EECS 242:
Volterra/Wiener Representation of Non-Linear Systems

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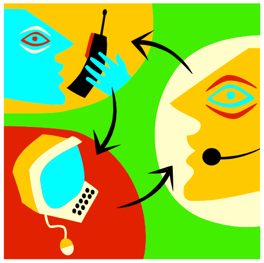


Linear Input/Output Representation

A linear system is completely characterized by its impulse response function:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \qquad \text{LTI}$$

$$\text{causality} \Rightarrow \quad h(t) = 0;\ t < 0$$

y(t) has memory since it depends on

$$x(t-\tau); \quad \tau \in [-\infty, \infty]$$

Non-Linear Order-N Convolution

Consider a degree-n system:

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underbrace{h_n(\tau_1, \tau_2, \ldots, \tau_n)}_{\text{kernel}} x(t-\tau_1) \ldots x(t-\tau_n)\, d\tau_1 \ldots d\tau_n$$

If $\quad x'(t) = \alpha x(t) \rightarrow y_n'(t) = \alpha^n y_n(t)$

Change of variables -

$$\alpha_j = t - \tau_j \qquad d\alpha_j = -d\tau_j$$

$$\tau_j = t - \alpha_j$$

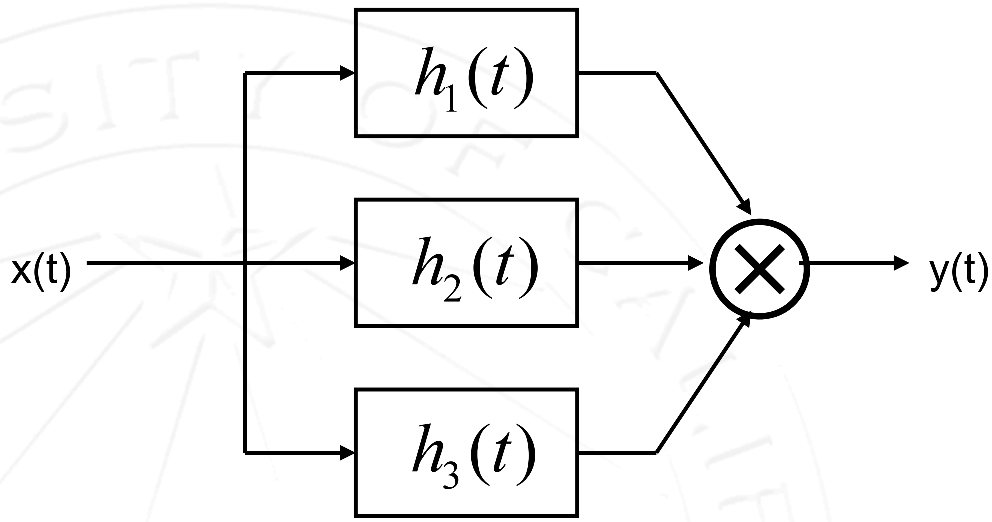
$\Rightarrow$

Generalized Convolution

Generalization of convolution integral of order *n*:

$$y_n(t) = \int_{-\infty}^{\infty} h_n(t-\tau_1,\ldots,t-\tau_n)u(\tau_1)\ldots u(\tau_n)d\tau_1\ldots d\tau_n$$

Non-Linear Example



$$y_j(t) = \int_{-\infty}^{\infty} h_j(t-\tau)x(\tau)d\tau$$

$$y(t) = y_1(t)y_2(t)y_3(t)$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \cdot \int_{-\infty}^{\infty} \textcircled{2}\, d\tau_2 \cdot \int_{-\infty}^{\infty} \textcircled{3}\, d\tau_3$$

Non-Linear Example (cont)

$$y(t) = y_1(t)y_2(t)y_3(t)$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \cdot \int_{-\infty}^{\infty} \boxed{2}\, d\tau_2 \cdot \int_{-\infty}^{\infty} \boxed{3}\, d\tau_3$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{h_1(\tau_1)h_2(\tau_2)h_3(\tau_3)}_{h(t_1,t_2,t_3) = h_1(t_1)h_2(t_2)h_3(t_3)} x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3$$

$$h_s(\,) = \frac{1}{6}\{h(t_1,t_2,t_3) + h(t_2,t_1,t_3) + h(t_2,t_3,t_1) + \ldots\}$$

Kernel is not in unique. We can define a unique "symmetric" kernel.

Symmetry of Kernel

Kernel h can be expressed as a symmetric function of its arguments: Consider output of a system where we permute any number of indices of *h*:

$$\int_{-\infty}^{\infty} h(\tau_2,\tau_1)x(t-\tau_2)x(t-\tau_1)d\tau_2 d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

$$h(\tau_1,\tau_2) \Leftrightarrow h(\tau_2,\tau_1)$$

For n arguments, $n!$ permutations

Symmetric kernel

We create a symmetric kernel by

$$h_{sym}(t_1,\ldots,t_n) = \frac{1}{n!}\sum h\left(t_{\pi(1)},\ldots,t_{\pi(n)}\right)$$

System output identical to original unsymmetrical kernel

Volterra Series: "Polynomial" of degree N

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)x(t-\tau_1)\cdots x(t-\tau_n)d\tau_1\ldots d\tau_n$$

Volterra Series

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) x(t-\tau_1) \ldots x(t-\tau_n) d\tau_1 \ldots d\tau_n$$

$If\ h_n(t_1, \ldots, t_n) = a_n \delta(t_1)\delta(t_2)\ldots\delta(t_n),$ we get ordinary power series:

$$y(t) = a_1 x(t) + a_2 x(t)^2 + \ldots. + a_N x(t)^N$$

It can be rigorously shown by the Stone-Weierstrass theorem that the above polynomial approximates a non-linear system to any desired precision if N is made sufficiently large.

