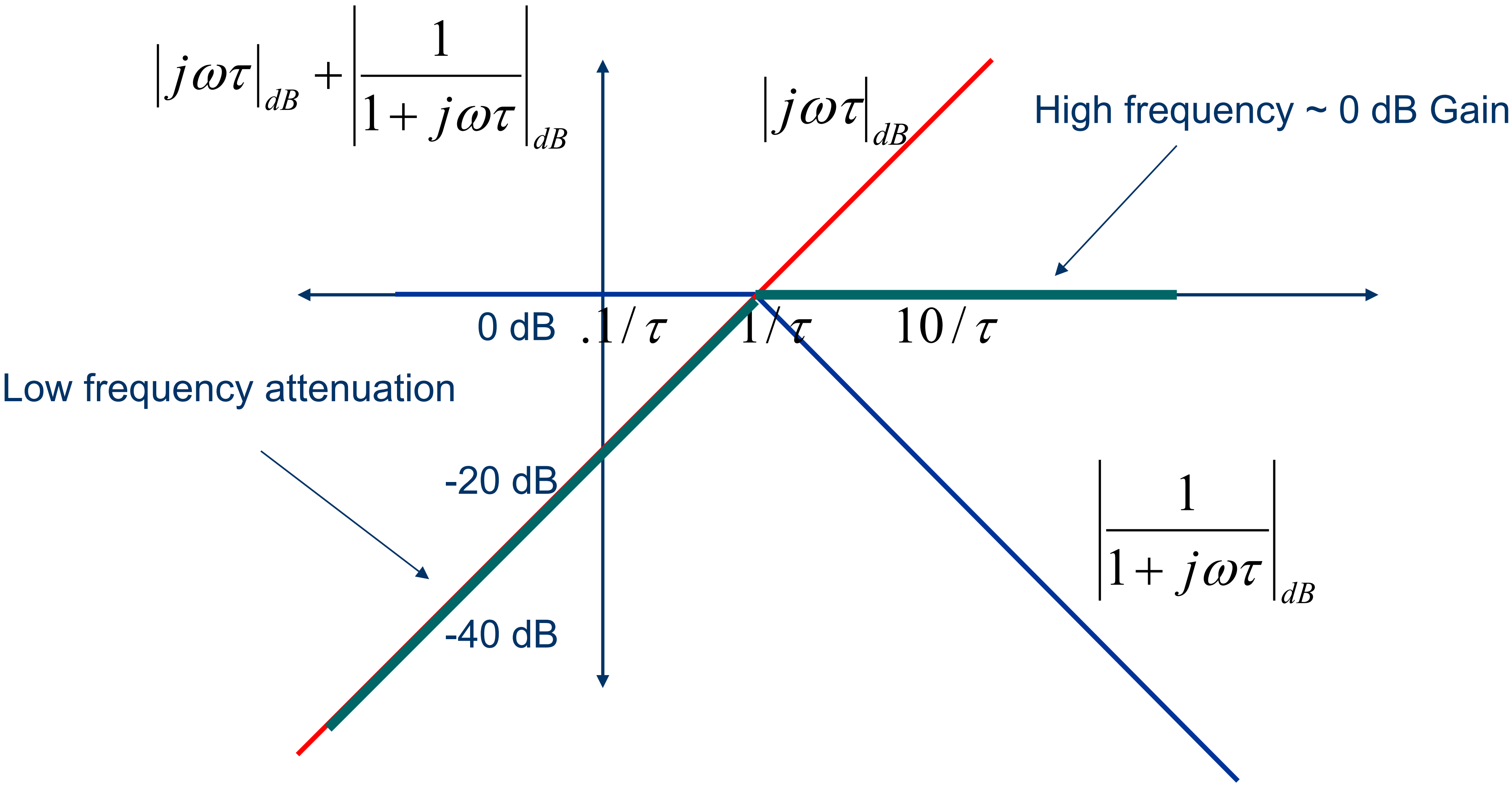
$\omega << \frac{1}{\tau}$ 0 dB

$\omega >> \frac{1}{\tau}$ -20 dB/dec

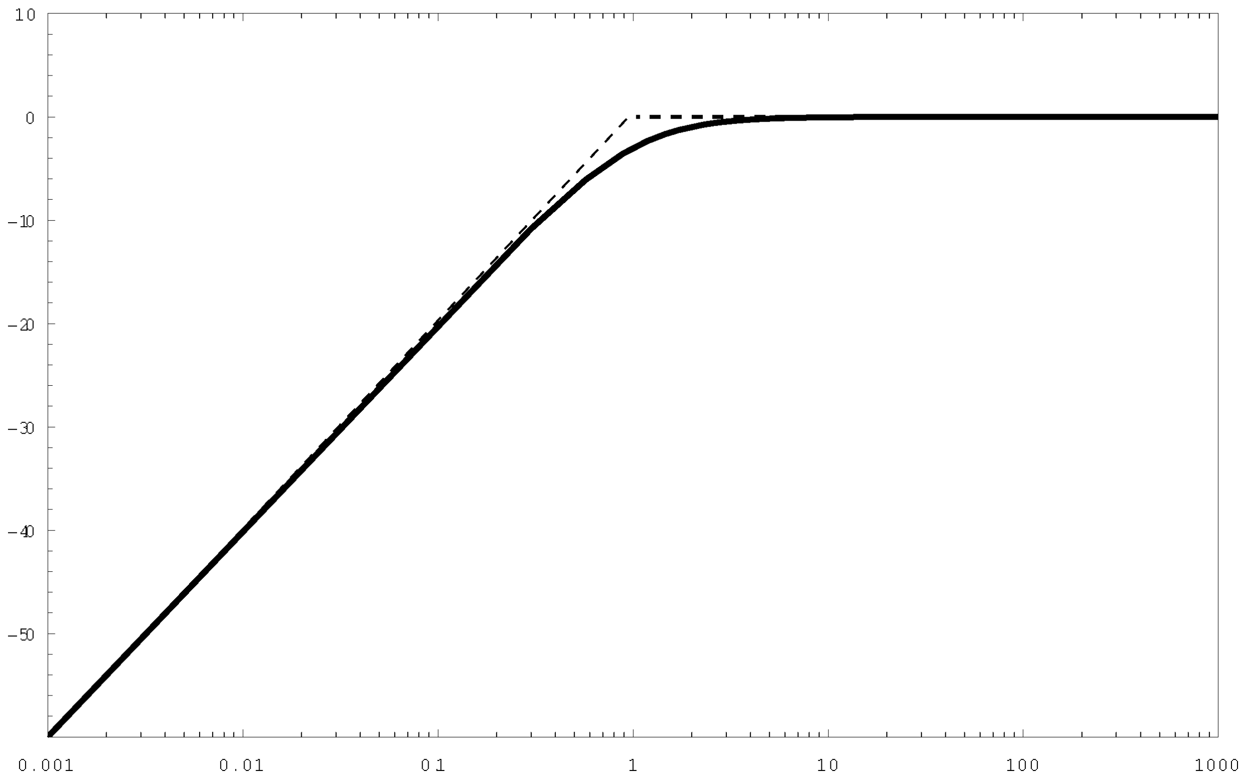
$\omega = \frac{1}{\tau}$ ~ -3dB

Composite Plot

