Power Gain and Stability

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Power Gain
We can define power gain in many different ways. The *power gain* $G_p$ is defined as follows

$$G_p = \frac{P_L}{P_{in}} = f(Y_L, Y_{ij}) \neq f(Y_S)$$

We note that this power gain is a function of the load admittance $Y_L$ and the two-port parameters $Y_{ij}$. 

\[ \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \]
The available power gain is defined as follows

\[ G_a = \frac{P_{av,L}}{P_{av,S}} = f(Y_S, Y_{ij}) \neq f(Y_L) \]

The available power from the two-port is denoted \( P_{av,L} \) whereas the power available from the source is \( P_{av,S} \).

Finally, the transducer gain is defined by

\[ G_T = \frac{P_L}{P_{av,S}} = f(Y_L, Y_S, Y_{ij}) \]

This is a measure of the efficacy of the two-port as it compares the power at the load to a simple conjugate match.
Bi-Conjugate Match

- When the input and output are simultaneously conjugately matched, or a *bi-conjugate match* has been established, we find that the transducer gain is maximized with respect to the source and load impedance

\[ G_{T,\text{max}} = G_{p,\text{max}} = G_{a,\text{max}} \]

- This is thus the recipe for calculating the optimal source and load impedance in to maximize gain

\[ Y_{in} = Y_{11} - \frac{Y_{12} Y_{21}}{Y_L + Y_{22}} = Y_S^* \]

\[ Y_{out} = Y_{22} - \frac{Y_{12} Y_{21}}{Y_L + Y_{11}} = Y_L^* \]

- Solution of the above four equations (real/imag) results in the optimal \( Y_{S,\text{opt}} \) and \( Y_{L,\text{opt}} \).
Another approach is to simply equate the partial derivatives of $G_T$ with respect to the source/load admittance to find the maximum point.

\[
\frac{\partial G_T}{\partial G_S} = 0; \quad \frac{\partial G_T}{\partial B_S} = 0
\]

\[
\frac{\partial G_T}{\partial G_L} = 0; \quad \frac{\partial G_T}{\partial B_L} = 0
\]
Again we have four equations. But we should be smarter about this and recall that the maximum gains are all equal. Since $G_a$ and $G_p$ are only a function of the source or load, we can get away with only solving two equations. For instance

$$\frac{\partial G_a}{\partial G_S} = 0; \quad \frac{\partial G_a}{\partial B_S} = 0$$

This yields $Y_{S, opt}$ and by setting $Y_L = Y_{out}^*$ we can find the $Y_{L, opt}$.

Likewise we can also solve

$$\frac{\partial G_p}{\partial G_L} = 0; \quad \frac{\partial G_p}{\partial B_L} = 0$$

And now use $Y_{S, opt} = Y_{in}^*$. 
Optimal Power Gain Derivation

Let’s outline the procedure for the optimal power gain. We’ll use the power gain $G_p$ and take partials with respect to the load. Let

$$Y_{jk} = m_{jk} + jn_{jk}$$

$$Y_L = G_L + jX_L$$

$$Y_{12}Y_{21} = P + jQ = Le^{j\phi}$$

$$G_p = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \frac{\Re(Y_L)}{\Re(Y_{in})} = \frac{|Y_{21}|^2}{D} G_L$$

$$\Re \left( Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} \right) = m_{11} - \frac{\Re(Y_{12}Y_{21}(Y_L + Y_{22})^*)}{|Y_L + Y_{22}|^2}$$

$$D = m_{11}|Y_L + Y_{22}|^2 - P(G_L + m_{22}) - Q(B_L + n_{22})$$

$$\frac{\partial G_p}{\partial B_L} = 0 = -\frac{|Y_{21}|^2 G_L}{D^2} \frac{\partial D}{\partial B_L}$$
Solving the above equation we arrive at the following solution

\[ B_{L,\text{opt}} = \frac{Q}{2m_{11}} - n_{22} \]

