Lecture 10:

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Consider the effect of an $m$’th order non-linearity on an input of $N$ tones.

\[ y_m = \left( \sum_{n=1}^{N} A_n \cos \omega_n t \right)^m \]

\[ y_m = \left( \sum_{n=1}^{N} \frac{A_n}{2} \left( e^{\omega_n t} + e^{-\omega_n t} \right) \right)^m \]

\[ y_m = \left( \sum_{n=-N}^{N} \frac{A_n}{2} e^{\omega_n t} \right)^m \]

where we assumed that $A_0 \equiv 0$ and $\omega_{-k} = -\omega_k$. 
Product of sums...

The product of sums can be written as lots of sums...

\[ = \sum() \times \sum() \times \sum() \times \cdots \times \sum() \]

\[ = \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} \cdots \sum_{k_m=-N}^{N} \frac{A_{k_1} A_{k_2} \cdots A_{k_m}}{2^m} e^{j(\omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_m}) t} \]

Notice that we generate frequency component \( \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_m} \), sums and differences between \( m \) non-distinct frequencies.

There are a total of \((2N)^m\) terms.
Example

Let’s take a simple example of $m = 3, \ N = 2$. We already know that this cubic non-linearity will generate harmonic distortion and $IM$ products.

We have $(2N)^m = 4^3 = 64$ combinations of complex frequencies. $\omega \in \{-\omega_2, -\omega_1, \omega_1, \omega_2\}$. There are 64 terms that looks like this ($HD_3$)

$$\omega_1 + \omega_1 + \omega_1 = 3\omega_1$$

$$\omega_1 + \omega_1 + \omega_2 = 2\omega_1 + \omega_2$$

($IM3$)

$$\omega_1 + \omega_1 - \omega_2 = 2\omega_1 - \omega_2$$

(Gain compression or expansion)

$$\omega_1 + \omega_1 - \omega_1 = \omega_1$$
Frequency Mix Vector

Let the vector \( \vec{k} = (k_{-N}, \ldots, k_{-1}, k_1, \ldots, k_N) \) be a \( 2N \)-vector where element \( k_j \) denotes the number of times a particular frequency appears in a given term.

As an example, consider the frequency terms

\[
\begin{align*}
\omega_2 + \omega_1 + \omega_2 \\
\omega_1 + \omega_2 + \omega_2 \\
\omega_2 + \omega_2 + \omega_1
\end{align*}
\]

\[\vec{k} = (0, 0, 1, 2)\]
Properties of $\vec{k}$

- First it’s clear that the sum of the $k_j$ must equal $m$

$$\sum_{j=-N}^{N} k_j = k_{-N} + \cdots + k_{-1} + k_1 + \cdots + k_N = m$$

- For a fixed vector $\vec{k}_0$, how many different sum vectors are there?

- We can sum $m$ frequencies $m!$ ways. But the order of the sum is irrelevant. Since each $k_j$ coefficient can be ordered $k_j!$ ways, the number of ways to form a given frequency product is given by

$$\binom{m}{\vec{k}} = \frac{m!}{(k_{-N})! \cdots (k_{-1})!(k_1)!) \cdots (k_N)!}$$
Extraction of Real Signal

- Since our signal is real, each term has a complex conjugate present. Hence there is another vector \( \vec{k}'_0 \) given by

\[
\vec{k}'_0 = (k_N, \ldots, k_1, k_{-1}, \ldots, k_{-N})
\]

- Notice that the components are in reverse order since \( \omega_{-j} = -\omega_j \). If we take the sum of these two terms we have

\[
2 \Re \left\{ e^{j(\omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_m})t} \right\} = 2 \cos(\omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_m})t
\]

- The amplitude of a frequency product is thus given by

\[
\frac{2 \times (m; \vec{k})}{2m} = \frac{(m; \vec{k})}{2^{m-1}}
\]
Example: $IM_3$ Again

- Using this new tool, let’s derive an equation for the $IM_3$ product more directly.

- Since we have two tones, $N = 2$. $IM_3$ is generated by a $m = 3$ non-linear term.

- A particular $IM_3$ product, such as $(2\omega_1 - \omega_2)$, is generated by the frequency mix vector $\vec{k} = (1, 0, 2, 0)$.

\[
(m; \vec{k}) = \frac{3!}{1! \cdot 2!} = 3 \quad 2^{m-1} = 2^2 = 4
\]

- So the amplitude of the $IM_3$ product is $3/4a_3s_i^3$. Relative to the fundamental

\[
IM_3 = \frac{3}{4} \frac{a_3s_i^3}{a_1s_i} = \frac{3}{4} \frac{a_3}{a_1} s_i^2
\]
Harder Example: Pentic Non-Linearity

Let’s calculate the gain expansion/compression due to the 5th order non-linearity. For a one tone, we have $N = 1$ and $m = 5$.