- Composite is simply the sum of each component:



Approximate versus Actual Plot



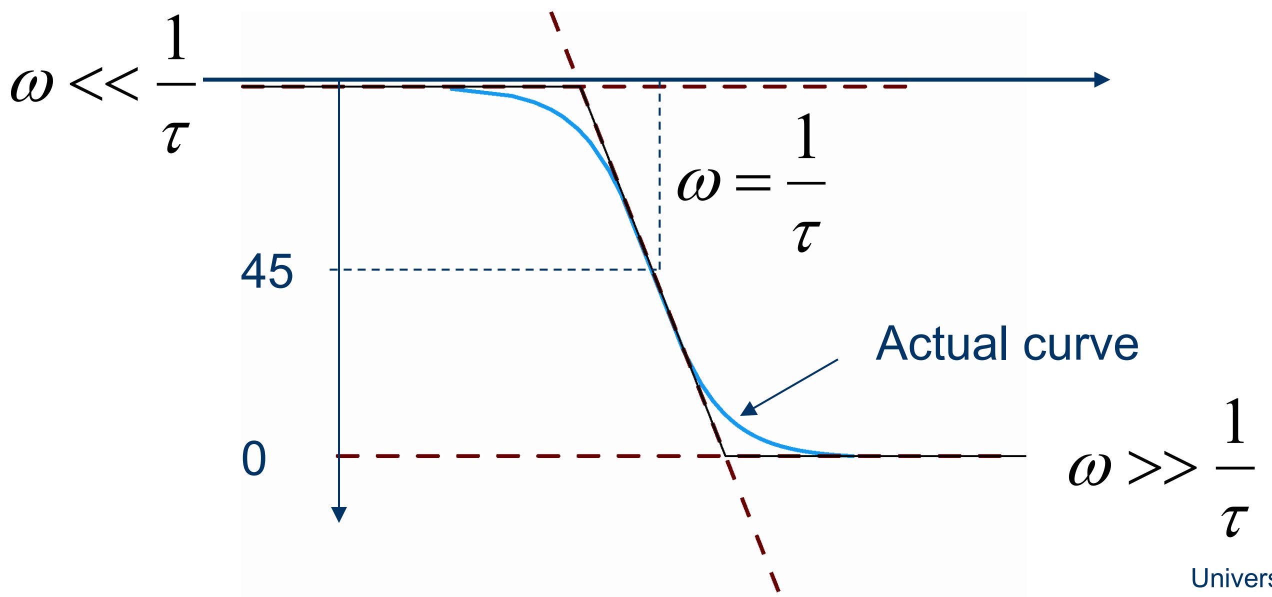
- Approximate curve accurate away from breakpoint
- At breakpoint there is a 3 dB error