In a similar fashion, solving for the optimal load conductance

\[ G_{L,\text{opt}} = \frac{1}{2m_{11}} \sqrt{(2m_{11}m_{22} - P)^2 - L^2} \]

If we substitute these values into the equation for \( G_p \) (lot’s of algebra ...), we arrive at

\[ G_{p,\text{max}} = \frac{|Y_{21}|^2}{2m_{11}m_{22} - P + \sqrt{(2m_{11}m_{22} - P)^2 - L^2}} \]
Notice that for the solution to exist, $G_L$ must be a real number. In other words

$$(2m_{11}m_{22} - P)^2 > L^2$$

$$(2m_{11}m_{22} - P) > L$$

$$K = \frac{2m_{11}m_{22} - P}{L} > 1$$

This factor $K$ plays an important role as we shall show that it also corresponds to an unconditionally stable two-port. We can recast all of the work up to here in terms of $K$

$$Y_{S, opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{22})}$$

$$Y_{L, opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{11})}$$

$$G_{p, max} = G_{T, max} = G_{a, max} = \frac{Y_{21}}{Y_{12}} \frac{1}{K + \sqrt{K^2 - 1}}$$
The maximum gain is usually written in the following insightful form

$$G_{\text{max}} = \frac{Y_{21}}{Y_{12}}(K - \sqrt{K^2 - 1})$$

For a reciprocal network, such as a passive element, $Y_{12} = Y_{21}$ and thus the maximum gain is given by the second factor

$$G_{r,\text{max}} = K - \sqrt{K^2 - 1}$$

Since $K > 1$, $|G_{r,\text{max}}| < 1$. The reciprocal gain factor is known as the efficiency of the reciprocal network.

The first factor, on the other hand, is a measure of the non-reciprocity.
For a unilateral network, the design for maximum gain is trivial. For a bi-conjugate match

\[ Y_S = Y_{11}^* \]

\[ Y_L = Y_{22}^* \]

\[ G_{T,\text{max}} = \frac{|Y_{21}|^2}{4m_{11}m_{22}} \]
The AC equivalent circuit for a MOSFET at low to moderate frequencies is shown above. Since $|S_{11}| = 1$, this circuit has infinite power gain. This is a trivial fact since the gate capacitance cannot dissipate power whereas the output can deliver real power to the load.
A more realistic equivalent circuit is shown above. If we make the unilateral assumption, then the input and output power can be easily calculated. Assume we conjugate match the input/output

\[ P_{avs} = \frac{|V_S|^2}{8R_i} \]

\[ P_L = \Re\left( \frac{1}{2} I_L V_L^* \right) = \frac{1}{2} \left| \frac{g_m V_1}{2} \right|^2 R_{ds} \]

\[ G_{TU,\text{max}} = g_m^2 R_{ds} R_i \left| \frac{V_1}{V_S} \right|^2 \]
At the center resonant frequency, the voltage at the input of the FET is given by

\[ V_1 = \frac{1}{j \omega C_{gs}} \frac{V_S}{2R_i} \]

\[ G_{TU,\text{max}} = \frac{R_{ds}}{R_i} \frac{(g_m/C_{gs})^2}{4\omega^2} \]

This can be written in terms of the device unity gain frequency \( f_T \)

\[ G_{TU,\text{max}} = \frac{1}{4} \frac{R_{ds}}{R_i} \left( \frac{f_T}{f} \right)^2 \]

The above expression is very insightful. To maximum power gain we should maximize the device \( f_T \) and minimize the input resistance while maximizing the output resistance.
Stability of a Two-Port
A two-port is unstable if the admittance of either port has a negative conductance for a passive termination on the second port. Under such a condition, the two-port can oscillate.

