1. Suppose the resonant frequency $\omega_0$ is equal to $(LC)^{0.5}$. The load impedance $Z_L$ is

$$Z_L = R + j\omega_0 L + \frac{1}{j\omega C} = R + j\left(\omega_0 L - \frac{1}{\omega C}\right)$$

If $\omega = \omega_0 + \Delta \omega$, $Z_L$ is equal to

$$Z_L = R + j(\omega_0 + \Delta \omega)L + \frac{1}{j(\omega_0 + \Delta \omega)C} = R + j\left[\omega_0 L\left(1 + \frac{\Delta \omega}{\omega_0}\right) - \frac{1}{\omega_0 C}\left(1 - \frac{\Delta \omega}{\omega_0}\right)\right] = R + \Delta \omega L + \frac{\Delta \omega}{\omega_0^2 C}$$

The last equality holds because $\omega_0 = (LC)^{0.5}$. Furthermore,

$$R + j\left[\Delta \omega L + \frac{\Delta \omega}{\omega_0^2 C}\right] = R + j\frac{\Delta \omega}{\omega_0}\left[\omega_0 L + \frac{1}{\omega_0 C}\right] = R + j\frac{\Delta \omega}{\omega_0} (2\omega_0 L) = R + j2\Delta \omega L$$

Using the values of the inductance and capacitance, the length of 2 cm corresponds to $1.5\pi$.

$$\beta_0 l = \frac{\omega_0 l}{v} l = \frac{2cm}{v\sqrt{LC}} \approx \frac{2 \times 10^{-3} \times 2 \times 10^{-3} \times 9.01 \times 10^{-3} = 1.5\pi}{10^6 \text{ m/s}}$$

In general, $\beta l = (\beta_0 + \Delta \beta) l = \frac{\omega_0 + \Delta \omega}{v} l = \beta_0 l (1 + \frac{\Delta \omega}{\omega_0})$. Thus $Z_{in}$ is

$$Z_{in} = Z_0 \left[\frac{R + j2\Delta \omega L + jZ_0 \tan(1.5\pi(1 + \Delta \omega/\omega_0))}{Z_0 + j(R + j2\Delta \omega L)\tan(1.5\pi(1 + \Delta \omega/\omega_0))}\right] \approx Z_0 \frac{jZ_0 \tan(1.5\pi(1 + \Delta \omega/\omega_0))}{j(R + j2\Delta \omega L)\tan(1.5\pi(1 + \Delta \omega/\omega_0))} = \frac{Z_0^2}{(R + j2\Delta \omega L)}$$

The approximation holds because $\tan(1.5\pi(1 + \Delta \omega/\omega_0)) >> R, \Delta \omega L, Z_0$. This expression has the same form as a parallel RLC circuit, with

$$R_{eq} = \frac{Z_0^2}{R}, \quad C_{eq} = \frac{L}{Z_0^2}, \quad L_{eq} = \frac{1}{\omega_0^2 \frac{L}{Z_0^2}} = Z_0^2 C$$
Therefore, the input impedance $Z_{\text{in}}$ is that of a second order circuit. Also, $L_{eq}C_{eq} = LC = \omega_0^{-2}$, so our assumption is correct, i.e., $\omega_0 = (LC)^{-0.5}$. The Q factor is

$$Q = \omega_0 R_{eq} C_{eq} = \frac{L}{R} \sqrt{\frac{C}{L}} = \sqrt{\frac{L}{C}} R = 94$$

The equivalent circuit is $L_{eq} = Z_0^2 C = 5 \mu F$, $C_{eq} = L/Z_0^2 = 0.3604 \text{ fH}$, and $R_{eq} = Z_0^2 / R = 1250$ in parallel.

