EECS 242:
Analysis of Memoryless Weakly Non-Linear Systems

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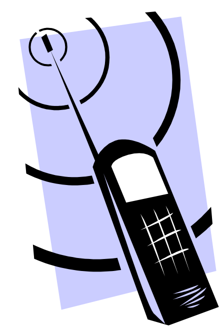


Review of Linear Systems

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)x(\tau)d\tau$$

Linear: $\hat{x} = \alpha\, x_1(t) + \beta\, x_2(t)$

$$y(t) = \alpha \underbrace{\int_{-\infty}^{\infty} h(t,\tau)x_1(\tau)d\tau}_{y_1(t)} + \beta \underbrace{\int_{-\infty}^{\infty} h(t,\tau)x_2(\tau)d\tau}_{y_2(t)}$$

$$= \alpha\, y_1(t) + \beta\, y_2(t)$$ ✓ Linear

- Complete description of a general time-varying linear system. Note output cannot have a DC offset!

Time-invariant Linear Systems

- Time-invariant Linear Systems has $h(t,\tau)=h(t-\tau)$
- Relative function of time rather than absolute
- The transfer function is "stationary"

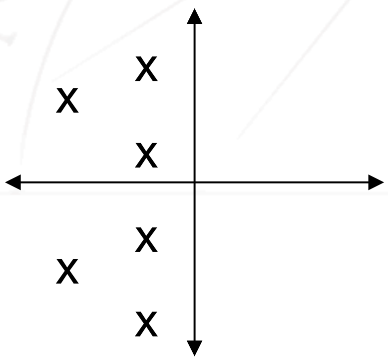
$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau = h(t) * x(t)$$

$$Y(j\omega) = H(j\omega) X(j\omega)$$

convution in time is
product in frequency

Stable Systems

- Linear, time invariant (LTI) system cannot generate frequency content not present in input



Poles of H are strictly in the left hand plane (LHP)

$$Y(j\omega_k) = 0 \quad \text{if} \quad X(j\omega_k) = 0$$

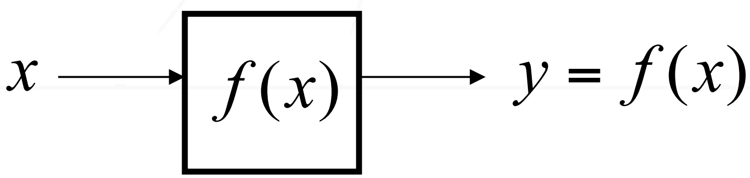
Memoryless Linear System

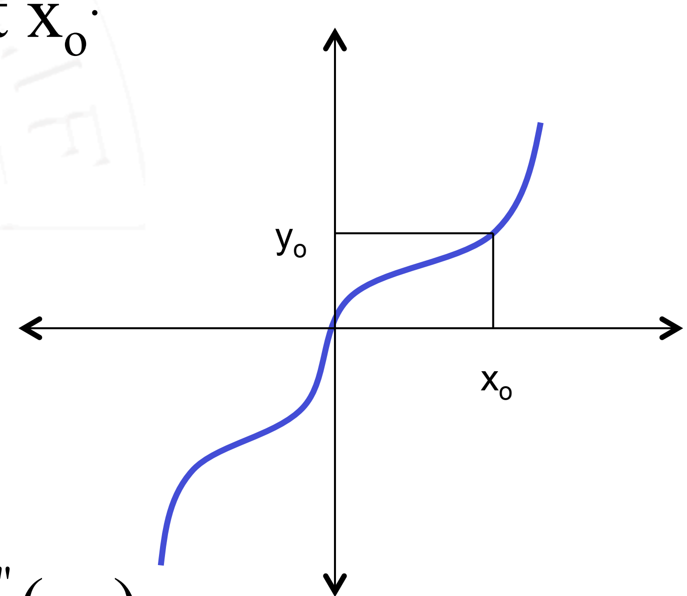
$$y(t) = \alpha x(t)$$

No “DC”