Consider the input admittance

\[ Y_{in} = G_{in} + jB_{in} = Y_{11} - \frac{Y_{12} Y_{21}}{Y_{22} + Y_L} \]

Using the following definitions

\[ Y_{11} = g_{11} + jb_{11} \]
\[ Y_{22} = g_{22} + jb_{22} \]
\[ Y_L = G_L + jB_L \]

Now substitute real/imag parts of the above quantities into \( Y_{in} \)

\[ Y_{in} = g_{11} + jb_{11} - \frac{P + jQ}{g_{22} + jb_{22} + G_L + jB_L} \]
\[ = g_{11} + jb_{11} - \frac{(P + jQ)(g_{22} + G_L - j(b_{22} + B_L))}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2} \]
Taking the real part, we have the input conductance

\[ \Re(Y_{in}) = G_{in} = g_{11} - \frac{P(g_{22} + G_L) + Q(b_{22} + B_L)}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2} \]

\[ = \frac{\left((g_{22} + G_L)^2 + (b_{22} + B_L)^2 - \frac{P}{g_{11}}(g_{22} + G_L) - \frac{Q}{g_{11}}(b_{22} + B_L)\right)}{D} \]

Since \( D > 0 \) if \( g_{11} > 0 \), we can focus on the numerator. Note that \( g_{11} > 0 \) is a requirement since otherwise oscillations would occur for a short circuit at port 2.

The numerator can be factored into several positive terms

\[ N = (g_{22} + G_L)^2 + (b_{22} + B_L)^2 - \frac{P}{g_{11}}(g_{22} + G_L) - \frac{Q}{g_{11}}(b_{22} + B_L) \]

\[ = \left(G_L + \left(g_{22} - \frac{P}{2g_{11}}\right)\right)^2 + \left(B_L + \left(b_{22} - \frac{Q}{2g_{11}}\right)\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} \]
Now note that the numerator can go negative only if the first two terms are smaller than the last term. To minimize the first two terms, choose $G_L = 0$ and $B_L = -\left(b_{22} - \frac{Q}{2g_{11}}\right)$ (reactive load)

$$N_{min} = \left(g_{22} - \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2}$$

And thus the above must remain positive, $N_{min} > 0$, so

$$\left(g_{22} + \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} > 0$$

$$g_{11}g_{22} > \frac{P + L}{2} = \frac{L}{2}(1 + \cos \phi)$$
Using the above equation, we define the Linvill stability factor

\[ L < 2g_{11}g_{22} - P \]

\[ C = \frac{L}{2g_{11}g_{22} - P} < 1 \]

The two-port is stable if \( 0 < C < 1 \).
It’s more common to use the inverse of $C$ as the stability measure

$$\frac{2g_{11}g_{22} - P}{L} > 1$$

The above definition of stability is perhaps the most common

$$K = \frac{2\Re(Y_{11})\Re(Y_{22}) - \Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} > 1$$

The above expression is identical if we interchange ports 1/2. Thus it’s the general condition for stability.

Note that $K > 1$ is the same condition for the maximum stable gain derived earlier. The connection is now more obvious. If $K < 1$, then the maximum gain is infinity!
We can also derive stability in terms of the input reflection coefficient. For a general two-port with load $\Gamma_L$ we have

\[ v_2^- = \Gamma_L^{-1} v_2^+ = S_{21} v_1^+ + S_{22} v_2^+ \]

\[ v_2^+ = \frac{S_{21}}{\Gamma_L^{-1} - S_{22}} v_1^- \]

\[ v_1^- = \left( S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}} \right) v_1^+ \]

\[ \Gamma = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - \Gamma_L S_{22}} \]

If $|\Gamma| < 1$ for all $\Gamma_L$, then the two-port is stable

\[ \Gamma = \frac{S_{11}(1 - S_{22} \Gamma_L) + S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} = \frac{S_{11} + \Gamma_L (S_{21} S_{12} - S_{11} S_{22})}{1 - S_{22} \Gamma_L} \]

\[ = \frac{S_{11} - \Delta \Gamma_L}{1 - S_{22} \Gamma_L} \]
To find the boundary between stability/instability, let’s set $|\Gamma| = 1$

$$\left| \frac{S_{11} - \Delta \Gamma_L}{1 - S_{22} \Gamma_L} \right| = 1$$

$$|S_{11} - \Delta \Gamma_L| = |1 - S_{22} \Gamma_L|$$

After some algebraic manipulations, we arrive at the following equation

$$\left| \Gamma_L - \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2} \right| = \frac{|S_{12} S_{21}|}{|S_{22}|^2 - |\Delta|^2}$$

This is of course an equation of a circle, $|\Gamma_L - C| = R$, in the complex plane with center at $C$ and radius $R$

Thus a circle on the Smith Chart divides the region of instability from stability.
In this example, the origin of the circle lies outside the stability circle but a portion of the circle falls inside the unit circle. Is the region of stability inside the circle or outside?