2. For a lossy line, the input impedance has a form

$$Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tanh(\alpha l + j\beta l)}{Z_0 + jZ_L \tanh(\alpha l + j\beta l)}$$

where $Z_L = 2j\Delta\omega L$. Also,

$$\tanh(\alpha l + j\beta l) = \frac{\sinh(2\alpha l)}{\cos(2\beta l) + \cosh(2\alpha l)} + j \frac{\sin(2\beta l)}{\cos(2\beta l) + \cosh(2\alpha l)}$$

$\sin(2\beta l) = \sin(3\pi(1 + \Delta\omega / \omega_0)) \approx \sin(3\pi) + \cos(3\pi) * (3\pi\Delta\omega / \omega_0) = -3\pi\Delta\omega / \omega_0$. Likewise,

$$\cos(2\beta l) \approx \cos(3\pi) - \sin(3\pi) * (3\pi\Delta\omega / \omega_0) = -1$$

So,

$$\tanh(\alpha l + j\beta l) = \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} - j \frac{3\pi\Delta\omega / \omega_0}{\cosh(2\alpha l) - 1}$$

The input impedance is then equal to

$$Z_{\text{in}} = \frac{2j\Delta\omega L + jZ_0 \left( \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} - j \frac{3\pi\Delta\omega / \omega_0}{\cosh(2\alpha l) - 1} \right)}{Z_0 + j2j\Delta\omega L \left( \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} - j \frac{3\pi\Delta\omega / \omega_0}{\cosh(2\alpha l) - 1} \right)}$$

$$= \frac{Z_0 \frac{3\pi\Delta\omega / \omega_0}{\cosh(2\alpha l) - 1} + j \left( 2\Delta\omega L + Z_0 \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} \right)}{Z_0 - 2\Delta\omega L \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} + j2\Delta\omega L \frac{3\pi\Delta\omega / \omega_0}{\cosh(2\alpha l) - 1}}$$

$$\approx Z_0 \frac{Z_0 \frac{3\pi\Delta\omega / \omega_0}{\cosh(2\alpha l) - 1} + j \left( 2\Delta\omega L (\cosh(2\alpha l) - 1) + Z_0 \sinh(2\alpha l) \right)}{Z_0 (\cosh(2\alpha l) - 1) - 2\Delta\omega L \sinh(2\alpha l)}$$

The third term in the denominator has $\Delta\omega^2$ dependence and is thus negligible.
3-31. Quarter-wave matching. (a) For the first circuit with the single quarter-wave transformer, we find

\[ Z_0 = \sqrt{Z_0 R_L} = \sqrt{(50)(400)} \approx 141.4\Omega \]

For the second circuit involving two quarter-wave sections cascaded together, the input impedance of the two transformers can be written as

\[ Z_{\text{in}2} = \frac{Z_{Q2}^2}{R_L} \quad \text{and} \]

\[ Z_{\text{in}1} = \frac{Z_{Q1}^2}{Z_{\text{in}2}} = \frac{Z_{Q1}^2}{Z_{Q2}^2 R_L} = Z_0 \]

resulting in \( Z_{Q1}/Z_{Q2} = \sqrt{Z_0/R_L} \). But it is also given that \( Z_{Q1} Z_{Q2} = Z_0 R_L \). Therefore, solving these two equations simultaneously, we have

\[ Z_{Q1}^2 = \sqrt{Z_0^3 R_L} = \sqrt{(50)^3 (400)} \]

yielding \( Z_{Q1} \approx 84.1\Omega \) and

\[ Z_{Q2} = \frac{Z_0 R_L}{Z_{Q1}} \approx \frac{(50)(400)}{84.1} \approx 238\Omega \]

respectively.

(b) At 15% above the design frequency we have for the first circuit:

\[ Z_{\text{in}} = Z_Q \frac{R_L + j Z_0 \tan[(2\pi)(1/4)(1.15)]}{Z_Q + j R_L \tan[(2\pi)(1/4)(1.15)]} \approx 60.21e^{j29.33^\circ} \]

and

\[ |\Gamma_{\text{in}}| = \left| \frac{Z_{\text{in}} - 50}{Z_{\text{in}} + 50} \right| \approx 0.278 \quad \rightarrow \quad S = \frac{1 + |\Gamma_{\text{in}}|}{1 - |\Gamma_{\text{in}}|} \approx 1.768 \]

