Transmission Line Menagerie

- T-Lines come in many shapes and sizes
- Coaxial usually 75Ω or 50Ω (cable TV, Internet)
- Microstrip lines are common on printed circuit boards (PCB) and integrated circuit (ICs)
- Coplanar also common on PCB and ICs
- Twisted pairs is almost a T-line, ubiquitous for phones/Ethernet
Waveguides and Transmission Lines

The transmission lines we’ve been considering have been propagating the “TEM” mode or Transverse Electro-Magnetic. Later we’ll see that they can also propagate other modes.

Waveguides cannot propagate TEM but propagation TM (Transverse Magnetic) and TE (Transverse Electric).

In general, any set of more than one lossless conductors with uniform cross-section can transmit TEM waves. Low loss conductors are commonly approximated as lossless.
Cascade of T-Lines (I)

Consider the junction between two transmission lines $Z_{01}$ and $Z_{02}$.

At the interface $z = 0$, the boundary conditions are that the voltage/current has to be continuous.

\[ v_1^+ + v_1^- = v_2^+ \]

\[ (v_1^+ - v_1^-) / Z_{01} = v_2^+ / Z_{02} \]
Cascade of T-Lines (II)

- Solve these equations in terms of $v_1^+$
- The reflection coefficient has the same form (easy to remember)
  \[
  \Gamma = \frac{v_1^-}{v_1^+} = \frac{Z_{02} - Z_{01}}{Z_{01} + Z_{02}}
  \]
- The second line looks like a load impedance of value $Z_{02}$
Transmission Coefficient

- The wave launched on the new transmission line at the interface is given by

\[ v_2^+ = v_1^+ + v_1^- = v_1^+ (1 + \Gamma) = \tau v_1^+ \]

- This “transmitted” wave has a coefficient

\[ \tau = 1 + \Gamma = \frac{2Z_{02}}{Z_{01} + Z_{02}} \]

- Note the incoming wave carries a power

\[ P_{in} = \frac{|v_1^+|^2}{2Z_{01}} \]
Conservation of Energy

- The reflected and transmitted waves likewise carry a power of

\[ P_{ref} = \frac{|v_1^-|^2}{2Z_{01}} = |\Gamma|^2 \frac{|v_1^+|^2}{2Z_{01}} \quad P_{tran} = \frac{|v_2^+|^2}{2Z_{02}} = |\tau|^2 \frac{|v_1^+|^2}{2Z_{02}} \]

- By conservation of energy, it follows that

\[ P_{in} = P_{ref} + P_{tran} \]

\[ \frac{1}{Z_{02}} \tau^2 + \frac{1}{Z_{01}} \Gamma^2 = \frac{1}{Z_{01}} \]

- You can verify that this relation holds!
Consider the bounce diagram for the following arrangement.
Junction of Parallel T-Lines

- Again invoke voltage/current continuity at the interface

\[ v_1^+ + v_1^- = v_2^+ = v_3^+ \]

\[ \frac{v_1^+ - v_1^-}{Z_{01}} = \frac{v_2^+}{Z_{02}} + \frac{v_3^+}{Z_{02}} \]

- But \( v_2^+ = v_3^+ \), so the interface just looks like the case of two transmission lines \( Z_{01} \) and a new line with char. impedance \( Z_{01} \parallel Z_{02} \).
Let’s analyze the problem intuitively first.

When a pulse first “sees” the inductance at the load, it looks like an open so \( \Gamma_0 = +1 \).

As time progresses, the inductor looks more and more like a short! So \( \Gamma_\infty = -1 \).
Reactive Terminations (II)

So intuitively we might expect the reflection coefficient to look like this:

The graph starts at $+1$ and ends at $-1$. In between we’ll see that it goes through exponential decay (1st order ODE)
Reactive Terminations (III)

Do equations confirm our intuition?

\[ v_L = L \frac{di}{dt} = L \frac{d}{dt} \left( \frac{v^+}{Z_0} - \frac{v^-}{Z_0} \right) \]

And the voltage at the load is given by \( v^+ + v^- \)

\[ v^- + \frac{L}{Z_0} \frac{dv^-}{dt} = \frac{L}{Z_0} \frac{dv^+}{dt} - v^+ \]

The right hand side is known, it’s the incoming waveform
Solution for Reactive Term

- For the step response, the derivative term on the RHS is zero at the load

\[ v^+ = \frac{Z_0}{Z_0 + R_s} V_s \]

- So we have a simpler case \( \frac{dv^+}{dt} = 0 \)

- We must solve the following equation

\[ v^- + \frac{L}{Z_0} \frac{dv^-}{dt} = -v^+ \]

- For simplicity, assume at \( t = 0 \) the wave \( v^+ \) arrives at load
Laplace Domain Solution I