Non-rigorous "proof"

Say y(t) is a non-linear function of $x(t-\tau)$ for all $\tau > 0$ (all past input)

Fix time t and say that $x(t-\tau)$ can be characterized by the set $\{x_1(t),...,x_n(t),...\}$ so that y(t) is some non-linear function:

$$y(t) = f(x_1(t), x_2(t),...)$$

Non-Rigorous Proof (cont)

Let $\{\phi_1(t), \phi_2(t), \ldots\}$ be an orthonormal basis for the space

$$\int_{-\infty}^{\infty} \phi_i(\tau)\phi_j(\tau)\,d\tau = \delta_{ij}$$

Thus

$$x(t-\tau) = \sum_{i=1}^{\infty} x_i(t)\phi_i(\tau)$$

$$x_i(t) = \int_{-\infty}^{\infty} x(t-\tau)\phi_i(\tau)\,d\tau \qquad \text{“inner product”}$$

Non-Rigorous Proof (cont)

Expand *f* into a Taylor series

$$f\left(x_1(t), x_2(t), \ldots\right)$$

$$y(t) = a_o + \sum_{i=1}^{\infty} a_i x_i(t) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j x_i(t) x_j(t) + \ldots$$

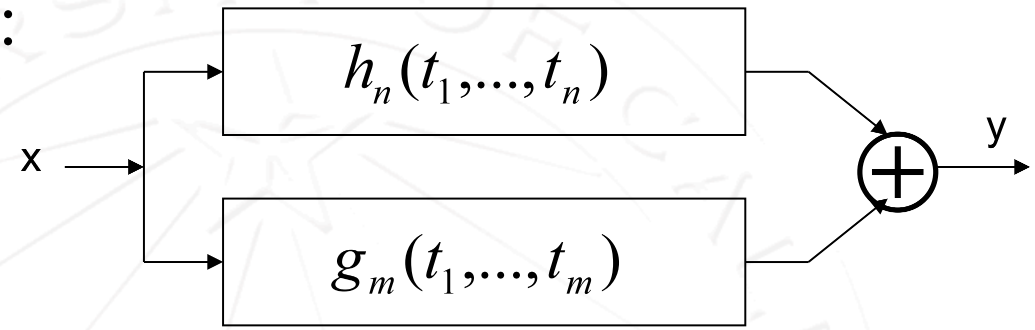
$$= a_o + \int_0^{\infty}\sum_{i=1}^{\infty} a_i \phi(\tau_1) x(t-\tau_1) d\tau_1 + \int_0^{\infty}\int_0^{\infty} \underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\phi_i(\tau_1)\phi_j(\tau_2)}_{h_2(\tau_1,\tau_2)} x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2$$

This is the Volterra/Wiener representation for a non-linear system

Sifting Property: $x(\sigma) = \int_{-\infty}^{\infty} \delta(t-\sigma) x(t) dt$

Interconnection of Non-Linear Systems

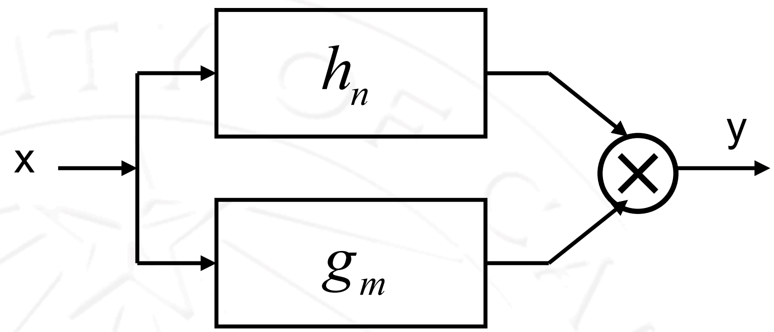
Sum:



$$f_n(t_1,t_2,\ldots,t_n) = h_n(t_1,\ldots,t_n) + g_m(t_1,\ldots,t_m)$$

Product Interconnection

Product:



$$y(t)=\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)x(t-\tau_1)\ldots x(t-\tau_n)d\tau_1\ldots d\tau_n \times$$

$$\int_{-\infty}^{\infty} g_m(\tau_1,\ldots,\tau_m)x(t-\tau_1)\ldots x(t-\tau_m)d\tau_1\ldots d\tau_m$$

$$=\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)g_m(\tau_{n+1},\ldots,\tau_{n+m})x(t-\tau_1)\ldots x(t-\tau_{n+m})d\tau_1\ldots d\tau_{n+m}$$

$$f_{n+m}(t_1,\ldots,t_{n+m})=h_n(t_1,\ldots,t_n)g_m(t_{n+1},\ldots,t_{n+m})$$

Volterra Series Laplace Domain

- Transform domain input/output representation
- Linear systems in time domain

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

- Define Generalized Laplace Transform:

$$F(s_1,\ldots,s_n) = L[f(t_1,\ldots,t_n)]$$

$$= \int_0^\infty f(t_1,\ldots,t_n)e^{-s_1t_1}\cdots e^{-s_nt_n}dt_1\cdots dt_n$$

Volterra Series Example

- Generalized transform of a function of two variables:

$$f(t_1, t_2) = t_1 - t_1 e^{-t_2} \qquad t_1, t_2 \geq 0$$

$$F(s_1, s_2) = \int_0^\infty \int_0^\infty t_1 e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2 - \int_0^\infty \int_0^\infty t_1 e^{-t_2} e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2$$

$$F(s_1, s_2) = \frac{1}{s_1^{\ 2}} \left(\underbrace{\int_0^\infty e^{-s_2 t_2} dt_2}_{\frac{1}{s_2}} - \underbrace{\int_0^\infty e^{-t_2} e^{-s_2 t_2} dt_2}_{\frac{1}{1+s_2}} \right)$$

$$= \frac{1}{s_1^{\ 2} s_2 (s_2 + 1)}$$

Properties of Transform

- Property 1: L is linear
- Property 2:

$$f(t_1,\ldots,t_n) = h(t_1,\ldots,t_k)g(t_{k+1},\ldots,t_n)$$

$$\Leftrightarrow$$

$$F(s_1,\ldots,s_n) = H(s_1,\ldots,s_k)G(s_{k+1},\ldots,s_n)$$

- Property 3: Convolution form #1

$$f(t_1,\ldots,t_n) = \int_0^\infty h(t)g(t_1-\tau,\ldots,t_n-\tau)d\tau$$

$$F(s_1,\ldots,s_n) = H(s_1+\cdots+s_n)G(s_1,\ldots,s_n)$$

Properties of Generalized Transform

- Property 4: Convolution Form #2:

$$f(t_1,\ldots,t_n) = \int_0^\infty h(t_1-\tau_1,\ldots,t_n-\tau_n)g(\tau_1,\ldots,\tau_n)\times d\tau_1\ldots d\tau_n$$