HPF Phase Plot

- Phase can be naturally decomposed as well:

$$\angle H(\omega) = \angle \frac{j\omega\tau}{1+j\omega\tau} = \angle j\omega\tau + \angle \frac{1}{1+j\omega\tau} = \frac{\pi}{2} - \tan^{-1}\omega\tau$$

- First term is simply a constant phase of 90 degrees
- The second term is the arctan function
- Estimate arctan function:



"s" Complex Plane

- You may see people talking about transfer functions as a function of complex "s" rather than frequency

$$H(s) = \frac{(z_1 - s)(z_2 - s)\cdots}{(p_1 - s)(p_2 - s)\cdots}$$

- This is a generalization (Laplace Domain) of frequency that you will learn about later. For now, just evaluate the function as follows

$$H(s = j\omega) = \frac{(z_1 - j\omega)(z_2 - j\omega)\cdots}{(p_1 - j\omega)(p_2 - j\omega)\cdots}$$

- This is why you may see people defining a function like: $H(j\omega)$

Power Flow

- The instantaneous power flow into any element is the product of the voltage and current: $P(t) = i(t)v(t)$
- For a periodic excitation, the average power is:

$$P_{av} = \int_T i(\tau)v(\tau)d\tau$$

- In terms of sinusoids we have

$$P_{av} = \int_T |I|\cos(\omega t + \varphi_i)|V|\cos(\omega t + \varphi_v)d\tau$$

$$= |I| \cdot |V| \int_T (\cos\omega t \cos\varphi_i - \sin\omega t \sin\varphi_i) \cdot (\cos\omega t \cos\varphi_v - \sin\omega t \sin\varphi_v)d\tau$$

$$= |I| \cdot |V| \int_T d\tau \cos^2\omega t \cos\varphi_i \cos\varphi_v + \sin^2\omega t \sin\varphi_i \sin\varphi_v + c \sin\omega t \cos\omega t$$

$$= \frac{|I| \cdot |V|}{2}(\cos\varphi_i \cos\varphi_v + \sin\varphi_i \sin\varphi_v) = \frac{|I| \cdot |V|}{2}\cos(\varphi_i - \varphi_v)$$

Power Flow with Phasors

$$P_{av} = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v)$$

Power Factor

- Note that if $(\phi_i - \phi_v) = \frac{\pi}{2}$, then $P_{av} = \frac{|I| \cdot |V|}{2} \cos(\pi / 2) = 0$
- Important: Power is a non-linear function so we can't simply take the real part of the product of the phasors:

$$P \neq \mathrm{Re}[I \cdot V]$$

- From our previous calculation:

$$P = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v) = \frac{1}{2} \mathrm{Re}[I \cdot V^*] = \frac{1}{2} \mathrm{Re}[I^* \cdot V]$$

More Power to You!

- In terms of the circuit impedance we have:

$$P = \frac{1}{2}\mathrm{Re}[I \cdot V^*] = \frac{1}{2}\mathrm{Re}[\frac{V}{Z} \cdot V^*] = \frac{|V|^2}{2}\mathrm{Re}[Z^{-1}]$$

$$= \frac{|V|^2}{2}\mathrm{Re}[\frac{Z^*}{|Z|^2}] = \frac{|V|^2}{2|Z|^2}\mathrm{Re}[Z^*] = \frac{|V|^2}{2|Z|^2}\mathrm{Re}[Z]$$

- Check the result for a real impedance (resistor)
- Also, in terms of current:

$$P = \frac{1}{2}\mathrm{Re}[I^* \cdot V] = \frac{1}{2}\mathrm{Re}[I^* \cdot I \cdot Z] = \frac{|I|^2}{2}\mathrm{Re}[Z]$$

Summary

- Complex exponentials are eigen-functions of LTI systems
  - Steady-state response of LCR circuits are LTI systems
  - Phasor analysis allows us to treat all LCR circuits as simple “resistive” circuits by using the concept of impedance (admittance)
- Frequency response allows us to completely characterize a system
  - Any input can be decomposed into either a continuum or discrete sum of frequency components
  - The transfer function is usually plotted in the log-log domain (Bode plot) – magnitude and phase
  - Location of poles/zeros is key