Lecture 2: Transmission Line Discontinuities

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Energy to “Charge” Transmission Line

The power flow into the line is given by

\[
P^+_{\text{line}} = i^+(0, t) v^+(0, t) = \frac{(v^+(0, t))^2}{Z_0}
\]

Or in terms of the source voltage

\[
P^+_{\text{line}} = \left( \frac{Z_0}{Z_0 + R_s} \right)^2 \frac{V_s^2}{Z_0} = \frac{Z_0}{(Z_0 + R_s)^2} V_s^2
\]
Energy Stored in Inds and Caps (I)

But where is the power going? The line is lossless!

Energy stored by a cap/ind is $\frac{1}{2}CV^2/\frac{1}{2}LI^2$

At time $t_d$, a length of $\ell = vt_d$ has been “charged”:

$$\frac{1}{2}CV^2 = \frac{1}{2}\ell C' \left( \frac{Z_0}{Z_0 + R_s} \right)^2 V_s^2$$

$$\frac{1}{2}LI^2 = \frac{1}{2}\ell L' \left( \frac{V_s}{Z_0 + R_s} \right)^2$$

The total energy is thus

$$\frac{1}{2}LI^2 + \frac{1}{2}CV^2 = \frac{1}{2} \frac{\ell V_s^2}{(Z_0 + R_s)^2} \left( L' + C' Z_0^2 \right)$$
Energy Stored (II)

Recall that \( Z_0 = \sqrt{\frac{L'}{C'}} \). The total energy stored on the line at time \( t_d = \ell/v \):

\[
E_{\text{line}}(\ell/v) = \ell L' \frac{V_s^2}{(Z_0 + R_s)^2}
\]

And the power delivered onto the line in time \( t_d \):

\[
P_{\text{line}} \times \ell = \frac{l}{v} Z_0 V_s^2 \frac{1}{(Z_0 + R_s)^2} = \ell \sqrt{\frac{L'}{C'}} \sqrt{L'C'} \frac{V_s^2}{(Z_0 + R_s)^2}
\]

As expected, the results match (conservation of energy).
Consider a finite transmission line with a termination resistance.

At the load we know that Ohm’s law is valid: $I_L = \frac{V_L}{R_L}$

So at time $t = \frac{\ell}{v}$, our pulse reaches the load. Since the current on the T-line is $i^+ = \frac{v^+}{Z_0} = \frac{V_s}{(Z_0 + R_s)}$ and the current at the load is $\frac{V_L}{R_L}$, a discontinuity is produced at the load.
Reflections

- Thus a reflected wave is created at discontinuity

\[ V_L(t) = v^+(\ell, t) + v^-(\ell, t) \]

\[ I_L(t) = \frac{1}{Z_0} v^+(\ell, t) - \frac{1}{Z_0} v^-(\ell, t) = V_L(t)/R_L \]

- Solving for the forward and reflected waves

\[ 2v^+(\ell, t) = V_L(t)(1 + Z_0/R_L) \]

\[ 2v^-(\ell, t) = V_L(t)(1 - Z_0/R_L) \]
Reflection Coefficient

And therefore the reflection from the load is given by

$$\Gamma_L = \frac{V^-(\ell, t)}{V^+(\ell, t)} = \frac{R_L - Z_0}{R_L + Z_0}$$

Reflection coefficient is a very important concept for transmission lines: $-1 \leq \Gamma_L \leq 1$

- $\Gamma_L = -1$ for $R_L = 0$ (short)
- $\Gamma_L = +1$ for $R_L = \infty$ (open)
- $\Gamma_L = 0$ for $R_L = Z_0$ (match)

Impedance match is the proper termination if we don’t want any reflections.
Propagation of Reflected Wave (I)

- If $\Gamma_L \neq 0$, a new reflected wave travels toward the source and unless $R_s = Z_0$, another reflection also occurs at source!

- To see this consider the wave arriving at the source. Recall that since the wave PDE is linear, a superposition of any number of solutions is also a solution.

- At the source end the boundary condition is as follows

\[ V_s - I_s R_s = v_1^+ + v_1^- + v_2^+ \]

- The new term $v_2^+$ is used to satisfy the boundary condition.
Propagation of Reflected Wave (II)

- The current continuity requires \( I_s = i_1^+ + i_1^- + i_2^+ \)

\[
V_s = (v_1^+ - v_1^- + v_2^+) \frac{R_s}{Z_0} + v_1^+ + v_1^- + v_2^+
\]

- Solve for \( v_2^+ \) in terms of known terms

\[
V_s = \left( 1 + \frac{R_s}{Z_0} \right) (v_1^+ + v_2^+) + \left( 1 - \frac{R_s}{Z_0} \right) v_1^- +
\]

- But \( v_1^+ = \frac{Z_0}{R_s + Z_0} V_s \)

\[
V_s = \frac{R_s + Z_0}{Z_0} \frac{Z_0}{R_s + Z_0} V_s + \left( 1 - \frac{R_s}{Z_0} \right) v_1^- + \left( 1 + \frac{R_s}{Z_0} \right) v_2^+
\]
So the source terms cancel out and

\[ v_2^+ = \frac{R_s - Z_0}{Z_0 + R_s} v_1^- = \Gamma_s v_1^- \]

The reflected wave bounces off the source impedance with a reflection coefficient given by the same equation as before

\[ \Gamma(R) = \frac{R - Z_0}{R + Z_0} \]

The source appears as a short for the incoming wave

Invoke superposition! The term \( v_1^+ \) took care of the source boundary condition so our new \( v_2^+ \) only needed to compensate for the \( v_1^- \) wave ... the reflected wave is only a function of \( v_1^- \)
We can track the multiple reflections with a “bounce diagram”
If we freeze time and look at the line, using the bounce diagram we can figure out how many reflections have occurred.