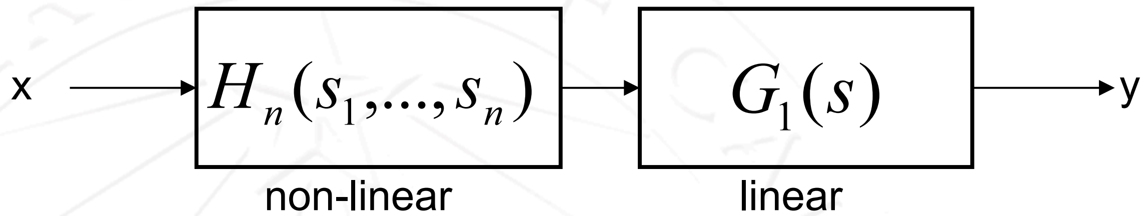
$$F(s_1,\ldots,s_n) = H(s_1,\ldots,s_n)G(s_1,\ldots,s_n)$$

- Property 5: Time delay $\tau_j > 0$

$$L[f(t_1-\tau_1,\ldots,t_n-\tau_n)] = F(s_1,\ldots,s_n)e^{-s_1\tau_1\ldots s_n\tau_n}$$

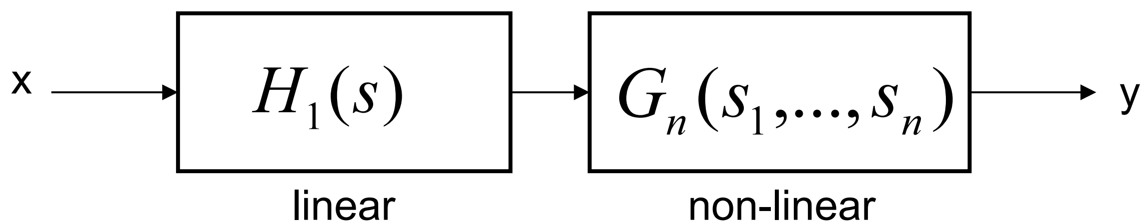
Cascades of Systems

- Cascade #1:



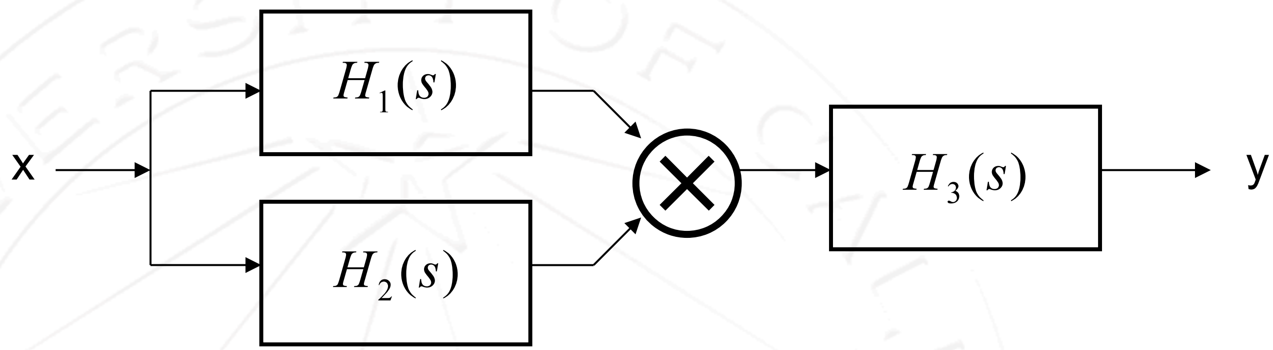
$$F_n(s_1,\ldots,s_n) = H_n(s_1,\cdots s_n)G_1(s_1 + \cdots + s_n)$$

- Cascade #2:



$$F_n(s_1,\ldots,s_n) = H_1(s_1)\cdots H_1(s_n)G_n(s_1,\ldots,s_n)$$

Cascade Example



$$F(s_1, s_2) = \underbrace{H_1(s_1)H_2(s_2)}_{\text{property \#1}}\underbrace{H_3(s_1 + s_2)}_{\text{property \#2}}$$

property #1

property #2

not symmetric

Exp Response of *n*-th Order System

$$y_n(t) = \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)x(t-\tau_1)\cdots x(t-\tau_n)d\tau_1\ldots d\tau_n$$

$$x(t) = \sum_{i=1}^{P} \alpha_i e^{\lambda_i t}$$

$$y_n(t) = \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\prod_{j=1}^{n}[\alpha_1 e^{\lambda_1(t-\tau_j)} + \cdots + \alpha_p e^{\lambda_p(t-\tau_j)}]\times d\tau_1 \cdots d\tau_n$$

$$\left(\sum_{k_1=1}^{P} \alpha_{k_1} e^{\lambda_{k_1}(t-\tau_1)}\right) \quad \cdots \quad \left(\sum_{k_n=1}^{P} \alpha_{k_n} e^{\lambda_{k_n}(t-\tau_n)}\right)$$

continued ⇨

Exponential Response (cont)

$$\sum_{k_1=1}^{P}\cdots\sum_{k_n=1}^{P}\left(\prod_{j=1}^{n}\alpha_{k_j}\right)\exp\{\underbrace{\lambda_{k_1}(t-\tau_1)+\ldots+\lambda_{k_n}(t-\tau_n)}_{\exp\{\sum_{j=1}^{n}\lambda_{k_j}(t-\tau_j)\}}\}$$

$$y_n(t)=\sum_{k_1=1}^{P}\cdots\sum_{k_n=1}^{P}\left(\prod_{j=1}^{n}\alpha_{k_j}\right)\exp\{\sum_{j=1}^{n}\lambda_{k_j}t\}\underbrace{\int_{-\infty}^{\infty}h_n(\tau_1,\ldots\tau_n)\exp\{-\sum_{j=1}^{n}\lambda_{k_j}\tau_j\}d\tau_1\ldots d\tau_n}_{H_n(\lambda_{k_1},\ldots,\lambda_{k_n})}$$