and for the second circuit we have

\[ Z_{\text{in}} = Z_{Q1} \frac{Z_{\text{in}}' + j Z_{Q1} \tan[(2\pi)(1/4)(1.15)]}{Z_{Q1} + j Z_{\text{in}}' \tan[(2\pi)(1/4)(1.15)]} \]

where

\[ Z_{\text{in}}' = Z_{Q2} \frac{R_L + j Z_{Q2} \tan[(2\pi)(1/4)(1.15)]}{Z_{Q2} + j R_L \tan[(2\pi)(1/4)(1.15)]} \]

substituting values we find \( Z_{\text{in}} \approx 44.55e^{j3.98^\circ} \) and

\[ |\Gamma_{\text{in}}| = \left| \frac{Z_{\text{in}} - 50}{Z_{\text{in}} + 50} \right| \approx 0.0673 \quad \rightarrow \quad S = \frac{1 + |\Gamma_{\text{in}}|}{1 - |\Gamma_{\text{in}}|} \approx 1.144 \]

Similar analysis for 15% below the design frequency gives \( S \approx 1.22 \) for the first circuit and \( S \approx 1.02 \) for the second circuit.
4. This problem is similar to the example 3-16 in the textbook. For matching of the load with either a short or an open shunt stub, we have the following circuit:

Where \( Z_M \) is either 0 or infinity for a short or an open stub, respectively.

Since we are dealing with shunt stub, admittance would simplify the calculation. The equivalent admittance at point A (excluding the stub for a moment) is given by

\[
Y_{in} = Y_0 \frac{Y_L + jY_0 \tan(\beta l)}{Y_0 + jY_L \tan(\beta l)}
\]

where \( Y_0 = [Z_0]^{-1} = 1/50 \), and \( Y_L = [Z_0]^{-1} = 1/100 + j3/100 \).

Separating the expression into the real and imaginary parts gives

\[
Y_{in} = \frac{1 + x^2}{(10 + 15x)^2 + 25x^2} + j \frac{3x^2 - 3x - 3}{(10 + 15x)^2 + 25x^2}
\]

where \( x = \tan(\beta l) \).

Since a short or an open stub can behave as a purely reactive element, the real part of \( Y_{in} \) above should be equal to \( Z_0 \) in order for the matching to take place. Thus, we have,

\[
\frac{1 + x^2}{(10 + 15x)^2 + 25x^2} = Y_0 = \frac{1}{50}
\]
The solution of this equation is: \( \tan(\beta l) = \frac{-3 - \sqrt{5}}{4} \) or \( \frac{-3 + \sqrt{5}}{4} \). Both of them are negative. This means that \( \beta l \) is larger than \( \pi/2 \). The more negative value represents a point closer to the load and this value will be used in the following calculation.

\[
\tan(\beta l) = \frac{-3 - \sqrt{5}}{4} \iff \beta l = 2.22315. 
\]

Substitute this value into the imaginary part of \( Yin \) and get

\[
j \frac{3x^2 - 3x - 3}{(10 + 15x)^2 + 25x^2} \bigg|_{x = -3 - \sqrt{5}} = 0.044721 j 
\]

The stub needs to have an impedance opposite to the value above in order to cancel the reactive part of the \( Yin \) for matching. For a short stub, \( Y = -jY_0 \cot(\beta l') \). Therefore, we want

\[
\frac{0.044721}{Y_0} = \cot(\beta l') \iff \beta l' = 0.420534. 
\]

The input impedance at point A including the stub is

\[
Y_{tot,in} = \frac{1 + \tan^2(2.22315)}{(10 + 15 \tan(2.22315))^2 + 25 \tan^2(2.22315)} \\
+ j \left( \frac{3 \tan^2(2.22315) - 3 \tan(2.22315) - 3}{(10 + 15 \tan(2.22315))^2 + 25 \tan^2(2.22315)} - \frac{Y_0}{\tan(0.420534)} \right)
\]

The imaginary part is equal to zero. The SWR is given by

\[
S = \frac{1 + \left| \frac{Y_0 - Y_{in}}{Y_0 + Y_{in}} \right|}{1 - \left| \frac{Y_0 - Y_{in}}{Y_0 + Y_{in}} \right|}
\]