For instance, at time $2.5t_d = 2.5\ell/v$ three waves have been excited ($v_1^+, v_1^-, v_2^+$), but $v_2^+$ has only travelled a distance of $\ell/2$.

To the left of $\ell/2$, the voltage is a summation of three components: $v = v_1^+ + v_1^- + v_2^+ = v_1^+(1 + \Gamma L + \Gamma L \Gamma_s)$.

To the right of $\ell/2$, the voltage has only two components: $v = v_1^+ + v_1^- = v_1^+(1 + \Gamma L)$. 

We can also pick at arbitrary point on the line and plot the evolution of voltage as a function of time.

For instance, at the load, assuming $R_L > Z_0$ and $R_S > Z_0$, so that $\Gamma_{s,L} > 0$, the voltage at the load will will increase with each new arrival of a reflection.

\[ v_{L}^+(t) = v_{L}^-(t) = 0.4 \]

\[ R_s = 75\Omega \quad R_L = 150\Omega \]

\[ \Gamma_s = 0.2 \quad \Gamma_L = 0.5 \]

\[ v_{ss} = \frac{2}{3}V \]
Steady-State Voltage on Line (I)

- To find steady-state voltage on the line, we sum over all reflected waves:

\[ v_{ss} = v_1^+ + v_1^- + v_2^+ + v_2^- + v_3^+ + v_3^- + v_4^+ + v_4^- + \cdots \]

- Or in terms of the first wave on the line

\[ v_{ss} = v_1^+ (1 + \Gamma_L + \Gamma_L \Gamma_s + \Gamma_L^2 \Gamma_s + \Gamma_L^2 \Gamma_s^2 + \Gamma_L^3 \Gamma_s^2 + \Gamma_L^3 \Gamma_s^3 + \cdots) \]

- Notice geometric sums of terms like \( \Gamma_L^k \Gamma_s^k \) and \( \Gamma_L^{k+1} \Gamma_s^k \).

Let \( x = \Gamma_L \Gamma_s \):

\[ v_{ss} = v_1^+ (1 + x + x^2 + \cdots + \Gamma_L (1 + x + x^2 + \cdots)) \]
The sums converge since \( x < 1 \)

\[
v_{ss} = v_1^+ \left( \frac{1}{1 - \Gamma_L \Gamma_s} + \frac{\Gamma_L}{1 - \Gamma_L \Gamma_s} \right)
\]

Or more compactly

\[
v_{ss} = v_1^+ \left( \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_s} \right)
\]

Substituting for \( \Gamma_L \) and \( \Gamma_s \) gives

\[
v_{ss} = V_s \frac{R_L}{R_L + R_s}
\]
What Happend to the T-Line?

- For steady state, the equivalent circuit shows that the transmission line has disappeared.
- This happens because if we wait long enough, the effects of propagation delay do not matter.
- Conversely, if the propagation speed were infinite, then the T-line would not matter.
- But the presence of the T-line will be felt if we disconnect the source or load!
- That’s because the T-line stores reactive energy in the capacitance and inductance.
- Every real circuit behaves this way! Circuit theory is an abstraction.
PCB Interconnect

Suppose $\ell = 3\text{cm}$, $v = 3 \times 10^8 \text{m/s}$, so that

$$t_p = \frac{\ell}{v} = 10^{-10} \text{s} = 100\text{ps}$$

On a time scale $t < 100\text{ps}$, the voltages on interconnect

act like transmission lines!

Fast digital circuits need to consider T-line effects
Example: Open Line (I)

- Source impedance is $Z_0/4$, so $\Gamma_s = -0.6$, load is open so $\Gamma_L = 1$

- As before a positive going wave is launched $v_1^+$

- Upon reaching the load, a reflected wave of of equal amplitude is generated and the load voltage overshoots $v_L = v_1^+ + v_1^- = 1.6V$

- Note that the current reflection is negative of the voltage

$$\Gamma_i = \frac{i^-}{i^+} = -\frac{v^-}{v^+} = -\Gamma_v$$

- This means that the sum of the currents at load is zero (open)
Example: Open Line (II)

- At source a new reflection is created $v_2^+ = \Gamma_L \Gamma_s v_1^+$, and note $\Gamma_s < 0$, so $v_2^+ = -0.6 \times 0.8 = -0.48$.

- At a time $3t_p$, the line charged initially to $v_1^+ + v_1^-$ drops in value

$$v_L = v_1^+ + v_1^- + v_2^+ + v_2^- = 1.6 - 2 \times 0.48 = 0.64$$

- So the voltage on the line undershoots $< 1$

- And on the next cycle $5t_p$ the load voltage again overshoots

- We observe ringing with frequency $2t_p$
Example: Open Line Ringing

Observed waveform as a function of time.
Physical Intuition: Shorted Line (I)

- The initial step charges the “first” capacitor through the “first” inductor since the line is uncharged.
- There is a delay since on the rising edge of the step, the inductor is an open.
- Each successive capacitor is charged by “its” inductor in a uniform fashion ... this is the forward wave $v_1^+$.
Physical Intuition: Shorted Line (II)

- The voltage on the line goes up from left to right due to the delay in charging each inductor through the inductors.
- The last inductor, though, does not have a capacitor to charge.
- Thus the last inductor is discharged ... the extra charge comes by discharging the last capacitor.
- As this capacitor discharges, so does its neighboring capacitor to the left.
- Again there is a delay in discharging the caps due to the inductors.
- This discharging represents the backward wave $v_1$. 