

EECS 117

Lecture 11: Capacitance and Energy

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Parallel Plate Capacitor

- A very convenient idealization is a parallel plate capacitor. Here we take two conducting plates of area A and separate them by a distance t . If $t \ll \min(dx, dy)$, then the field is largely independent of x and y and only a function of z . This is true especially in regions far from the edges of the capacitor.
- Now charge the upper plate to a voltage of V_0 relative to the bottom plate. For convenience take the reference of zero potential at the bottom plate.

Potential in Parallel Plate Cap

- If we neglect the edge effects, then the solution of the potential in the charge free region between the plates is governed by a one dimensional differential equation.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial z^2} = 0$$

- subject to the boundary conditions $\phi(0) = 0$ and $\phi(t) = V_0$. The solution is clearly a linear function (integrate the above equation twice) $\phi(z) = C_1 z + C_2$ and applying the boundary conditions to solve for C_1 and C_2 we arrive at

$$\phi(z) = \frac{V_0}{t} z$$

Field and Charge in ||-Plate Cap

- The electric field $E = -\nabla\phi$ can thus be computed by a simple derivative and is a constant within the capacitor

$$\mathbf{E} = -\hat{\mathbf{z}}\frac{V_0}{t}$$

- By Gauss' law, the normal component of the electric flux density is equal to the charge density on the plates.

$$\mathbf{D} \cdot \hat{\mathbf{n}} = \rho$$

- Since the field is in fact everywhere perpendicular to the conductor, we simply have

$$\rho_{\text{top}} = -\hat{\mathbf{z}} \cdot \mathbf{D} = \epsilon\frac{V_0}{t} \quad \rho_{\text{bot}} = \hat{\mathbf{z}} \cdot \mathbf{D} = -\epsilon\frac{V_0}{t}$$

||-Plate Capacitance

- To find the capacitance we simply note that we're interested in the coefficient $q = CV_0$. To get the total charge, we multiply the constant charge density by the area of the plate $q = \rho_{\text{top}}A = \frac{\epsilon A}{t}V_0 = CV_0, C = \frac{\epsilon A}{t}$
- This equation is intuitively satisfying. The capacitance goes up with A since for a fixed charge on the plates, the charge density drops and so does the potential giving a larger capacitance.
- Likewise, if we increase t the capacitance drops since now there is less motivation for positive (negative) charge to flow onto the top (bottom) plate! The charges are more distant from their beloved negative (positive) charges.

General Capacitance Equation

- For two conductors of any shape, the capacitance is defined as

$$C \triangleq \frac{Q}{\phi_{12}}$$

- The potential difference ϕ_{12} is the line integral of \mathbf{E} over *any* path from conductor 1 to conductor 2.
- By Gauss' law, the positive charge on conductor 1 is equal to the electric flux crossing *any* surface enclosing the conductor.
- The capacitance is therefore written as

$$C = \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{-\int_C \mathbf{E} \cdot d\boldsymbol{\ell}}$$

Multiple-Conductors

- The concept of capacitance can be generalized to multiple conductors

$$q_1 = C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + \dots$$

$$q_2 = C_{21}V_1 + C_{22}V_2 + C_{23}V_3 + \dots$$

$$\vdots$$

- Each coefficient C_{ii} represents the self capacitance. It can be computed by applying $V_i = 1V$ to conductor i while grounding all other conductors. Then C_{ii} is simply the total charge the conductor.
- Likewise, to find C_{ij} , we apply a voltage of $V_j = 1V$ to conductor j while grounding all other conductors. Then C_{ij} is again simply the total charge on conductor i .

Grounding

- When we say we “ground” all other conductors, we mean that we connect them to a voltage source of zero volts
- Recall that voltage is always defined relative to a reference. For instance, we may take one of the conductors as the “ground” reference and then measure all absolute voltages relative to this conductor.
- Otherwise we may also connect the conductors to a much larger body, one with infinite capacitance. Then charge can be freely removed or added to the “ground” without changing its potential.

Coefficients of Potential

- We may also express the voltage on each conductor in terms of the total charge on each conductor in the system in the following manner

$$v_1 = P_{11}q_1 + P_{12}q_2 + P_{13}q_3 + \dots$$

$$v_2 = P_{21}q_1 + P_{22}q_2 + P_{23}q_3 + \dots$$

$$\vdots$$

- To find P_{ij} , we add a charge of 1C to conductor j and leave all other conductors neutral. Then we observe the voltage at conductor i

Coefficient of Potentials (cont)

- This applies by direct application of superposition to the equation

$$\phi(r) = \int_V \frac{\rho(r')}{4\pi\epsilon|r - r'|} dV' = \sum_i \int_{V_i} \frac{\rho(r')}{4\pi\epsilon|r - r'|} dV'$$