No Delay

- If function is continuous at x_o, then we can do a Taylor Series expansion about x_o:





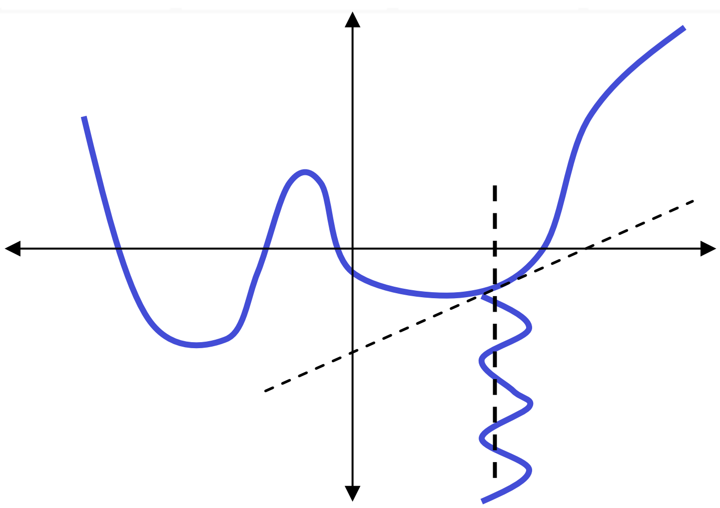
$$\hat{y}(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \ldots$$

Taylor Series Expansion

- This expansion has a certain radius of convergence. If we truncate the series, we can compute a bound on the error $y - \hat{y}$

$$s_o(x) = \hat{y}(x) - y_o = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

- Let's assume: $x(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t)$



Maximum excursion must be less than radius of convergence. Certainly the max A_k has to be smaller than the radius of convergence.

Sinusoidal Exciation

$$y_m = \left(\sum_{n=1}^{N} A_n \cos \omega_n t \right)^m = \left(\sum_{n=1}^{N} \frac{A_n}{2} \left(e^{j\omega_n t} + e^{-j\omega_n t} \right) \right)^m$$

$$= \left(\sum_{n=-N}^{N} \frac{A_n}{2} e^{j\omega_n t} \right)^m$$

$$A_o \equiv 0$$

$$\omega_{-k} = -\omega_k$$

$$= \underbrace{\sum(\ )\sum(\ )\sum(\ )\sum(\ )}_{\text{m-times}}$$

$$= \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} \cdots \sum_{k_m=-N}^{N} \frac{A_{k_1} A_{k_2} \dots A_{k_m}}{2^m} e^{j(\omega_{k1} + \omega_{k2} + \dots + \omega_{km})t}$$

General Mixing Product

We have frequency components: $\omega_{k_1} + \omega_{k_2} + \ldots + \omega_{k_m}$

where k_p ranges over $2N$ values

Terms in summation: $(2N)^m$

Example: Take m=3, N=2 $\begin{cases} \omega_1, \omega_{-1} \\ \omega_2, \omega_{-2} \end{cases}$ $(2N)^m = 4^3 = 64$

64 Terms in summation!

$$\omega \in \{-\omega_2, -\omega_1, \omega_1, \omega_2\}$$

64 Terms

$$\omega_1 + \omega_1 + \omega_1 \rightarrow 3\omega_1 \quad \leftarrow \text{HD}_3$$

$$\omega_1 + \omega_1 + \omega_2 \rightarrow 2\omega_1 + \omega_2$$

$$\omega_1 + \omega_1 - \omega_2 \rightarrow 2\omega_1 - \omega_2 \quad \leftarrow \text{IM}_3$$

$$\omega_1 + \omega_1 - \omega_1 \rightarrow \omega_1$$

← gain expression or compression

Vector Frequency Notation

Define $\vec{k} = (k_{-N}, \ldots, k_{-1}, k_1, \ldots, k_N)$

2N-vector where k_j denotes the number of times a particular frequency appears in a give summation:

No DC terms

$$\left.\begin{array}{l} \omega_2 + \omega_1 + \omega_2 \\ \omega_1 + \omega_2 + \omega_2 \\ \omega_2 + \omega_2 + \omega_1 \end{array}\right\} \qquad \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array}$$

$$\vec{k} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\omega_1 + \omega_1 - \omega_2 \qquad \vec{k} = \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}$$

Sum= order of non-linearity

$$\sum_{k=-N}^{k=+N} k_j = k_{-N} + \ldots + k_{-1} + k_1 + \ldots + k_N = m$$

Multinomial Coefficient

For a fixed vector $\vec{k}_o$, how many different sum vectors are there?

m frequencies can be summed $m!$ different ways, but order is immaterial.

Each coefficient k_j can be ordered $k_j!$ ways. Therefore, we have:

$$(m;k) = \frac{m!}{(k_{-N})!\cdots(k_{-1})!(k_{+1})!\cdots(k_N)!}$$

Multinomial coefficient

Game of Cards (example)

- 3 Cards: 3! or six ways to order cards

❶
$R_1 \; R_2 \; B_1$
$R_2 \; R_1 \; B_1$

❷
$R_1 \; B_1 \; R_2$
$R_2 \; B_1 \; R_1$

❸
$B_1 \; R_1 \; R_2$
$B_1 \; R_2 \; R_1$

$$\frac{3!}{1!\,2!} = 3$$ ways to order

Since $R_1 = R_2$,

Reds not distinguished

Making Conjugate Pairs

Usually, we only care about a particular frequency mix generated by certain order non-linearity

Since our signal is <u>real</u>, each term has a complex conjugate. Hence, there is another:

$$\vec{k} = \underbrace{\left(k_N, \ldots, k_1, k_{-1}, \ldots, k_{-N}\right)}_{\text{reverse order}}$$

Taking the complex conjugates in pairs:

$$2\,\mathrm{Re}\left(e^{j(\omega_{k1}+\omega_{k2}+\ldots+\omega_{km})t}\right) = 2\cos\left(\omega_{k_1} + \omega_{k_2} + \ldots + \omega_{k_m}\right)t$$

Amplitude of Mix

Thus the amplitude of any particular frequency component is:

$$\frac{2\times\left(m;\vec{k}\right)}{2^m}=\frac{\left(m;\vec{k}\right)}{2^{m-1}}$$

<u>Ex</u>: IM_3 product generated by the cubic term

$$\begin{matrix} & & -2 & -1 & 1 & 0 \end{matrix}$$

IM_3: $\left(2\omega_1-\omega_2\right)\Rightarrow\vec{k}=\begin{pmatrix}1 & 0 & 2 & 0\end{pmatrix}$

$m=3$

$$\left(m;\vec{k}\right)=\frac{3!}{1!\,2!}=3 \qquad 2^{m-1}=2^2=4$$

$N=2$

Amplitude of IM_3 relative to fundamental:

$$IM_3=\frac{3}{4}\frac{a_3 s_i^{\,3}}{a_1 s_i}=\frac{3}{4}\frac{a_3}{a_1}s_i^{\,2}$$

Gain Compression/Expansion

- How much gain compression occurs due to cubic and pentic (x^5) terms?

<u>cubic</u>: m=3, N=1 $\omega_1 + \omega_1 - \omega_1 = \omega_1$

$\vec{k} = (1, 2)$ — $1 \to -\omega_1$ appear anywhere; $2 \to +\omega_1$ This to appear twice anywhere

$$\frac{(m;k)}{2^{m-1}} = \frac{\frac{3!}{2!}}{2^2} = \frac{3}{4}$$

amp. of fund: $a_1 s_i + \frac{3}{4} a_3 s_i^3$

App. Gain: $a_i + \frac{3}{4} a_3 s_i^2$ Gain depends on signal amplitude

<u>pentic</u>: m=5, N=1 $\not\omega_1 - \not\omega_1 + \not\omega_1 - \not\omega_1 + \omega_1 = \omega_1$

$$\vec{k} = \begin{pmatrix} 2 & 3 \end{pmatrix} \quad (\text{-1},\ 1)$$

$$(m;\vec{k}) = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2} = 10 \qquad 2^{m-1} = 2^4 = 16$$