In general, for \( \beta' \neq \beta \), i.e., \( \beta l' = \beta l * n \), where \( n \) is the ratio of \( \beta' \) to \( \beta \), the input impedance shown above is
\[ Y_{tot,\text{in}}(n) = \frac{1 + \tan^2(2.22315n)}{(10 + 15 \tan(2.22315n))^2 + 25 \tan^2(2.22315n)} \]
\[ + j\left(\frac{3 \tan^2(2.22315n) - 3 \tan(2.22315n) - 3}{(10 + 15 \tan(2.22315n))^2 + 25 \tan^2(2.22315n)} - \frac{Y_0}{\tan(0.420534^* n)}\right) \]

and the SWR is equal to
\[ S = \frac{1 + \left|\frac{Y_0 - Y_{in}(n)}{Y_0 + Y_{in}(n)}\right|}{1 - \left|\frac{Y_0 - Y_{in}(n)}{Y_0 + Y_{in}(n)}\right|} \]

SWR can be plotted as a function of n, representing the amount of shift in the signal frequency / phase constant from the ones used in calculating the numbers above.

The values of n where S = 1.2 are 0.989 and 1.01, for a short stub.

Following the same procedure, we have these for an open stub

\[ Y = jY_0 \tan(\beta l') \Leftrightarrow \frac{0.044721}{Y_0} = -\tan(\beta l') \Leftrightarrow \beta l' = 1.99133. \]
\[
Y_{\text{tot,n}}(n) = \frac{1 + \tan^2(2.22315n)}{(10 + 15\tan(2.22315n))^2 + 25\tan^2(2.22315n)} \\
+ j\left(\frac{3\tan^2(2.22315n) - 3\tan(2.22315n) - 3}{(10 + 15\tan(2.22315n))^2 + 25\tan^2(2.22315n)} - Y_0 \tan(1.99133*n)\right)
\]

The plot of S vs. n:

![Graph](image)

The values of n where S = 1.2 are 0.993 and 1.01.

For impedance matching with a lumped element, we have the following circuit:

![Circuit Diagram](image)

To keep the problem simple, let’s make the lumped element \(Z_M\) a purely reactive component. With this constraint, the real part of \(Y_{\text{in}}\) needs to match the characteristic impedance of the transmission line, just like the cases of short and open stubs above. The
previous calculation gives $\beta l = 2.22315$ or $2.95288$. Just as before, we pick the shortest distance $\beta l = 2.22315$. At this length, $Y_{in}$ is equal to

$$Y_{in} = 0.02 + 0.0447214 j$$

The positive imaginary part implies that the lumped element needs to be an inductor in order to make the impedance matching work. So,

$$Y_M = 1/Z_M = -j/\omega L = -j/(\beta v L) = -0.0447214 j$$

where $v$ is the propagation velocity of the transmission line. The velocity is property of the transmission line, and does not depend on the signal frequency.

In general, $Y_{tot,in}$ is

$$Y_{tot,in}(n) = \frac{1 + \tan^2(2.22315n)}{(10 + 15 \tan(2.22315n))^2 + 25 \tan^2(2.22315n)}$$

$$+ j \left( \frac{3 \tan^2(2.22315n) - 3 \tan(2.22315n) - 3}{(10 + 15 \tan(2.22315n))^2 + 25 \tan^2(2.22315n)} \right) \frac{0.0447214}{n}$$

The plot of $S$ vs. $n$:

The values of $n$ where $S = 1.2$ are 0.989 and 1.01.
In summary,

<table>
<thead>
<tr>
<th></th>
<th>Bandwidth for $S \leq 1.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short</td>
<td>$0.989 \omega_0$ to $1.01 \omega_0$</td>
</tr>
<tr>
<td>Open</td>
<td>$0.993 \omega_0$ to $1.01 \omega_0$</td>
</tr>
<tr>
<td>Lumped</td>
<td>$0.989 \omega_0$ to $1.01 \omega_0$</td>
</tr>
</tbody>
</table>
3-38. Quarter-wave matching. (a) We start by writing the input impedance of the 50Ω transmission line of length \( l \) on the right looking toward the load \( Z_L = 40 + j30 \ Ω \) as

\[
Z_1 = (50) \frac{(40 + j30) + j50T}{50 + j(40 + j30)T} = (50) \frac{40 + j(30 + 50T)}{(50 - 30T) + j40T} \\
= (50) \frac{[40 + j(30 + 50T)](50 - 30T) - j40T}{(50 - 30T)^2 + (40T)^2}
\]

where \( T = \tan(\beta l) \) and \( \beta = 2\pi/\lambda \). Note that the input impedance of the 50Ω line at the location of the quarter-wave transformer must be purely real so that a match can be achieved. Therefore, to find the location of the quarter-wave transformer with respect to the load, we equate the imaginary part of \( Z_1 \) to zero, i.e.,

\[
\Im\{Z_1\} = 0 \quad \rightarrow \quad (30 + 50T)(50 - 30T) - (40)^2T = 0
\]

\[
\rightarrow \quad 15T^2 = 15 \quad \rightarrow \quad T = \tan(\beta l) = \pm 1
\]
From $\tan(\beta l) = +1$, we find $l/\lambda = 0.125$ and from $\tan(\beta l) = -1$, we find $l/\lambda = 0.375$. If we choose the nearest location (i.e., $l = 0.125\lambda$) for the quarter-wave transformer design, then $T = +1$, and substituting this value into the $Z_1$ expression above yields

$$Z_1 = (50\frac{40 + j30}{50 + j(40 + j30)}) = (50\frac{40 + j80}{20 + j40}) = 100\Omega$$

which is a purely resistive impedance, as expected. Using this value of $Z_1$, we can now determine the characteristic impedance of the quarter-wave transformer inserted at a distance of $l = 0.125\lambda$ away from the load to match the load impedance $Z_L = 40 + j30\ \Omega$ to the $Z_0 = 50\ \Omega$ line as

$$Z_Q = \sqrt{Z_0Z_1} = \sqrt{(50)(100)} \approx 70.7\Omega$$

(Note that if the other location was chosen for the design, then $T = -1$, and the input impedance of the $50\ \Omega$ line at that location is $Z_1 = 25\ \Omega$ and therefore for a quarter-wave transformer introduced at that position, the characteristic impedance would be $Z_Q = \sqrt{(50)(25)} \approx 35.4\Omega$)

(b) Following the same steps with $Z_L = 80 - j60\ \Omega$, we have

$$Z_1 = (50\frac{80 - j60 + j50T}{50 + j(80 - j60)T}) = (50\frac{80 + j(50T - 60)}{(50 + 60T) + j80T})$$

$$= (50\frac{[80 + j(50T - 60)][(50 + 60T) - j80T]}{(50 + 60T)^2 + (80T)^2})$$

Equating the imaginary part of $Z_1$ to zero yields

$$\Re \{Z_1\} = 0 \rightarrow (50T - 60)(50 + 60T) - (80)^2T = 0$$

$$\rightarrow 6T^2 - 15T - 6 = 0 \rightarrow T \approx 2.85, -0.351$$

From $\tan(\beta l) \approx 2.85$, we find $l/\lambda \approx 0.196$ and from $\tan(\beta l) \approx -0.351$, we find $l/\lambda \approx 0.446$. Choosing $l \approx 0.196\lambda$ (i.e., the nearest location with respect to the position of the load) for the design, the value of $Z_1$ at that position is

$$Z_1(l \approx 0.196\lambda) \approx (50\frac{80(50 + 60T) + 80T(50T - 60)}{(50 + 60T)^2 + (80T)^2})_{T=2.85} \approx 18.1\Omega$$

To match $Z_1 \approx 18.1\Omega$ to $50\Omega$, we need a quarter-wave transformer with characteristic impedance given by

$$Z_Q = \sqrt{Z_0Z_1} \approx \sqrt{(50)(18.1)} \approx 30.1\Omega$$