The Final Result…

$$y_n(t) = \sum_{k_1=1}^{P} \cdots \sum_{k_n=1}^{P} \left(\prod_{j=1}^{n} \alpha_{k_j} \right) \exp\{\sum_{l=1}^{n} \lambda_{k_l} t\} H_n(\lambda_{k_1}, \ldots, \lambda_{k_n})$$

- We've seen this before… A particular frequency mix $m_1\lambda_1 + m_2\lambda_2 + \ldots + m_p\lambda_p$ has response

$$\alpha_1^{m_1} \ldots \alpha_p^{m_p} G_{m_1,\ldots,m_p}(\lambda_1,\ldots,\lambda_n) e^{(m_1\lambda_1 + \ldots + m_p\lambda_p)t}$$

Frequency mix response

$$y_n(t) = \sum_{\vec{m}} \alpha_1^{m_1} \ldots \alpha_p^{m_p} G_{\vec{m}}(\vec{\lambda}) \exp\{\vec{m} \cdot \vec{x}\}$$

- Sum over all vectors $\vec{m}$ such that $0 \le m_i \le n$

$$\sum_{i=1}^{P} m_i = n$$

- If $H_n(s_1, \ldots, s_n)$ is symmetric, then we can group the terms as before

$$G_{\vec{m}}(\vec{\lambda}) = (n;\vec{m}) H_{n,sym}(\underbrace{\lambda_1, \ldots, \lambda_1}_{m_1}, \ldots, \underbrace{\lambda_p, \ldots, \lambda_p}_{m_p})$$

$$(n;\vec{m}) = \frac{n!}{m_1! m_2! \ldots m_p!}$$

Important special case $P=n$

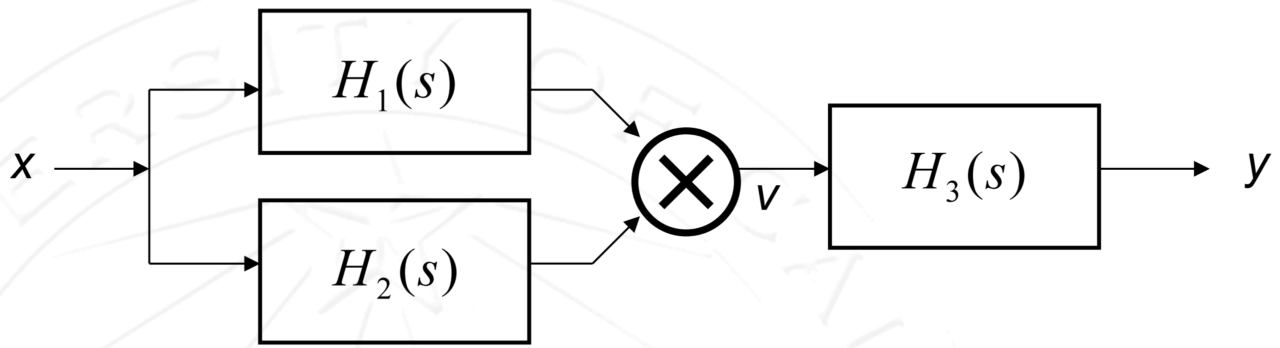
$$G_{1,\ldots,n}(\lambda_1,\ldots,\lambda_n) = n!\,H_{n,sym}(\lambda_1,\ldots,\lambda_n)$$

- To derive $H_{n,sym}(\lambda_1,\ldots,\lambda_n)$, we can apply n exponentials to a degree n system and the symmetric transfer function is given by $\frac{1}{n!}$ times the coefficient of

$$e^{\lambda_1 t+\ldots+\lambda_n t}$$

- We call this the "Growing Exponential Method"

Example 1



- Excite system with two-tones:

$$v(t) = \left(H_1(\lambda_1)e^{\lambda_1 t} + H_1(\lambda_2)e^{\lambda_2 t}\right) \times \left(H_2(\lambda_1)e^{\lambda_1 t} + H_2(\lambda_2)e^{\lambda_2 t}\right)$$

$$= H_1(\lambda_1)H_2(\lambda_1)e^{2\lambda_1 t} + H_1(\lambda_2)H_2(\lambda_2)e^{2\lambda_2 t}$$

$$+ H_1(\lambda_1)H_2(\lambda_2)e^{(\lambda_1+\lambda_2)t} + H_1(\lambda_2)H_2(\lambda_1)e^{(\lambda_1+\lambda_2)t}$$

Example 1 (cont)