In the Laplace domain

\[ V^-(s) + \frac{sL}{Z_0} V^-(s) - \frac{L}{Z_0} v^-(0) = -v^+/s \]

Solve for reflection \( V^-(s) \)

\[ V^-(s) = \frac{v^-(0)L/Z_0}{1 + sL/Z_0} - \frac{v^+}{s(1 + sL/Z_0)} \]

Break this into basic terms using partial fraction expansion

\[ \frac{-1}{s(1 + sL/Z_0)} = \frac{-1}{1 + sL/Z_0} + \frac{L/Z_0}{1 + sL/Z_0} \]
Laplace Domain Solution (II)

Invert the equations to get back to time domain \( t > 0 \)

\[
v^-(t) = (v^-(0) + v^+)e^{-t/\tau} - v^+
\]

Note that \( v^-(0) = v^+ \) since initially the inductor is an open

So the reflection coefficient is

\[
\Gamma(t) = 2e^{-t/\tau} - 1
\]

The reflection coefficient decays with time constant \( L/Z_0 \)
Time Harmonic Steady-State

- Compared with general transient case, sinusoidal case is very easy $\frac{\partial}{\partial t} \rightarrow j\omega$

- Sinusoidal steady state has many important applications for RF/microwave circuits

- At high frequency, T-lines are like interconnect for distances on the order of $\lambda$

- Shorted or open T-lines are good resonators

- T-lines are useful for impedance matching
Why Sinusoidal Steady-State?

- Typical RF system modulates a sinusoidal carrier (either frequency or phase)
- If the modulation bandwidth is much smaller than the carrier, the system looks like it’s excited by a pure sinusoid
- Cell phones are a good example. The carrier frequency is about 1 GHz and the voice digital modulation is about 200 kHz (GSM) or 1.25 MHz (CDMA), less than a 0.1% of the bandwidth/carrier
**Generalized Distributed Circuit Model**

- \( Z' \): impedance per unit length (e.g. \( Z' = j\omega L' + R' \))
- \( Y' \): admittance per unit length (e.g. \( Y' = j\omega C' + G' \))
- A lossy T-line might have the following form (but we’ll analyze the general case)
Time Harmonic Telegrapher’s Equations

- Applying KCL and KVL to a infinitesimal section

\[ v(z + \delta z) - v(z) = -Z'\delta z i(z) \]

\[ i(z + \delta z) - i(z) = -Y'\delta z v(z) \]

- Taking the limit as before \((\delta z \to 0)\)

\[ \frac{dv}{dz} = -Z i(z) \]

\[ \frac{di}{dz} = -Y v(z) \]
Sin. Steady-State (SSS) Voltage/Current

- Taking derivatives (notice \( z \) is the only variable) we arrive at

\[
\frac{d^2 v}{dz^2} = -Z \frac{di}{dz} = YZv(z) = \gamma^2 v(z)
\]

\[
\frac{d^2 i}{dz^2} = -Y \frac{dv}{dz} = YZi(z) = \gamma^2 i(z)
\]

- Where the propagation constant \( \gamma \) is a complex function

\[
\gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}
\]

- The general solution to \( D^2 G - \gamma^2 G = 0 \) is \( e^{\pm \gamma z} \)
The voltage and current are related (just as before, but now easier to derive)

\[ v(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z} \]

\[ i(z) = \frac{V^+}{Z_0} e^{-\gamma z} - \frac{V^-}{Z_0} e^{\gamma z} \]

Where \( Z_0 = \sqrt{\frac{Z'}{Y'}} \) is the characteristic impedance of the line (function of frequency with loss)

For a lossless line we discussed before, \( Z' = j \omega L' \) and \( Y' = j \omega C' \)

Propagation constant is imaginary

\[ \gamma = \sqrt{j \omega L' j \omega C'} = j \sqrt{L' C'} \omega \]
Recall that the real voltages and currents are the $\Re$ and $\Im$ parts of

$$v(z, t) = e^{\pm \gamma z} e^{j\omega t} = e^{j\omega t \pm \beta z}$$

Thus the voltage/current waveforms are sinusoidal in space and time.

Sinusoidal source voltage is transmitted unaltered onto T-line (with delay).

If there is loss, then $\gamma$ has a real part $\alpha$, and the wave decays or grows on the T-line:

$$e^{\pm \gamma z} = e^{\pm \alpha z} e^{\pm j\beta z}$$

The first term represents amplitude response of the T-line.
Passive T-Line/Wave Speed

- For a passive line, we expect the amplitude to decay due to loss on the line.

- The speed of the wave is derived as before. In order to follow a constant point on the wavefront, you have to move with velocity

\[
\frac{d}{dt} (\omega t \pm \beta z = \text{constant})
\]

- Or, \( v = \frac{dz}{dt} = \pm \frac{\omega}{\beta} = \pm \sqrt{\frac{1}{L'C''}} \)