- Notice that this equation scales linearly with the absolute amount of charge on conductor i . So we may perform the integral based solely on geometric calculations to obtain coefficient P_{ij}

$$\phi(r) = \sum_i q_i \int_{V_i} \frac{\rho(r')/q_i}{4\pi\epsilon|r - r'|} dV'$$

Capacitance Matrix (I)

- Or in matrix form, we may write $v = Pq$. If the matrix P is not singular, we may invert this equation to obtain $q = P^{-1}v$
- We may be tempted to call P_{ij}^{-1} a capacitance but notice that these coefficients are in terms of the potential V_i relative to a common reference

$$q_1 = c_{11}V_1 + c_{12}V_2 + c_{13}V_3 + \dots$$

$$q_2 = c_{21}V_1 + c_{22}V_2 + c_{23}V_3 + \dots$$

$$\vdots$$

- To relate c_{ij} to C_{ij} , simply equate the total charges

$$q_1 = C_{11}V_1 + C_{12}V_{12} + C_{13}V_{13} = c_{11}V_1 + c_{12}V_2 + c_{13}V_3 + \dots$$

Capacitance Matrix (II)

- Since $V_{1i} = V_1 - V_j$, we have

$$C_{11} = c_{11} + c_{12} + c_{13} + \dots$$

- And also $C_{ij} = -c_{ij}$
- So these matrices are related but not the same. Notice that $c_{ij} < 0$ is logical, since if we put a positive voltage on node j and observe the induced charge on node i , it should be negative.
- On the other hand $C_{ij} > 0$, since if we connect the positive terminal of a battery to node i and the negative node of the battery to node j , then the charge on node i should be positive
- Capacitors in SPICE are always of the C_{ij} form, and hence positive.

Electrostatic Energy of a Capacitor

- Consider the energy required to charge a capacitor
- The amount of work released to move a charge dQ from the positive terminal to the negative terminal is $dU = V dQ$. This work must be stored in the capacitor

$$dU = V dQ = CV dV$$

- Integrating over the volage on the capacitor we have

$$U = \frac{1}{2} CV^2$$

- We say that this energy is stored in teh field of the capacitor. This line of reasoning will become clear as we develop these ideas further

Energy in terms of the Field

- For a parallel plate capacitor, the field is constant and equal of $E = V/d$. If we substitute in the energy equation, we have

$$U = \frac{1}{2} C E^2 d^2$$

where $C = \epsilon A/d$. Substituting further

$$U = \frac{1}{2} \frac{\epsilon A}{d} E^2 d^2 = \frac{1}{2} \epsilon \mathcal{V} E^2$$

- Where $\mathcal{V} = dA$ is the volume of the region in between the plates. Since the fields are confined to this volume, we may speculate that the energy density is also so confined

Preview: Energy Density

- In this particular case, we have

$$u = \frac{U}{\mathcal{V}} = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}$$

- We will show later that this is true in general for any electrostatic field

Energy for Two Conductors

- Consider the energy required to charge two conductors to voltages ϕ_1 and ϕ_2 . Let's do the calculation in phases. First apply voltage ϕ_1 to conductor 1 but keep conductor 2 grounded. A charge $q_1 = c_{11}\phi_1$ flows onto conductor one whereas a charge $q_2 = c_{21}\phi_1$ flows onto conductor two. Since conductor 2 is grounded, there is no energy required to add or remove charge from it. For conductor 1, though, the energy required is $\frac{1}{2}c_{11}\phi_1^2$.
- Now raise the voltage on conductor 2 from zero to ϕ_2 . An additional work of $\frac{1}{2}c_{22}\phi_2^2$ is required. But an additional charge of $q_1 = c_{12}\phi_2$ also flows onto conductor one. The work required to do this is $c_{12}\phi_1\phi_2$ (no integration is required since the potential is fixed at ϕ_1)

Total Work for Two Conductors

- The total work is therefore the sum of the various terms

$$W = \frac{1}{2}c_{11}\phi_1^2 + \frac{1}{2}c_{22}\phi_2^2 + c_{12}\phi_1\phi_2$$

- But if we had reversed the order of charging the conductors, we would have arrived at the following result

$$W' = \frac{1}{2}c_{11}\phi_1^2 + \frac{1}{2}c_{22}\phi_2^2 + c_{21}\phi_2\phi_1$$

- But the energy of the system would surely be the same, or $W = W'$ which implies that $c_{12} = c_{21}$
- Thus the capacitance matrix is symmetric.
- A symmetric matrix with non-zero diagonal is invertible thus justifying why we could go freely from P to C

Energy Again ...

- Let's derive this result more directly as follows. The energy increment $dW = dQv$ where we add charge dQ at constant voltage v .