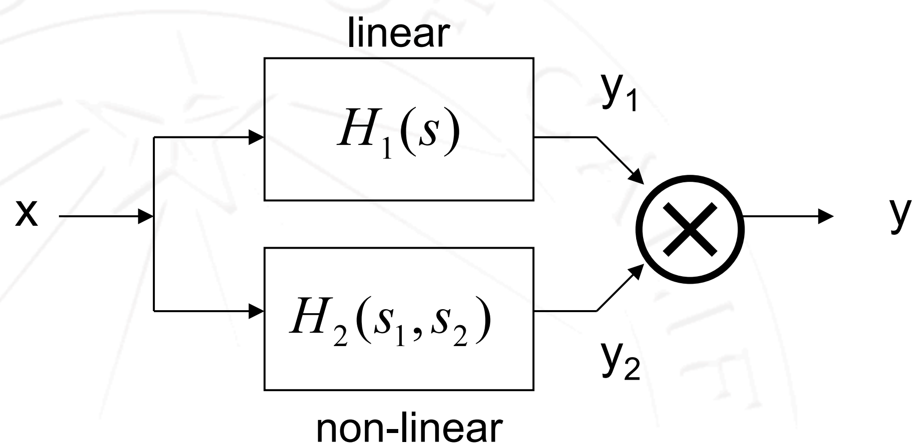
$$y(t) = H_1(\lambda_1)H_2(\lambda_1)H_3(2\lambda_1)e^{2\lambda_1 t} + H_1(\lambda_2)H_2(\lambda_2)H_3(2\lambda_2)e^{2\lambda_2 t}$$

$$+ \underbrace{[H_1(\lambda_1)H_2(\lambda_2) + H_1(\lambda_2)H_2(\lambda_1)]}_{2!\, H_{sym}(s_1, s_2)} \cdot H_3(\lambda_1 + \lambda_2)e^{(\lambda_1 + \lambda_2)t}$$

$$H_{sym}(s_1, s_2) = \tfrac{1}{2}[H_1(s_1)H_2(s_2) + H_1(s_2)H_2(s_1)]H_3(s_1 + s_2)$$

Example 2

- Non-linear system in parallel with linear system:



$$y_1 = \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \qquad y_2 = \int_{-\infty}^{\infty} h_2(\tau_2,\tau_3)x(t-\tau_2)x(t-\tau_3)d\tau_2 d\tau_3$$

$$y_1 \times y_2 = \int_{-\infty}^{\infty} \underbrace{h_1(\tau_1)h_2(\tau_2,\tau_3)}_{h_c(\tau_1,\tau_2,\tau_3)} x(t-\tau_1)\cdots x(t-\tau_3)d\tau_1 \ldots d\tau_3$$

composite → $h_c(\tau_1,\tau_2,\tau_3)$

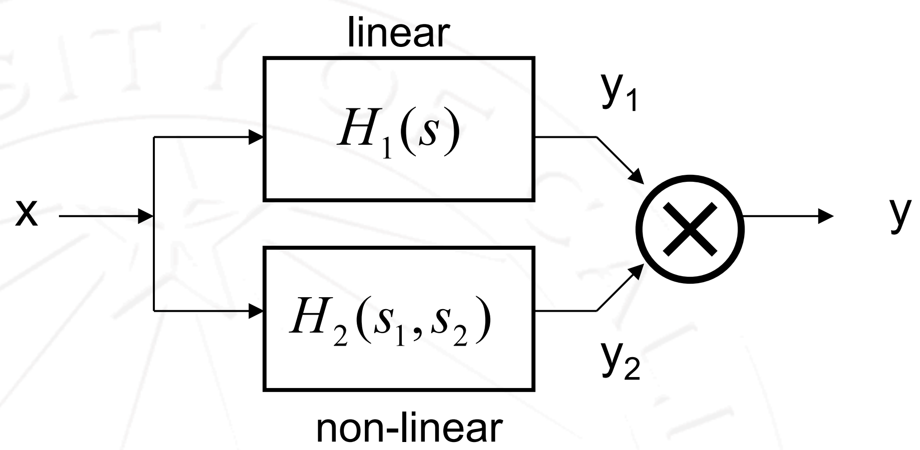
Example 2 (cont)

$$H_{sym}(s_1,s_2,s_3) = \tfrac{1}{3}\{H_1(s_1)H_2(s_2,s_3) + H_1(s_2)H_2(s_1,s_3) + H_1(s_3)\underbrace{H_2(s_1,s_2)}\}$$

assuming H_2 is symmetric

<u>Notation</u>: $H_{sym}(s_1,s_2,s_3) = \overline{H_1(s_1)H_2(s_2,s_3)}$

Example 2 Again



- Redo example with growing exponential method
- Overall system is third order, so apply sum of 3 exponentials to system

$$e^{\lambda_1 t} + e^{\lambda_2 t} + e^{\lambda_3 t}$$

Example 3

- We can drop terms that we don't care about
- We only care about the final term $e^{\lambda_1 t} + e^{\lambda_2 t} + e^{\lambda_3 t}$ so for now ignore terms except $e^{(\lambda_j + \lambda_k)t}$ where $j \neq k$
- Focus on terms in y_2 first

$$2e^{(\lambda_1+\lambda_2)t} H_{2s}(\lambda_1, \lambda_2)$$

$$2e^{(\lambda_1+\lambda_2)t} H_{2s}(\lambda_1, \lambda_3)$$

$$2e^{(\lambda_2+\lambda_3)t} H_{2s}(\lambda_2, \lambda_3)$$

$$H_{2s}(\lambda_1, \lambda_2) = H_{2s}(\lambda_2, \lambda_1)$$

symmetric kernel

Example 3 (cont)

- Now the product of $y_1(t)$ & $y_2(t)$ produces terms like $e^{(\lambda_1+\lambda_2+\lambda_3)t}$

$$
\begin{aligned}
&2H_{2s}(\lambda_1,\lambda_2)H_1(\lambda_3)e^{(\lambda_1+\lambda_2+\lambda_3)t} \\
&\quad +2H_{2s}(\lambda_1,\lambda_3)H_1(\lambda_2)e^{(\lambda_1+\lambda_2+\lambda_3)t} \\
&\quad +2H_{2s}(\lambda_2,\lambda_3)H_1(\lambda_1)e^{(\lambda_1+\lambda_2+\lambda_3)t} \\
&\qquad = 3!H_{3s}(\lambda_1,\lambda_2,\lambda_3)
\end{aligned}
$$

