EECS 117

Lecture 21: Wave Propagation in Lossy Media and Poynting’s Theorem

Prof. Niknejad

University of California, Berkeley
Time-Harmonic Wave Equation

Start by taking the curl of Faraday's Eq.

\[ \nabla \times (\nabla \times E) = -j\omega \nabla \times B \]

\[ \nabla \times H = \sigma E + j\omega \epsilon E \]

\[ \nabla \times (\nabla \times E) = -j\omega \mu (\sigma E + j\omega \epsilon E) \]

In a source free region, \( \nabla \cdot E = 0 \), and thus

\[ \nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E = -\nabla^2 E \]

We thus have Helmholtz’ equation

\[ \nabla^2 E - \gamma^2 E = 0 \]

Where \( \gamma^2 = j\omega \mu (\sigma + j\omega \epsilon) = \alpha + j\beta \)
Lossy Materials

- In addition to conductive losses $\sigma$, materials can also have dielectric and magnetic losses.

- A lossy dielectric is characterized by a complex permittivity $\varepsilon = \varepsilon_r + j\varepsilon_i$, where $\varepsilon_i$ arises due to the phase lag between the field and the polarization. Likewise $\mu = \mu_r + j\mu_i$.

- Most materials we study are weakly magnetic and thus $\mu \approx \mu_r$.

- For now assume that $\varepsilon, \mu, \text{ and } \sigma$ are real scalar quantities.

- Thus

\[
\gamma = \sqrt{(-\omega^2 \varepsilon \mu)(1 + \frac{\sigma}{j\omega\varepsilon})}
\]
Let’s compute the real and imaginary part of $\gamma$

$$\gamma = j\omega \sqrt{\varepsilon \mu} \left(1 - j \frac{\sigma}{\omega \varepsilon}\right)^{\frac{1}{2}}$$

Consider $(1 - jh) = re^{-j\theta}$, so that

$$y = \sqrt{1 - jh} = \sqrt{re^{-j\theta/2}}$$

Note that $\tan \theta = -h$, and $r = \sqrt{1 + h^2}$. Finally

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 + \frac{1}{r}}{2}} = \sqrt{\frac{r + 1}{2r}}$$
Similarly

\[
\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{r - 1}{2r}}
\]

\[
y = \sqrt{r} e^{-j\theta/2} = \sqrt{\frac{r + 1}{2}} - j \sqrt{\frac{r - 1}{2}} = a + jb
\]

Using the above manipulations, we can now break \( \gamma \) into real and imaginary components

\[
\gamma = j\omega \sqrt{\mu \varepsilon} (a + jb) = -\omega \sqrt{\mu \varepsilon} b + j\omega \sqrt{\mu \varepsilon} a = \alpha + j\beta
\]

\[
\alpha = -\omega \sqrt{\mu \varepsilon} \left( -\frac{\sqrt{r - 1}}{\sqrt{2}} \right)
\]
We have now finally shown that

\[ \alpha = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]^{1/2} \]

\[ \beta = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} + 1 \right]^{1/2} \]

It’s easy to show that the imaginary part of \( \epsilon \) can be lumped into an effective conductivity term.

In practice, most materials are either low loss, such that \( \frac{\sigma_{\text{eff}}}{\omega \epsilon} \ll 1 \), or good conductors, such that \( \frac{\sigma_{\text{eff}}}{\omega \epsilon} \gg 1 \).

In these extreme cases, simplified versions of the above equations are applicable.
Effective Dielectric Constant

We can also lump the conductivity into an effective dielectric constant

\[ \nabla \times \mathbf{H} = \sigma \mathbf{E} + j\omega \varepsilon \mathbf{E} = j\omega \varepsilon_{\text{eff}} \mathbf{E} \]

where \( \varepsilon_{\text{eff}} = \varepsilon - j\sigma/\omega \). In the low loss case, this is a good way to include the losses

When \( \varepsilon \) or \( \mu \) become complex, the wave impedance is no longer real and the electric and magnetic field fall out of phase. Since \( H = E/\eta_c \)

\[ \eta_c = \sqrt{\frac{\mu}{\varepsilon_{\text{eff}}}} = \sqrt{\frac{\mu}{\varepsilon - j\sigma/\omega}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}} \]
Propagation in Low Loss Materials

- If $\frac{\sigma}{\omega \varepsilon} \ll 1$, then our equations simplify

\[ \gamma = j \omega \sqrt{\mu \varepsilon} \left( 1 - j \frac{1}{2} \frac{\sigma}{\omega \varepsilon} \right) \]

- To first order, the propagation constant is unchanged by the losses ($\sigma_{\text{eff}} = \sigma + \omega \varepsilon''$)

\[ \beta = \omega \sqrt{\mu \varepsilon} \quad \alpha = \frac{1}{2} \sigma_{\text{eff}} \sqrt{\frac{\mu}{\varepsilon}} \]

- A more accurate expression can be obtained with a 1st order expansion of $(1 + x)^n$

\[ \beta = \omega \sqrt{\mu \varepsilon} \left( 1 + \frac{1}{8} \left( \frac{\sigma_{\text{eff}}}{\omega \varepsilon'} \right)^2 \right) \]
Propagation in Conductors

As we saw in the previous lecture, this approximation is valid when \( \frac{\sigma}{\omega \epsilon} \gg 1 \)

\[
\gamma = \alpha + j\beta = \sqrt{j\omega \mu \sigma} = \omega \mu \sigma e^{j45^\circ}
\]

\[
\alpha = \beta = \sqrt{\frac{\omega \mu \sigma}{2}}
\]