A pentic term generates fundamental as follows

$$\omega_1 + \omega_1 + \omega_1 - \omega_1 - \omega_1 = \omega_1$$

In terms of the $\vec{k}$ vector, this is captured by $\vec{k} = (2, 3)$. The amplitude of this term is given by

$$\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4}{2} = 10 \quad 2^{m-1} = 2^4 = 16$$

So the fundamental amplitude generated is $a_5 \frac{10}{16} S_i^5$. 
Apparent Gain Due to Pentic

- The apparent gain of the system, including the 3rd and 5th, is thus given by

\[ \text{AppGain} = a_1 + \frac{3}{4} a_3 S_i^2 + \frac{10}{16} a_5 S_i^4 \]

- At what signal level is the 5th order term as large as the 3rd order term?

\[ \frac{3}{4} a_3 S_i^2 = \frac{10}{16} a_5 S_i^4 \]

\[ S_i = \sqrt{1.2 \frac{a_3}{a_5}} \]

- For a bipolar amplifier, we found that \( a_3 = 1/(3! V_t^3) \) and \( a_5 = 1/(5! V_t^5) \). Solving for \( S_i \), we have

\[ S_i = V_t \sqrt{1.2 \times 5 \times 4} \approx 127 \text{ mV} \]
We usually implement the feedback with a passive network.

Assume that the only distortion is in the forward path $a$

\[ s_o = a_1 s_\epsilon + a_2 s_\epsilon^2 + a_3 s_\epsilon^3 + \cdots \]

\[ s_\epsilon = s_i - f s_o \]

\[ s_o = a_1 (s_i - f s_o) + a_2 (s_i - f s_o)^2 + a_3 (s_i - f s_o)^3 + \cdots \]
Feedback and Disto (cont)

- We’d like to ultimately derive an equation as follows

\[ s_o = b_1 s_i + b_2 s_i^2 + b_3 s_i^3 + \cdots \]

- Substitute this solution into the equation to obtain

\[
\begin{align*}
    b_1 s_i + b_2 s_i^2 + b_3 s_i^3 + \cdots &= a_1 (s_i - f b_1 s_i - f b_2 s_i^2 - \cdots) \\
    &+ a_2 (b_1 s_i + b_2 s_i^2 + b_3 s_i^3 + \cdots)^2 \\
    &+ a_3 (b_1 s_i + b_2 s_i^2 + b_3 s_i^3 + \cdots)^3 + \cdots
\end{align*}
\]

- Solve for the first order terms

\[ b_1 s_i = a_1 (s_i - f b_1 s_i) \]

\[ b_1 = \frac{a_1}{1 + a_1 f} = \frac{a_1}{1 + T} \]
The above equation is the same as linear analysis (loop gain $T = a_1 f$)

Now let’s equate second order terms

$$b_2 s_i^2 = -a_1 f b_2 s_i^2 + a_2 (s_i - f b_1 s_i)^2$$

$$b_2 (a + a_1 f) = a_2 \left( 1 - \frac{fa_1}{1 + T} \right)^2$$

$$b_2 (1 + T)^3 = a_2 (1 + T - T)^2 = a_2$$

$$b_2 = \frac{a_2}{(1 + T)^3}$$

Same equation holds at high frequency if we replace with $T(j\omega)$
Equating third-order terms

\[ b_3 s_i^3 = a_1(-fb_3 s_i^3) + a_2(-fb_2 s_i^3) + a_3(s_i - fb_1 s_i)^3 + \cdots \]

\[ b_3(1 + a_1 f) = -2a_2 b_2 f \frac{1}{1 + T} + \frac{a_3}{(1 + T)^3} \]

\[ b_3(1 + T) = \frac{-2a_2 f}{1 + T} \frac{a_2}{(1 + T)^3} + \frac{a_3}{(1 + T)^3} \]

\[ b_3 = \frac{a_3(1 + a_1 f) - 2a_2^2 f}{(1 + a_1 f)^5} \]
Second Order Interaction

- The term $2a_2^2 f$ is the second order interaction
- Second order disto in fwd path is fed back and combined with the input linear terms to generate third order disto
- Can get a third order null if

$$a_3(1 + a_1 f) = 2a_2^2 f$$
$H D_2$ in Feedback Amp

$$H D_2 = \frac{1}{2} b_1^2 s_{om}$$

$$= \frac{1}{2} \frac{a_2}{(1 + T)^3} \frac{(1 + T)^2}{a_1^2} s_{om}$$

$$= \frac{1}{2} \frac{a_2}{a_1^2} \frac{s_{om}}{1 + T}$$

- Without feedback $H D_2 = \frac{1}{2} \frac{a_2}{a_1^2} s_{om}$
- For a given output signal, the negative feedback reduces the second order distortion by $\frac{1}{1 + T}$
$HD_3$ in Feedback Amp