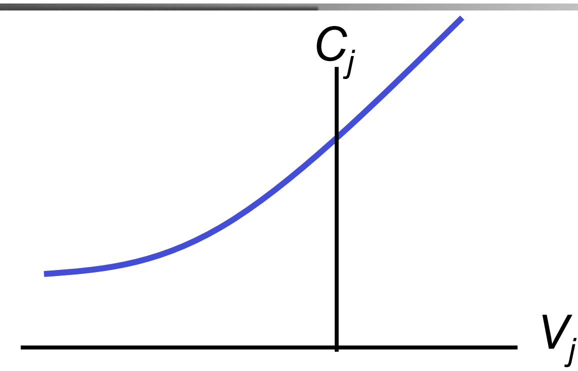
$$H_{3s}(s_1,s_2,s_3) = \frac{2}{3!}(\qquad\qquad)$$

Capacitive non-linearity

- Non-linear capacitors:

C_μ & C_{cs} BJT

C_{db} & C_{sb} MOSFET



- Small signal (incremental) capacitance

$$C_j = \frac{dQ}{dV_j} = \frac{K}{(\Phi + V_j)^{\frac{1}{n}}} \qquad n \approx 2-3$$

Let $V_j = V_Q + v$

$$C_j = \frac{K}{(\Phi + V_j)^{\frac{1}{n}}(1 + \frac{v}{\Phi + V_Q})^{\frac{1}{n}}} \approx C_{\mu_o} + C_{\mu_1} v + C_{\mu_2} v^2 + \ldots$$

small signal cap

cap/V

cap/V²

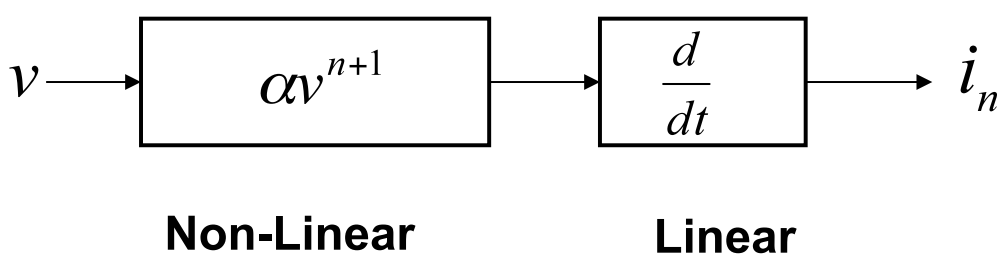
Cap Non-Linearity (cont)

$$i = \frac{dQ}{dt} = \frac{dQ}{dv}\frac{dv}{dt} = C_j(v)\frac{dv}{dt}$$

$$i = C_{\mu 0}\frac{dv}{dt} + C_{\mu_1} v \frac{dv}{dt} + C_{\mu_2} v^2 \frac{dv}{dt} + \cdots$$

$$= C_{\mu 0}\frac{dv}{dt} + \frac{C_{\mu_1}}{2}\frac{dv^2}{dt} + \frac{C_{\mu_2}}{3}\frac{dv^3}{dt} + \cdots$$

Model:

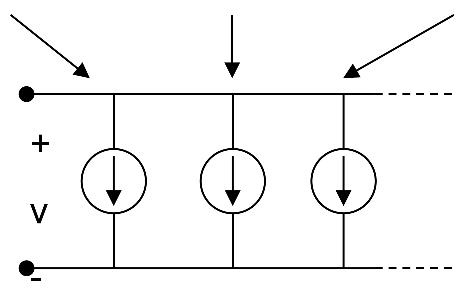


Overall Model

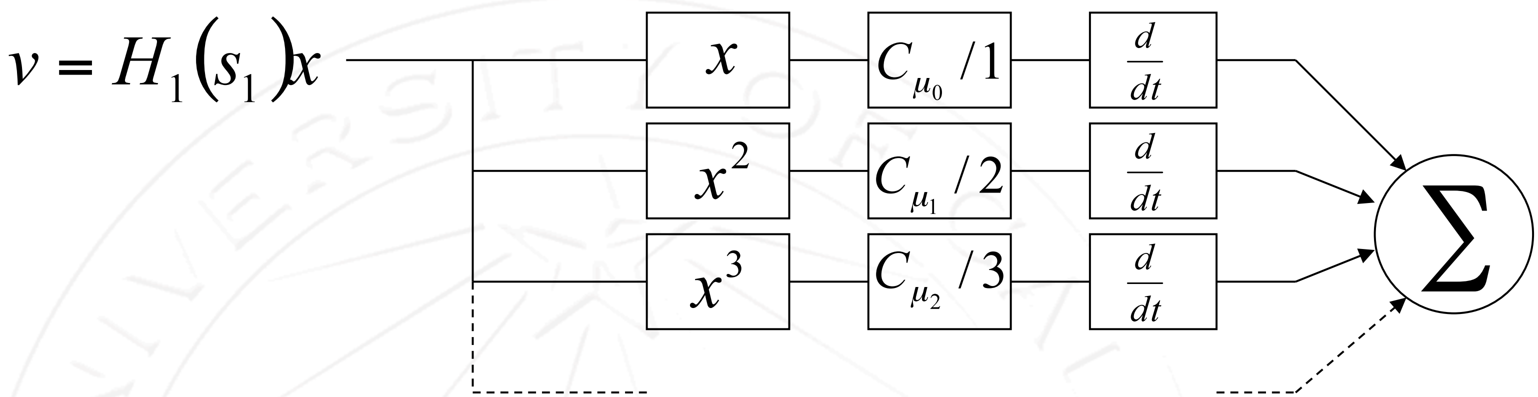
$$i = \frac{dQ}{dt} = \underbrace{\frac{dQ}{dv}}_{C_j(v)}\frac{dv}{dt} = C_j(v)\frac{dv}{dt}$$

$$C_j(v) = C_{\mu_o} + C_{\mu_1} v + C_{\mu_2} v^2 + \ldots$$

$$i = C_{\mu_o}\frac{dv}{dt} + \frac{1}{2}C_{\mu_1}\frac{dv^2}{dt} + \frac{1}{3}C_{\mu_2}\frac{dv^3}{dt} + \ldots$$