App. Gain: $= a_1 + \frac{3}{4} a_3 s_i^2 + \frac{10}{16} a_5 s_i^4$

Who wins? Pentic or Cubic?

$$R = \frac{\text{Gain Reduction due to Cubic}}{\text{Gain Reduction due to Pentic}} = \frac{\frac{3}{4} a_3 s_i^2}{\frac{5}{8} a_5 s_i^4}$$

$$R = \frac{1}{s_i^2} \frac{a_3}{a_5} \frac{6}{5}$$

Take an exponential transfer function and consider gain compression: $I_C = I_S e^{V_{BE}/V_T}$ $\quad V_{BE} = V_Q + v_s$

$$I_C = \underbrace{I_S e^{V_Q/V_T}} e^{v_s/V_T}$$

$$= I_Q\left(1 + \left(\frac{v_s}{V_T}\right) + \frac{\left(\frac{v_s}{V_T}\right)^2}{2!} + \ldots\right)$$

Compression for Exp (BJT)

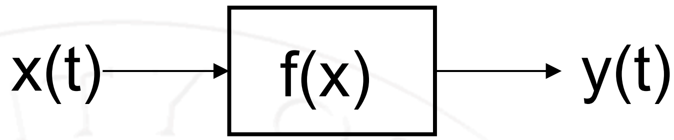
$$a_3 = \frac{\left(\frac{1}{V_T}\right)^3}{3!} \qquad a_5 = \frac{\left(\frac{1}{V_T}\right)^5}{5!}$$

$$R = \frac{1}{s_i^2}\frac{a_3}{a_5}\frac{6}{5} = \frac{1}{s_i^2}\left(\frac{1}{V_T}\right)^3 \frac{1}{3!}\frac{5!}{\left(\frac{1}{V_T}\right)^5}\frac{6}{5} = \frac{\frac{1}{s_i^2}}{\left(\frac{1}{V_T}\right)^2}\frac{5\cdot 4\cdot 6}{5} = 24\left(\frac{V_T}{s_i}\right)^2$$

- When R=1, pentic non-linearity contributes equally to gain compression…

$$\text{R=1} \quad \Rightarrow \quad s_i = V_T\sqrt{24} \approx \underline{127\text{mV}}$$

Summary of Distortion



$$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

- Due to non-linearity, y(t) has frequency components not present in input. For sinusoidal excitation by *N* tones, we *M* tones in output:

$$M = \frac{(2N + m - 1)!}{m!\,(2N - 1)!}$$

<u>m:</u> Order of highest term in non-linearity (Taylor exp.)

Amplitude of Frequency Mix

Particular frequency mix $\vec{k}$ has frequency

$$f_{\vec{k}} = \sum_{\substack{j=-N \\ j\neq 0}}^{N} k_j f_j = (k_1 - k_{-1})f_1 + \ldots + (k_N - k_{-N})f_N$$

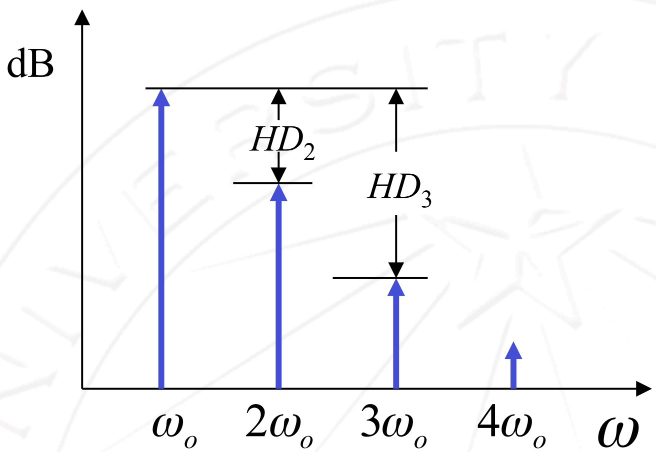
$$\sum_{j=-N}^{+N} k_j = k_{-N} + \ldots + k_{-1} + k_1 + \ldots + k_N = m$$

