EECS 117

*Lecture 1: Transmission Lines*

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Problem: A long cable – the trans-Atlantic telephone cable – is laid out connecting NY to London. We would like analyze the electrical properties of this cable.

For simplicity, assume the cable has a uniform cross-sectional configuration (shown as two wires here)
Can we do it with circuit theory?

Fundamental problem with circuit theory is that it assumes that the speed of light is infinite. So all signals are in phase: \[ V(z) = V(z + \ell) \]

Consequently, all variations in space are ignored:
\[ \frac{\partial}{\partial z} \to 0 \]

This allows the \textit{lumped} circuit approximation.
Lumped Circuit Properties of Cable

- **Shorted Line:** The long loop has *inductance* since the magnetic flux $\psi$ is not negligible (long cable) ($\psi = LI$)

- **Open Line:** The cable also has substantial capacitance ($Q = CV$)
Sectional Model (I)

- So do we model the cable as an inductor or as a capacitor? Or both? How?

- Try a *distributed* model: Inductance and capacitance occur together. They are intermingled.

- Can add loss (series and shunt resistors) but let’s keep it simple for now.

- Add more sections and solution should converge
More sections → The equiv $LC$ circuit represents a smaller and smaller section and therefore lumped circuit approximation is more valid.

This is an easy problem to solve with SPICE.

But the people 1866 didn’t have computers ... how did they analyze a problem with hundreds of inductors and capacitors?
Go to a fully distributed model by letting the number of sections go to infinity.

- Define inductance and capacitance per unit length:
  \[ L' = \frac{L}{\ell}, \quad C' = \frac{C}{\ell} \]

For an infinitesimal section of the line, circuit theory applies since signals travel instantly over an infinitesimally small length.
KCL and KVL for a small section

- **KCL:** \( i(z) = \delta z C' \frac{\partial v(z)}{\partial t} + i(z + \delta z) \)

- **KVL:** \( v(z) = \delta z L' \frac{\partial i(z+\delta z)}{\partial t} + v(z + \delta z) \)

- Take limit as \( \delta z \to 0 \)

We arrive at “Telegrapher’s Equations”

\[
\lim_{\delta z \to 0} \frac{i(z) - i(z + \delta z)}{\delta z} = -\frac{\partial i}{\partial z} = C' \frac{\partial v}{\partial t}
\]

\[
\lim_{\delta z \to 0} \frac{v(z) - v(z + \delta z)}{\delta z} = -\frac{\partial v}{\partial z} = L' \frac{\partial i}{\partial t}
\]
Derivation of Wave Equations

We have two coupled equations and two unknowns \((i\) and \(v)\) ... can reduce it to two de-coupled equations:

\[
\frac{\partial^2 i}{\partial t \partial z} = -C' \frac{\partial^2 v}{\partial t^2}
\]

\[
\frac{\partial^2 v}{\partial z^2} = -L' \frac{\partial^2 i}{\partial z \partial t}
\]

note order of partials can be changed (at least in EE)

\[
\frac{\partial^2 v}{\partial z^2} = L' C' \frac{\partial^2 v}{\partial t^2}
\]

Same equation can be derived for current:

\[
\frac{\partial^2 i}{\partial z^2} = L' C' \frac{\partial^2 i}{\partial t^2}
\]
The Wave Equation

We see that the currents and voltages on the transmission line satisfy the one-dimensional wave equation. This is a partial differential equation. The solution depends on boundary conditions and the initial condition.

\[
\frac{\partial^2 i}{\partial z^2} = L'C' \frac{\partial^2 i}{\partial t^2}
\]
Wave Equation Solution

Consider the function \( f(z, t) = f(z \pm vt) = f(u) \):

\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} = \frac{\partial f}{\partial u} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \pm v \frac{\partial f}{\partial u}
\]

\[
\frac{\partial^2 f}{\partial^2 z} = \frac{\partial^2 f}{\partial u^2} \quad \frac{\partial^2 f}{\partial t^2} = \pm v \frac{\partial}{\partial u} \left( \frac{\partial f}{\partial t} \right) = v^2 \frac{\partial^2 f}{\partial u^2}
\]

\[
\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}
\]

It satisfies the wave equation!
Wave Motion

General voltage solution: \( v(z, t) = f^+(z - vt) + f^-(z + vt) \)

Where \( v = \sqrt{\frac{1}{LC}} \)
Wave Speed

- Speed of motion can be deduced if we observe the speed of a point on the waveform

\[ z \pm vt = \text{constant} \]

- To follow this point as time elapses, we must move the \( z \) coordinate in step. This point moves with velocity

\[ \frac{dz}{dt} \pm v = 0 \]

- This is the speed at which we move with speed \( \frac{dz}{dt} = \pm v \)

- \( v \) is the velocity of wave propagation
Since the current also satisfies the wave equation

\[ i(z, t) = g^+(z - vt) + g^-(z + vt) \]

Recall that on a transmission line, current and voltage are related by

\[ \frac{\partial i}{\partial z} = -C' \frac{\partial v}{\partial t} \]

For the general function this gives

\[ \frac{\partial g^+}{\partial u} + \frac{\partial g^-}{\partial u} = -C' \left( -v \frac{\partial f^+}{\partial u} + v \frac{\partial f^-}{\partial u} \right) \]
Since the forward waves are independent of the reverse waves

\[ \frac{\partial g^+}{\partial u} = C'' v \frac{\partial f^+}{\partial u} \quad \frac{\partial g^-}{\partial u} = -C'' v \frac{\partial f^-}{\partial u} \]

Within a constant we have

\[ g^+ = \frac{f^+}{Z_0} \quad g^- = -\frac{f^-}{Z_0} \]

Where \( Z_0 = \sqrt{\frac{L'}{C'}} \) is the “Characteristic Impedance” of the line
Example: Step Into Infinite Line

- Excite a step function onto a transmission line
- The line is assumed uncharged: \( Q(z, 0) = 0, \psi(z, 0) = 0 \) or equivalently \( v(z, 0) = 0 \) and \( i(z, 0) = 0 \)
- By physical intuition, we would only expect a forward traveling wave since the line is infinite in extent
- The general form of current and voltage on the line is given by
  \[
  v(z, t) = v^+(z - vt)
  \]
  \[
  i(z, t) = i^+(z - vt) = \frac{v^+(z - vt)}{Z_0}
  \]
- The T-line looks like a resistor of \( Z_0 \) ohms!
Example 1 (cont)

- We may therefore model the line with the following simple equivalent circuit

\[ i_s = R_s \]

\[ Z_0 \]

\[ i^+ = \frac{v^+}{Z_0} \]

\[ V_s \]

\[ Z_0 \]

- Since \( i_s = i^+ \), the excited voltage wave has an amplitude of

\[ v^+ = \frac{Z_0}{Z_0 + R_s} V_s \]

- It’s surprising that the voltage on the line is not equal to the source voltage
Example 1 (cont)

The voltage on the line is a delayed version of the source voltage

\[ V_s \frac{Z_0}{Z_0 + R_s} \]

\[ v(z, t = \frac{\ell}{v}) \]

\[ \ell \quad z \]