Cap Model Decomposition



- Let

$$v = H_1(s_1)x$$

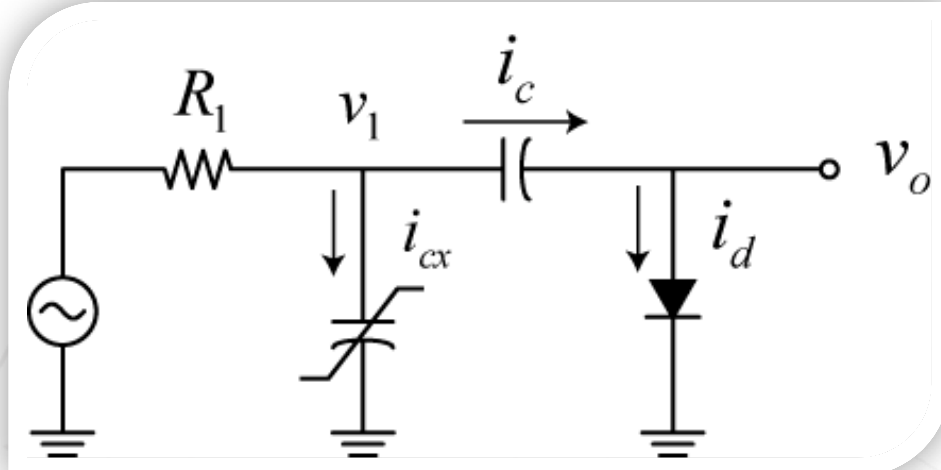
$$v^2 = H_1(s_1)H_1(s_2)x^2$$

$$i_2 = (s_1 + s_2)H_1(s_1)H_1(s_2)\frac{1}{2}C_{\mu_1}x^2$$

$$\vdots$$

$$i_n = (s_1 + \ldots + s_n)H_1(s_1)\ldots H_1(s_n)\frac{1}{n}C_{\mu,n-1}x^n$$

A Real Circuit Example



(Note: DC Bias not shown)

- Find distortion in v_o for sinusoidal steady state response

$$v_o = B_1(j\omega_1) \circ v_i + B_2(j\omega_1, j\omega_2) \circ v_i^2 + \ldots$$

- Need to also find

$$v_1 = A_1(j\omega_1) \circ v_i + A_2(j\omega_1, j\omega_2) \circ v_i^2 + \ldots$$

Circuit Example (cont)

- Setup non-linearities
- Diode: $$i_d = I_S e^{(v_o + V_Q)/V_T} - I_Q$$

$$= \underbrace{I_S e^{V_Q/V_T}}_{I_Q} e^{v_o/V_T} - I_Q \quad = I_Q\left(e^{v_o/V_T} - 1\right)$$

$$= g_1 v_o + g_2 v_o^{\,2} + \ldots \qquad g_1 = \frac{qI_Q}{kT}$$

- Capacitor:

$$C_x = \frac{dQ}{dv_1} = C_o + C_1 v_1 + C_2 v_1^{\,2} + \ldots$$

$$i_{cx} = C_o \frac{dv_1}{dt} + \frac{C_1}{2}\frac{dv_1^{\,2}}{dt} + \frac{C_2}{3}\frac{dv_1^{\,3}}{dt}$$

Second-Order Terms

(1)

$$0 = \frac{A_2}{R_1} + j(\omega_a + \omega_b)C(A_2 - B_2) +$$

$$j(\omega_a + \omega_b)C_o A_2 + j(\omega_a + \omega_b)\frac{C_1}{2} A_1(j\omega_o)A_1(j\omega_b)$$

(2)

$$-j(\omega_a + \omega_b)C(A_2 - B_2) + g_1 B_2(j\omega_a, j\omega_b) +$$

$$g_2 B_1(j\omega_a)B_1(j\omega_b) = 0$$

- Solve for A and B

Third-Order Terms

(1) $$\frac{A_3}{R_1} + j(\omega_a + \omega_b + \omega_c)C(A_3 - B_3) + j(\omega_a + \omega_b + \omega_c)C_o A_3 + j(\omega_a + \omega_b + \omega_c)\frac{C_1}{2} 2\overline{A_1(j\omega_a)A_2(j\omega_a, j\omega_b)} + j(\omega_a + \omega_b + \omega_c)\frac{C_2}{3} A_1(j\omega_a)A_1(j\omega_b)A_1(j\omega_c) = 0$$

(2) $$-j(\omega_a + \omega_b + \omega_c)C(A_3 - B_3) + g_1 B_3 + g_2 2\overline{B_1 B_2} + g_3 B_1 B_1 B_1 = 0$$

- Solve for A_3 & B_3

Distortion Calc at High Freq

$$s_o = H_1(j\omega_a)\circ s_i + H_2(j\omega_a, j\omega_b)\circ s_i^{\,2} + \ldots$$

Compute IM_3 at $2\omega_2$-ω_1 only generated by n ≥3

$$\begin{matrix} & -3 & -2 & -1 & +1 & +2 & +3 \\ \vec{k}_{IM_3} = & (0 & 0 & 1 & 0 & 2 & 0) \end{matrix}$$

H_3 is symmetric so we can group all terms producing this frequency mix by H_3

$$\frac{(3;\vec{k}_{IM_3})}{2^{3-1}} = \frac{3!}{2!\cdot 4} = \frac{3}{4}$$

$$\frac{3}{4} H_3(j\omega_2, j\omega_2, -j\omega_1)s_1 s_2^2$$

For equal amp o/p signal, we adjust each input amp so that:

$$s_o = |H_1(j\omega_1)|s_1 = |H_1(j\omega_2)|s_2$$

$$IM_3 = \frac{3}{4}\frac{s_1 s_2^{\,2}|H_3(j\omega_2, j\omega_2, -j\omega_1)|}{|H_1(j\omega_1)|s_1}$$