- On conductor one we have

$$q_1 = c_{11}V_1 + C_{12}V_2 + \dots$$

$$dq_1 = c_{11}dV_1 + C_{12}dV_2 + \dots$$

- So the energy increment is given by

$$dW = dq_1V_1 = c_{11}V_1dV_1 + C_{12}V_1dV_2 + \dots$$

- Integration yields

$$W = \frac{1}{2}C_{11}V_1^2 + C_{12}V_1V_2 + \dots$$

Parallel Capacitors

- From circuit theory we know how to add capacitors in series or parallel
- Parallel caps are easier. If we connect two conductors and connect them to a potential V_0 , the charge is simply the total charge

$$Q = Q_1 + Q_2 = C_1 V_0 + C_2 V_0 = (C_1 + C_2) V_0$$

- In general we have

$$C_{||} = C_1 + C_2 + C_3 + \dots$$

Series Capacitors

- For series capacitors, note that the applied voltage is divided between the capacitors

$$V_0 = V_1 + V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$

- The central observation is that the charge on each capacitor is the same, $|Q_1| = |Q_2|$
- This is because the floating node must have zero net charge and thus $-Q_1 = Q_2$
- The result can be easily generalized

$$\frac{1}{C_{\text{series}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$

Energy for a Two Point Charges

- Let's find the total energy for a distribution of point charges. We can imagine building up the distribution one charge at a time
- The energy to bring in the first charge is naturally zero since the field is zero
- The second charge, though, is repelled (or attracted) to the first charge so it requires more (less) energy to bring it in. In general the work required is given by

$$W_2 = Q_2\phi_{12}$$

- where ϕ_{12} is the potential due to charge 1

Energy Three Point Charge Distribution

- Since the electrostatic field is *conservative*, it does not matter how we bring in the second charge. Only its final position relative to the first charge is important.

- In terms of

$$W_2 = Q_2\phi_{12} = \frac{Q_2Q_1}{4\pi\epsilon R_{12}}$$

- where R_{12} is the final distance between the point charges.
- Likewise, when we bring in the third charge, the extra work required is

$$W_3 = Q_3\phi_{13} + Q_3\phi_{23} = \frac{Q_3Q_1}{4\pi\epsilon R_{13}} + \frac{Q_3Q_2}{4\pi\epsilon R_{23}}$$

Energy for Point Charge Distribution

- We can therefore write in general that the electrostatic energy takes on the following form

$$4\pi\epsilon W = \frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_1 Q_4}{R_{14}} + \dots + \frac{Q_2 Q_3}{R_{23}} + \frac{Q_2 Q_4}{R_{24}} + \dots$$

- The general term has the form $\frac{Q_i Q_j}{4\pi R_{ij}}$ where i and j sum over all the particles in the system

$$W = \frac{1}{2} \sum \sum_{i \neq j} \frac{Q_i Q_j}{4\pi R_{ij}}$$

- The factor of $\frac{1}{2}$ takes care of the double counting and enforcing $i \neq j$ ensures that we don't try to include the "self" energy of the particles.

Static Energy in terms of Potential

- We can rewrite the double sum into a more general form by observing that the inner sum is simply the potential due to all the particles evaluated at position of particle i

$$W = \frac{1}{2} \sum_{i \neq j} Q_i \sum_j \frac{Q_j}{4\pi R_{ij}} = \frac{1}{2} \sum_{i \neq j} Q_i \phi_i$$

- If we now consider a charge distribution $\rho(r)$, it's easy to see how the above sum turns into an integral

$$W = \frac{1}{2} \int_V \rho \phi dV$$

Static Energy in terms of Fields

- The derived expression can be modified if we substitute $\rho = \nabla \cdot \mathbf{D}$ for charge

$$W = \frac{1}{2} \int_V \nabla \cdot \mathbf{D} \phi dV$$

- and employ the chain rule

$$\nabla \cdot \phi \mathbf{D} = \phi \nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla \phi$$

- Since $\nabla \phi = -\mathbf{E}$, we have two volume integrals

$$W = \frac{1}{2} \int_V \nabla \cdot \phi \mathbf{D} dV + \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} dV$$

Surface Integral Terms

- We can show that the first integral vanishes as follows. First apply the divergence theorem

$$\int_V \nabla \cdot \phi \mathbf{D} dV = \oint_S \phi \mathbf{D} \cdot \hat{\mathbf{n}} dS$$

- Now take a surface S that is very large. In fact take a large sphere. If the sphere is very large and the charge distribution is of finite extent, then at some great distance from the source the actual charge distribution is immaterial. Only the net charge matters. We know that the radial potential and fields for a charge density take on the following limiting forms $\phi \sim \frac{1}{r}$ $D \sim \frac{1}{r^2}$
- Since the surface area $S \sim r^2$, the integrand vanishes

Final Expression for Electrostatic Energy

- Therefore the electrostatic energy takes on the following form

$$W = \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} dV$$

- If we define the energy density w , we have

$$W = \frac{1}{2} \int_V w dV$$

$$w = \frac{1}{2} \mathbf{D} \cdot \mathbf{E}$$

- We found this to be true for an ideal capacitor but now we see this is true in general