$$\theta_{\vec{k}} = \sum \theta_j k_j = (k_1 - k_{-1})\theta_1 + \ldots + (k_N - k_{-N})\theta_N$$

The amplitude of any particular frequency mix

$$\underbrace{\left(\frac{(m;\vec{k})}{2^{m-1}} |A_1|^{k_1 + k_{-1}} \ldots |A_N|^{k_N + k_{-N}} \right)}_{\text{amplitude}} \times \cos(\omega_k t + \theta_k)$$

Harmonic Distortion



- For an input frequency ω_j, each order non-linearity (power) produces a jth order harmonic in output

$$HD_2 = \frac{1}{2}\frac{a_2}{a_1}s_i = \frac{1}{2}\frac{a_2}{a_1^2}s_o$$

$HD_2 \propto$ Signal amplitude

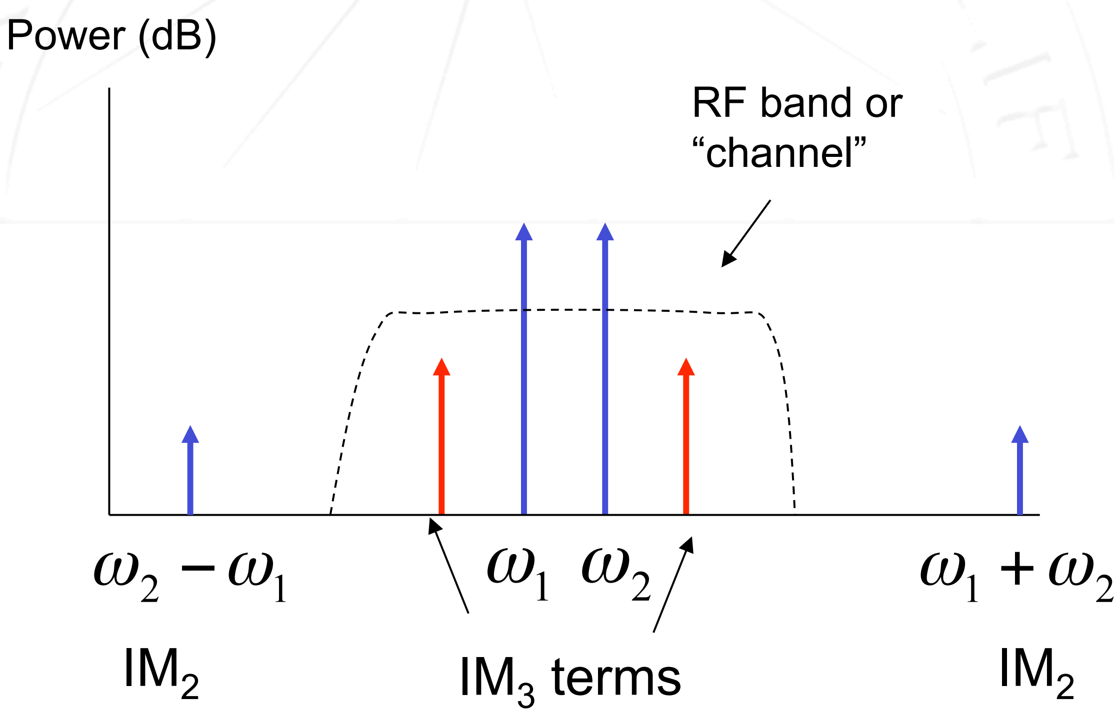
$$HD_3 = \frac{1}{4}\frac{a_3}{a_1}s_i^2 = \frac{1}{4}\frac{a_3}{a_1^3}s_o^2$$

$HD_3 \propto$ (Signal amplitude)^2

2 dB increase for 1 dB signal increase

Intermodulation

- For a two-tone input to a memoryless non-linearity, output contains $(2\omega_1 - \omega_2)$ & $(2\omega_2 - \omega_1)$ due to cubic power and $(\omega_2 - \omega_1)$ & $(\omega_2 + \omega_1)$ due to second order power.