Disto Calc at High Freq (2)

$$IM_3 = \frac{3}{4} \frac{\left|H_3(j\omega_2, j\omega_2, -j\omega_1)\right|}{\left|H_1(j\omega_1)\right|\left|H_1(j\omega_2)\right|^2} s_o^{\ 2}$$

At low frequency: $IM_3 = \frac{3}{4} \frac{a_3}{a_1^{\ 3}} s_o^{\ 2}$

- Conclude that at high frequency all third order distortion (fractional) ∝ (signal level)^2 for small distortion all second order ∝ (signal level)

Disto Calc at High Freq (3)

- Similarly

$$HD_3 = \frac{s_1^{\;3}}{4}\frac{\left|H_3(j\omega_1, j\omega_1, j\omega_1)\right|}{s_o}$$

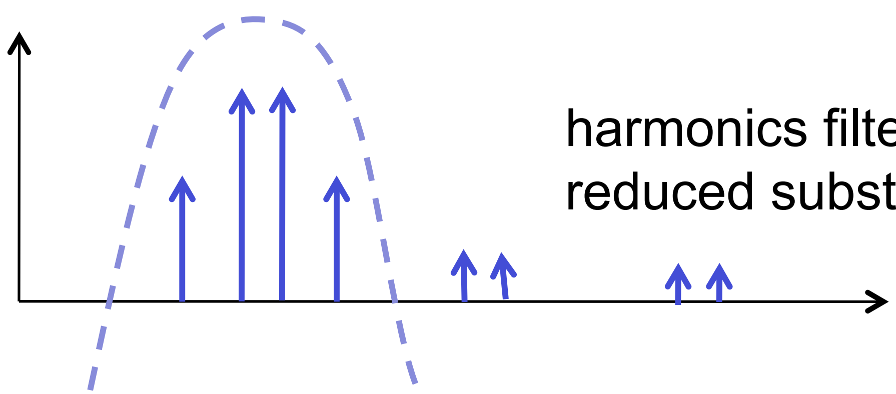
$$s_o = \left|H_1(j\omega_1)\right| s_1$$

$$HD_3 = \frac{1}{4}\frac{\left|H_3(j\omega_1, j\omega_1, j\omega_1)\right|}{\left|H_1(j\omega_1)\right|^3} s_o^{\;2}$$

- Low Freq:

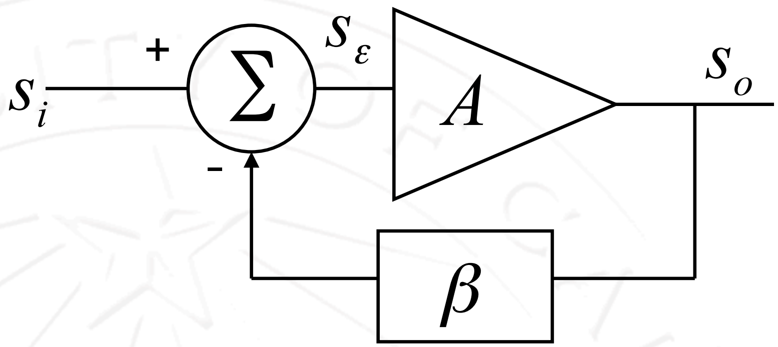
$$HD_3 = \frac{1}{4}\frac{a_3}{a_1^{\;3}} s_o^{\;2}$$

- No fixed relation between HD_3 and IM_3



harmonics filtered and
reduced substantially

High Freq Distortion & Feedback



- Let

$$s_o = A_1(j\omega_a) \circ s_2 + A_2(j\omega_a, j\omega_b) \circ s_\varepsilon{}^2 + \ldots$$

$$s_{fb} = \beta(j\omega_a) \circ s_o \qquad s_\varepsilon = s_i - s_{fb}$$

- Look for

$$s_o = B_1(j\omega_a) \circ s_i + B_2(j\omega_a, j\omega_b) \circ s_i{}^2 + \ldots$$

$$B_1(j\omega_a) \circ s_i + B_2 \circ s_i{}^2 + \ldots = A_1\Big(s_i - \beta(j\omega_a) \circ \big(B_1(j\omega_a) \circ s_i + B_2 \circ s_i{}^2 + \ldots\big) + \ldots\Big)$$

$$+ A_2 \circ (\ )^2 + \ldots$$

High Freq Disto & FB (2)

- First-order: $B_1(j\omega_a) = \dfrac{A_1(j\omega_a)}{1 + \underbrace{A_1(j\omega_a)\beta(j\omega_a)}_{T(j\omega_a)}}$ Frequency dependent loop gain
- Second-order:

$$B_2 = -A_1(j\omega_a + j\omega_b)\beta(j\omega_a + j\omega_b)B_2(j\omega_a, j\omega_b) + A_2(j\omega_a, j\omega_b)B_1(j\omega_a)B_1(j\omega_b)$$

$$B_2(j\omega_a, j\omega_b) = \frac{A_2(j\omega_a, j\omega_b)B_1(j\omega_a)B_1(j\omega_b)}{(1 + A_1(j\omega_a + j\omega_b)\beta(j\omega_a + j\omega_b)B_2(j\omega_a, j\omega_b))}$$

$$B_2(j\omega_a, j\omega_b) = \frac{A_2(j\omega_a, j\omega_b)}{[1 + A_1(j\omega_a + j\omega_b)\beta(j\omega_a + j\omega_b)] \times [1 + A_1(j\omega_a)\beta(j\omega_a)] \times [1 + A_1(j\omega_b)\beta(j\omega_b)]}$$

Comments about HF/LF Disto

- Feedback reduces distortion at low frequency and high frequency $\times\frac{1}{1+T}$ for a fixed output signal level
- True at high frequency if we use $\left|\frac{1}{1+T(j\omega)}\right|$ where ω is evaluated at the frequency of the distortion product
- While IM/HD no longer related, CM, TB, P-1dB, PBL are related since frequencies close together
- Most circuits (90%) can be analyzed with a power series

References

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