The phase velocity is given by \( v_p = \omega / \beta \)

\[
v_p = \sqrt{\frac{2\omega}{\mu \sigma}}
\]

This is a function of frequency! This is a very dispersive medium.
Waves in Conductors

- The wavelength is given by

\[ \lambda = \frac{v_p}{f} = 2\sqrt{\frac{\pi}{f \mu \sigma}} \]

Example: Take \( \sigma = 10^7 \text{ S/m} \) and \( f = 100 \text{ MHz} \). Using the above equations

\[ \lambda = 10^{-4} \text{ m} \]

\[ v_p = 10^4 \text{ m/s} \]

- Note that \( \lambda_0 = 3 \text{ m} \) in free-space, and thus the wave is very much smaller and much slower moving in the conductor
Energy Storage and Loss in Fields

We have learned that the power density of electric and magnetic fields is given by

\[ w_m = \frac{1}{2} E \cdot D = \frac{1}{2} \epsilon E^2 \]

\[ w_m = \frac{1}{2} H \cdot B = \frac{1}{2} \mu H^2 \]

Also, the power loss per unit volume due to Joule heating in a conductor is given by

\[ p_{\text{loss}} = \mathbf{E} \cdot \mathbf{J} \]

Using \( \mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \), this can be expressed as

\[ \mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot \nabla \times \mathbf{H} - \frac{\partial}{\partial t} (\nabla \times \mathbf{D}) \]
Poynting Vector

We will demonstrate that the Poynting vector \( \mathbf{E} \times \mathbf{H} \) plays an important role in the energy of an EM field.

\[
\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})
\]

\[
\mathbf{E} \cdot \mathbf{J} = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{J}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}
\]

\[
= \mathbf{H} \cdot (-\frac{\partial \mathbf{B}}{\partial t}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H})
\]

\[
\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \left(\frac{\partial \mu \mathbf{H}}{\partial t}\right) = \frac{1}{2} \frac{\partial \mu \mathbf{H} \cdot \mathbf{H}}{\partial t} = \frac{1}{2} \frac{\partial \mu |\mathbf{H}|^2}{\partial t}
\]

\[
\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \left(\frac{\partial \varepsilon \mathbf{E}}{\partial t}\right) = \frac{1}{2} \frac{\partial \varepsilon \mathbf{E} \cdot \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial \mu |\mathbf{E}|^2}{\partial t}
\]
Poynting’s Theorem

Collecting terms we have shown that

\[ \mathbf{E} \cdot \mathbf{J} = - \frac{\partial}{\partial t} \left( \frac{1}{2} \mu |\mathbf{H}|^2 \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon |\mathbf{E}|^2 \right) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \]

Applying the Divergence Theorem

\[ \int_V \mathbf{E} \cdot \mathbf{J} dV = - \frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \mu |\mathbf{H}|^2 + \frac{1}{2} \epsilon |\mathbf{E}|^2 \right) dV - \int_S \mathbf{E} \times \mathbf{H} dV \]

The above equation can be re-stated as

power
dissipated in
volume V (heat) = rate of change of energy
storage in
volume V – a surface
integral over the
volume of \( \mathbf{E} \times \mathbf{H} \).
Interpretation of the Poynting Vector

- We now have a physical interpretation of the last term in the above equation. By the conservation of energy, it must be equal to the energy flow into or out of the volume.
- We may be so bold, then, to interpret the vector \( S = \mathbf{E} \times \mathbf{H} \) as the energy flow density of the field.
- While this seems reasonable, it’s important to note that the physical meaning is only attached to the integral of \( S \) and not to discrete points in space.
Current Carrying Wire

Consider the above wire carrying a uniform current $I$

From circuit theory we know that the power loss in the wire is simply $I^2 R$. This is easily confirmed

$$P_L = \int_V E \cdot J dV = \int_V \frac{1}{\sigma} J \cdot J dV = \frac{1}{A^2 \sigma} \int_V I^2 dV$$

$$P_L = \frac{A \cdot \ell}{A^2 \sigma} I^2 = \frac{\ell}{A \sigma} I^2$$
Energy Stored around a Wire Section

Let’s now apply Poynting’s Theorem. Since the current is dc, we can neglect all time variation $\frac{\partial}{\partial t} = 0$ and thus the energy storage of the system is fixed in time.

The magnetic field around the wire is simply given by

$$H = \hat{\phi} \frac{I}{2\pi r}$$

The electric field is proportional to the current density. At the surface of the wire

$$E = \frac{1}{\sigma} J = \frac{I}{\sigma A} \hat{z}$$
Power Loss in Wire

- The Poynting vector at the surface thus points into the conductor

\[ S = E \times H = \frac{I}{\sigma A} \hat{z} \times \frac{I}{2\pi r} = \frac{-\hat{r}I^2}{2\pi r\sigma A} \]

- The power flow into the wire is thus given by

\[ \int_S S \cdot ds = \int_0^\ell \int_0^{2\pi} \frac{I^2}{2\pi r\sigma A} r\,d\theta\,dz = I^2R \]

- This result confirms that the energy flowing into the wire from the field is heating up the wire.
Sources and Fields

This result is surprising because it hints that the signal in a wire is carried by the fields, and not by the charges.

In other words, if a signal propagates down a wire, the information is carried by the fields, and the current flow is impressed upon the conductor from the fields.

We know that the sources of EM fields are charges and currents. But we also know that if the configuration of charges changes, the fields “carry” this information.