This is easily determined if we note that if $\Gamma_L = 0$, then $\Gamma = S_{11}$. So if $S_{11} < 1$, the origin should be in the stable region. Otherwise, if $S_{11} > 1$, the origin should be in the unstable region.
Stability: Unilateral Case

- Consider the stability circle for a unilateral two-port

\[ C_S = \frac{S_{11}^* - (S_{11}^* S_{22}^*) S_{22}}{|S_{11}|^2 - |S_{11} S_{22}|^2} = \frac{S_{11}^*}{|S_{11}|^2} \]

\[ R_S = 0 \quad |C_S| = \frac{1}{|S_{11}|} \]

- The center of the circle lies outside of the unit circle if \(|S_{11}| < 1\). The same is true of the load stability circle. Since the radius is zero, stability is only determined by the location of the center.

- If \(S_{12} = 0\), then the two-port is unconditionally stable if \(S_{11} < 1\) and \(S_{22} < 1\).

- This result is trivial since

\[ \Gamma_S |_{S_{12}=0} = S_{11} \]

- The stability of the source depends only on the device and not on the load.
If we want to determine if a two-port is unconditionally stable, then we should use the $\mu$ test

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} > 1$$

The $\mu$ test not only is a test for unconditional stability, but the magnitude of $\mu$ is a measure of the stability. In other words, if one two port has a larger $\mu$, it is more stable.

The advantage of the $\mu$ test is that only a single parameter needs to be evaluated. There are no auxiliary conditions like the $K$ test derivation earlier.

The derivation of the $\mu$ test proceeds as follows. First let $\Gamma_S = |\rho_s|e^{j\phi}$ and evaluate $\Gamma_{out}$

$$\Gamma_{out} = \frac{S_{22} - \Delta|\rho_s|e^{j\phi}}{1 - S_{11}|\rho_s|e^{j\phi}}$$
Next we can manipulate this equation into the following circle:

\[ |\Gamma_{out} - C| = R \]

\[
\left| \Gamma_{out} + \frac{\rho_s |S_{11}^* \Delta - S_{22}|}{1 - |\rho_s| |S_{11}|^2} \right| = \frac{\sqrt{|\rho_s| |S_{12} S_{21}|}}{(1 - |\rho_s| |S_{11}|^2)}
\]

For a two-port to be unconditionally stable, we’d like \( \Gamma_{out} \) to fall within the unit circle:

\[ ||C| + R| < 1 \]

\[ ||\rho_s| S_{11}^* \Delta - S_{22}| + \sqrt{|\rho_s| |S_{21} S_{12}|} < 1 - |\rho_s| |S_{11}|^2 \]

\[ ||\rho_s| S_{11}^* \Delta - S_{22}| + \sqrt{|\rho_s| |S_{21} S_{12}|} + |\rho_s| |S_{11}|^2 < 1 \]

The worse case stability occurs when \( |\rho_s| = 1 \) since it maximizes the left-hand side of the equation. Therefore we have:

\[ \mu = \frac{1 - |S_{11}|^2}{|S_{11}^* \Delta - S_{22}| + |S_{12} S_{21}|} > 1 \]
The $K$ stability test has already been derived using $Y$ parameters. We can also do a derivation based on $S$ parameters. This form of the equation has been attributed to Rollett and Kurokawa.

The idea is very simple and similar to the $\mu$ test. We simply require that all points in the instability region fall outside of the unit circle.

