Problem Set 6
Solutions

Please note that these are merely suggested solutions. Many of these problems can be approached in different ways.

1. (a) First we note that we simply have three impedances in series, where \( Z_C = \frac{1}{j\omega C} \) and \( Z_R = R \). Then we derive \( V_o \) using a voltage divider:

\[
H(\omega) = \frac{V_o}{V_i} = \frac{Z_C}{2Z_C + Z_R} = \frac{1/j\omega C}{2/j\omega C + R} = \frac{1}{2 + j\omega RC} = \frac{1}{2 + j\omega(0.02)}
\]

(b) The magnitude is found by considering the magnitude of the numerator divided by that of the denominator. Recall that the magnitude of a complex number in rectangular form can be found using the Pythagorean theorem.

\[
|H(\omega)| = \left|1\right| \left|2 + j\omega(0.02)\right| = \frac{1}{\sqrt{4 + (0.02\omega)^2}}
\]

For the phase, we look at the phase of the numerator minus the phase of the denominator. Note that the phase is just the angle of the complex number above the real axis. In general, this is given by the inverse tangent of the imaginary part divided by the real part.

\[
\angle H(\omega) = \angle 1 - \angle (2 + j\omega(0.02)) = 0 - \tan^{-1}\left(\frac{0.02\omega}{2}\right) = -\tan^{-1}(0.01\omega)
\]

(c) The phasor representation compacts the signal’s amplitude and phase as follows:

\( V_i = 5\angle \frac{\pi}{4} \) V. Note that the frequency is not expressed in phasor form.

(d) For any LTI system in general (and thus the circuits that we analyze in this class), the output in response to an input of the form \( v_i(t) = A\cos(\omega t + \phi) \) is \( v_o(t) = |H(\omega)|A\cos(\omega t + \phi + \angle H(\omega)) \). In our case, \( \omega = 100 \). So

\[
v_o(t) = |H(100)|5\cos(100t + \frac{\pi}{4} + \angle H(100)) = (0.354)5\cos(100t + \frac{\pi}{4} - \frac{\pi}{4}) = 1.77\cos(100t)
\]

In phasor notation, this is simply \( V_o = 1.77\angle 0 \) V.

Note that you could also have found the solution by first considering

\[
V_o = H(\omega)V_i = H(100)5\angle \frac{\pi}{4} = \frac{5e^{j\pi/4}}{2 + j2} = 1.77e^0 = 1.77\angle 0 \text{ V}
\]

Then you can convert this back to the time domain by writing this as a cosine function with the same frequency as the input, namely \( \omega = 100 \).
2. Turning off the current source will make it an open circuit. Then $V_{1a}$ can be found using a voltage divider:

$$V_{1a} = \frac{-j10 + 100}{10 - j10 + 100} \angle 90^\circ = \frac{100 - j10}{110 - j10} \angle 20^\circ = 0.164 + j18.2 \text{ V}$$

Turning off the voltage source makes it a short. So we can write a single KCL equation at $V_{1b}$ only taking into account the current source:

$$\frac{V_{1b}}{10} + \frac{V_{1b}}{100 - j10} + 2 = 0 \Rightarrow V_{1b} = \frac{j20 - 200}{11 - j} = -18.2 + j0.164 \text{ V}$$

Notice that we kept the two solutions in rectangular form for easier addition. The final solution is

$$V_1 = V_{1a} + V_{1b} = -18 + j18.4 = 25.7 \angle 134^\circ \text{ V}$$

3. Starting with $V_{oc}$, notice that $I_x$ must remain 0 for KCL to hold at the bottom right node. Since no current flows across the capacitor, there is no voltage drop across it, and $V_{oc} = 20 \angle 45^\circ \text{ V}$. For $I_{sc}$, adding a short between terminals $a$ and $b$ no longer automatically forces all the current to be 0. By KCL, we infer that $I_{sc} = 2I_x$.

The most straightforward way to solve for $I_x$ is to write a KVL loop. We start from the reference node and work our way clockwise (note that there is no voltage drop across the current source as it is shorted).

$$-20 \angle 45^\circ - j10I_x + 100I_x = 0 \Rightarrow I_x = \frac{20 \angle 45^\circ}{100 - j10} = 0.2 \angle 50.7^\circ \text{ A}$$

Thus, the short-circuit current is $I_{sc} = 2I_x = 0.4 \angle 50.7^\circ \text{ A}$.

We thus have $V_{Th} = V_{oc} = 20 \angle 45^\circ \text{ V}$ and $I_N = I_{sc} = 0.4 \angle 50.7^\circ \text{ A}$. The last step is to find the Thévenin/Norton impedance, which is given by Ohm’s law:

$$Z_{Th} = Z_N = \frac{V_{Th}}{I_N} = \frac{20 \angle 45^\circ \text{ V}}{0.4 \angle 50.7^\circ \text{ A}} = 50 - j5 \Omega = 50.2 \angle -5.71^\circ \Omega$$

4. If we view the components in terms of their complex impedances, we simply have an inverting amplifier. Recall that the transfer function is given by

$$H(\omega) = \frac{V_o}{V_i} = -\frac{Z_2}{Z_1} = -\frac{R}{R + 1/j\omega C} = -\frac{j\omega RC}{1 + j\omega RC}$$

This is a high-pass filter. As $\omega \to 0$ at low frequencies, the magnitude of the transfer function goes to 0; as $\omega \to \infty$ at higher frequencies, the transfer function approaches 1 and faithfully passes the input signal.