\[ HD_3 = \frac{1}{4} \frac{b_3}{b_1^3} s_{om}^2 \]

\[ = \frac{1}{4} \frac{a_3 (1 + T) - 2a_2^2 f}{(1 + T)^3} \frac{(1 + T)^3}{a_1^3} s_{om}^2 \]

\[ = \frac{1}{4} \frac{a_3}{a_1^3} s_{om}^2 \frac{1}{1 + T} \left[ 1 - \frac{2a_2^2 f}{a_3 (1 + T)} \right] \]

\[ \text{disto with no fb} \]
Feedback versus Input Attenuation

Notice that the distortion is improved for a given output signal level. Otherwise we can see that simply decreasing the input signal level improves the distortion.

Say $s_{o1} = fs_i$ with $f < 1$. Then

$$s_o = a_1s_{o1} + a_2s_{o1}^2 + a_3s_{o1}^3 + \cdots = a_1f s_i + a_2f^2 s_i^2 + a_3f^3 s_i^3 + \cdots$$

But the distortion is unchanged for a given output signal

$$HD_2 = \frac{1}{2} \frac{b_2}{b_1^2} s_{om} = \frac{1}{2} \frac{a_2}{a_1^2} s_{om}$$
The total input signal applied to the base of the amplifier is

\[ v_i + V_Q = V_{BE} + I_E R_E \]

- The \( V_{BE} \) and \( I_E \) terms can be split into DC and AC currents (assume \( \alpha \approx 1 \))

\[ v_i + V_Q = V_{BE,Q} + v_{be} + (I_Q + i_c)R_E \]

- Subtracting bias terms we have a separate AC and DC equation

\[ V_Q = V_{BE,Q} + I_Q R_E \]

\[ v_i = v_{be} + i_C R_E \]
The AC equation can be put into the following form

\[ v_{be} = v_i - i_c R_E \]

Compare this to our feedback equation

\[ s_\epsilon = s_i - f s_o \]

The equations have the same form with the following substitutions

\[ s_\epsilon = v_{be} \]
\[ s_o = i_c \]
\[ s_i = v_i \]
\[ f = R_E \]
Now we know that

\[ i_c = a_1 v_{be} + a_2 v_{be}^2 + a_3 v_{be}^3 + \cdots \]

where the coefficients \( a_{1,2,3,\ldots} \) come from expanding the exponential into a Taylor series

\[ a_1 = g_m \quad a_2 = \frac{1}{2} \frac{I_Q}{V_t^2} \quad \cdots \]

With feedback we have

\[ i_c = b_1 v_i + b_2 v_i^2 + b_3 v_i^3 + \cdots \]
Emitter Degeneration (cont)

The loop gain \( T = a_1 f = g_m R_E \)

\[
\begin{align*}
    b_1 &= \frac{g_m}{1 + g_m R_E} \\
    b_2 &= \frac{\frac{1}{2} \left( \frac{q}{kT} \right)^2 I_Q}{(1 + g_m R_E)^3} \\
    b_3 &= \cdots
\end{align*}
\]
Harmonic Distortion with Feedback

Using our previously derived formulas we have

\[ HD_2 = \frac{1}{2} \frac{b_2}{b_1^2} s_{om} \]

\[ = \frac{1}{4} \frac{i_c}{i_q} \frac{1}{1 + g_m R_E} \]

\[ HD_3 = \frac{1}{4} \frac{b_3}{b_1^3} s_{om}^2 \]

\[ = \frac{1}{24} \left( \frac{i_c}{I_Q} \right)^2 \frac{1 - \frac{3g_m R_E}{1 + g_m R_E}}{1 + g_m R_E} \]
Harmonic Distortion Null

- We can adjust the feedback to obtain a null in $HD_3$
- $HD_3 = 0$ can be achieved with

$$\frac{3gmR_E}{1 + gmR_E} = 1$$

or

$$R_E = \frac{1}{2gm}$$
Example: For $I_Q = 1\, \text{mA}$, $R_E = 13\, \Omega$
BJT with Finite Source Resistance

Assuming that $\alpha \approx 1$, $\beta = \beta_0$ (constant). Let $R_B = R_S + r_b$ represent the total resistance at the base.

$$v_i + V_Q - I_B R_B = V_{BE} + I_E R_E$$

- The formula is the same as the case of a BJT with emitter degeneration with $R'_E = R_E + R_B / \beta_0$
Emitter Follower

\[ v_i \quad \downarrow \quad I_C \quad \downarrow \quad v_o \]

\[ V_Q \quad R_L \]

The same equations as before with \( R_E = R_L \)
Same equation as CE with $R_E$ feedback

$$v_i - V_Q + I_C R_E = -V_{BE}$$