The stability circle will intersect with the unit circle if

$$|C_L| - R_L > 1$$

or

$$\frac{|S_{22}^* - \Delta^* S_{11}| - |S_{12} S_{21}|}{|S_{22}|^2 - |\Delta|^2} > 1$$

This can be recast into the following form (assuming $|\Delta| < 1$)

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}| |S_{21}|} > 1$$
The power flowing into a two-port can be represented by

\[ P_{in} = \frac{|V_1^+|^2}{2Z_0} (1 - |\Gamma_{in}|^2) \]

The power flowing to the load is likewise given by

\[ P_L = \frac{|V_2^-|^2}{2Z_0} (1 - |\Gamma_L|^2) \]

We can solve for \( V_1^+ \) using circuit theory

\[ V_1^+ + V_1^- = V_1^+ (1 + \Gamma_{in}) = \frac{Z_{in}}{Z_{in} + Z_S} V_S \]

In terms of the input and source reflection coefficient

\[ Z_{in} = \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} Z_0 \]
\[ Z_S = \frac{1 + \Gamma_S}{1 - \Gamma_S} Z_0 \]
Two-Port Incident Wave

- Solve for $V_1^+$

$$V_1^+(1 + \Gamma_{in}) = \frac{V_S(1 + \Gamma_{in})(1 - \Gamma_S)}{(1 + \Gamma_{in})(1 - \Gamma_S) + (1 + \Gamma_S)(1 - \Gamma_{in})}$$

$$V_1^+ = \frac{V_S}{2} \frac{1 - \Gamma_S}{1 - \Gamma_{in}\Gamma_S}$$

- The voltage incident on the load is given by

$$V_2^- = S_{21} V_1^+ + S_{22} V_2^+ = S_{21} V_1^+ + S_{22} \Gamma_L V_2^-$$

$$V_2^- = \frac{S_{21} V_1^+}{1 - S_{22} \Gamma_L}$$

$$P_L = \frac{|S_{21}|^2 |V_1^+|^2}{|1 - S_{22} \Gamma_L|^2} \frac{1 - |\Gamma_L|^2}{2Z_0}$$
The operating power gain can be written in terms of the two-port s-parameters and the load reflection coefficient

\[ G_p = \frac{P_L}{P_{in}} = \frac{|S_{21}|^2 \left(1 - |\Gamma_L|^2\right)}{|1 - S_{22}\Gamma_L|^2 \left(1 - |\Gamma_{in}|^2\right)} \]

The available power can be similarly derived from \( V_1^+ \)

\[ P_{avs} = P_{in}|_{\Gamma_{in}=\Gamma_s^*} = \frac{|V_1^a|^2}{2 Z_0} \left(1 - |\Gamma_s^*|^2\right) \]

\[ V_{1a}^+ = V_1^+|_{\Gamma_{in}=\Gamma_s^*} = \frac{V_S}{2} \frac{1 - \Gamma_s^*}{1 - |\Gamma_s|^2} \]

\[ P_{avs} = \frac{|V_S|^2}{8 Z_0} \frac{1 - |\Gamma_S|^2}{1 - |\Gamma_S|^2} \]
The transducer gain can be easily derived

\[ G_T = \frac{P_L}{P_{avS}} = \frac{|S_{21}|^2 (1 - |\Gamma_L|^2)(1 - |\Gamma_S|^2)}{|1 - \Gamma_{in}\Gamma_S|^2 |1 - S_{22}\Gamma_L|^2} \]

Note that as expected, \( G_T \) is a function of the two-port s-parameters and the load and source impedance.

If the two port is connected to a source and load with impedance \( Z_0 \), then we have \( \Gamma_L = \Gamma_S = 0 \) and

\[ G_T = |S_{21}|^2 \]
If $S_{12} \approx 0$, we can simplify the expression by just assuming $S_{12} = 0$. This is the *unilateral* assumption

$$G_{TU} = \frac{1 - |\Gamma_S|^2}{|1 - S_{11} \Gamma_S|^2} |S_{21}|^2 \frac{1 - |\Gamma_L|^2}{|1 - S_{22} \Gamma_L|^2} = G_S |S_{21}|^2 G_L$$

The gain partitions into three terms, which can be interpreted as the gain from the source matching network, the gain of the two port, and the gain of the load.
We know that the maximum gain occurs for the biconjugate match

$$\Gamma_S = S_{11}^*$$

$$\Gamma_L = S_{22}^*$$

$$G_{S,max} = \frac{1}{1 - |S_{11}|^2}$$

$$G_{L,max} = \frac{1}{1 - |S_{22}|^2}$$

$$G_{TU,max} = \frac{|S_{21}|^2}{(1 - |S_{11}|^2)(1 - |S_{22}|^2)}$$

Note that if $|S_{11}| = 1$ or $|S_{22}| = 1$, the maximum gain is infinity. This is the unstable case since $|S_{ii}| > 1$ is potentially unstable.
So far we have only discussed power gain using bi-conjugate matching. This is possible when the device is unconditionally stable. In many cases, though, we’d like to design with a potentially unstable device.

Moreover, we would like to introduce more flexibility in the design. We can trade off gain for
- bandwidth
- noise
- gain flatness
- linearity
- etc.

We can make this tradeoff by identifying a range of source/load impedances that can realize a given value of power gain. While maximum gain is achieved for a single point on the Smith Chart, we will find that a lot more flexibility if we back-off from the peak gain.
Unilateral Design

- No real transistor is unilateral. But most are predominantly unilateral, or else we use cascades of devices (such as the cascode) to realize such a device.

- The *unilateral figure of merit* can be used to test the validity of the unilateral assumption

\[
U_m = \frac{|S_{12}|^2 |S_{21}|^2 |S_{11}|^2 |S_{22}|^2}{(1 - |S_{11}|^2)(1 - |S_{22}|^2)}
\]

- It can be shown that the transducer gain satisfies the following inequality

\[
\frac{1}{(1 + U)^2} < \frac{G_T}{G_{TU}} < \frac{1}{(1 - U)^2}
\]

- Where the actual power gain $G_T$ is compared to the power gain under the unilateral assumption $G_{TU}$. If the inequality is tight, say on the order of 0.1 dB, then the amplifier can be assumed to be unilateral with negligible error.
We now can plot gain circles for the source and load. Let

\[ g_S = \frac{G_S}{G_{S,\text{max}}} \]

\[ g_L = \frac{G_L}{G_{L,\text{max}}} \]

By definition, \(0 \leq g_S \leq 1\) and \(0 \leq g_L \leq 1\). One can show that a fixed value of \(g_S\) represents a circle on the \(\Gamma_S\) plane

\[ \left| \Gamma_S - \frac{S_{11}^* g_S}{|S_{11}|^2 (g_S - 1) + 1} \right| = \left| \frac{\sqrt{1 - g_S(1 - |S_{11}|^2)}}{|S_{11}|^2 (g_S - 1) + 1} \right| \]

More simply, \(|\Gamma_S - C_S| = R_S\). A similar equation can be derived for the load. Note that for \(g_S = 1\), \(R_S = 0\), and \(C_S = S_{11}^*\) corresponding to the maximum gain.
All gain circles lie on the line given by the angle of $S_{ii}^*$. We can select any desired value of source/load reflection coefficient to achieve the desired gain. To minimize the impedance mismatch and thus maximize the bandwidth, we should select a point closest to the origin.
For $|\Gamma| > 1$, we can still employ the Smith Chart if we make the following mapping. The reflection coefficient for a negative resistance is given by

$$\Gamma(-R + jX) = \frac{-R + jX - Z_0}{-R + jX + Z_0} = \frac{(R + Z_0) - jX}{(R - Z_0) - jX}$$

$$\frac{1}{\Gamma^*} = \frac{(R - Z_0) + jX}{(R + Z_0) + jX}$$

We see that $\Gamma$ can be mapped to the unit circle by taking $1/\Gamma^*$ and reading the resistance value (and noting that it’s actually negative).
For a unilateral two-port with $|S_{11}| > 1$, we note that the input impedance has a negative real part. Thus we can still design a stable amplifier as long as the source resistance is larger than $\mathcal{R}(Z_{in})$

$$\mathcal{R}(Z_S) > |\mathcal{R}(Z_{in})|$$

The same is true of the load impedance if $|S_{22}| > 1$. Thus the design procedure is identical to before as long as we avoid source or load reflection coefficients with real part less than the critical value.
Pot. Unstable Unilateral Amp Example

- Consider a transistor with the following S-Parameters

\[ S_{11} = 2.02 \angle -130.4^\circ \]
\[ S_{22} = 0.50 \angle -70^\circ \]
\[ S_{12} = 0 \]
\[ S_{21} = 5.00 \angle 60^\circ \]

- Since \(|S_{11}| > 1\), the amplifier is potentially unstable. We begin by plotting \(1/S_{11}^*\) to find the negative real input resistance.

- Now any source inside this circle is stable, since \(\Re(Z_S) > \Re(Z_{in})\).

- We also draw the source gain circle for \(G_S = 5\) dB.
The input impedance is read off the Smith Chart from $1/S_{11}^*$. Note the real part is interpreted as negative

$$Z_{in} = 50(-0.4 - 0.4j)$$

The $G_S = 5\,\text{dB}$ gain circle is calculated as follows

$$g_S = 3.15(1 - |S_{11}|^2)$$

$$R_S = \frac{\sqrt{1 - g_S(1 - |S_{11}|^2)}}{1 - |S_{11}|^2 (1 - g_S)} = 0.236$$

$$C_S = \frac{g_S S_{11}^*}{1 - |S_{11}|^2 (1 - g_S)} = -0.3 + 0.35j$$

We can select any point on this circle and obtain a stable gain of $5\,\text{dB}$. In particular, we can pick a point near the origin (to maximize the BW) but with as large of a real impedance as possible:

$$Z_S = 50(0.75 + 0.4j)$$
In the bilateral case, we will work with the power gain $G_p$. The transducer gain is not used since the source impedance is a function of the load impedance. $G_p$, on the other hand, is only a function of the load.

$$ G_p = \frac{|S_{21}|^2 (1 - |\Gamma_L|^2)}{\left(1 - \left|\frac{S_{11} - \Delta \Gamma}{1 - S_{22} \Gamma_L}\right|^2\right) |1 - S_{22} \Gamma_L|^2} = |S_{21}|^2 g_p $$

It can be shown that $g_p$ is a circle on the $\Gamma_L$ plane. The radius and center are given by

$$ R_L = \sqrt{1 - 2K |S_{12} S_{21}| g_p + |S_{12} S_{21}|^2 g_p^2} $$

$$ C_L = \frac{g_p (S_{22}^* - \Delta^* S_{11})}{1 + g_p (|S_{22}|^2 - |\Delta|^2)} $$
Bilateral Amp (cont)

- We can also use this formula to find the maximum gain. We know that this occurs when $R_L = 0$, or

$$1 - 2K|S_{12}S_{21}|g_{p,\text{max}} + |S_{12}S_{21}|^2g_{p,\text{max}}^2 = 0$$

$$g_{p,\text{max}} = \frac{1}{|S_{12}S_{21}|} \left( K - \sqrt{K^2 - 1} \right)$$

$$G_{p,\text{max}} = \frac{|S_{21}|}{|S_{12}|} \left( K - \sqrt{K^2 - 1} \right)$$

- The design procedure is as follows
  1. Specify $g_p$
  2. Draw operating gain circle.
  3. Draw load stability circle. Select $\Gamma_L$ that is in the stable region and not too close to the stability circle.
  5. To maximize gain, calculate $\Gamma_{in}$ and check to see if $\Gamma_S = \Gamma_{in}^*$ is in the stable region. If not, iterate on $\Gamma_L$ or compromise.