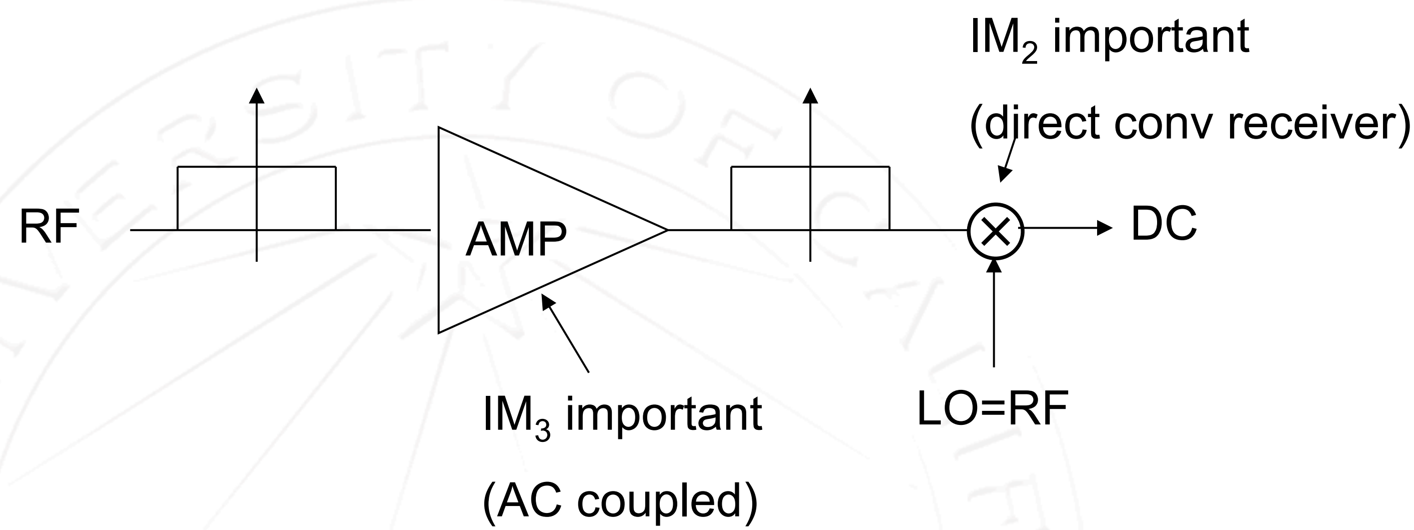


$$\omega_1 \approx \omega_2 \approx \omega_o$$

$$IM_2 \approx DC, 2\omega_o$$

$$IM_3 \approx \omega_o$$

Filtering Intermodulation



- IM_2 products fall at much lower (DC) and higher frequencies ($2\omega_o$). These signals appear as interference to others, but can be attenuated by filtering
- IM_3 products cannot be filtered for close tones.
- In a direct conversion receiver, IM2 is important due to DC.

IM/Harmonic Relations

$$IM_2 = \frac{a_2}{a_1} s_i = \frac{a_2}{a_1^2} s_o = HD_2(dB) + 6dB$$

$$IM_2 = 2HD_2$$

$IM_2 \propto$ Signal level

$$IM_3 = \frac{3}{4}\frac{a_3}{a_1} s_i^2 = \frac{3}{4}\frac{a_3}{a_1^3} s_o^2 = HD_3(dB) + 9.5dB$$

$$IM_3 = 3HD_3$$

$IM_3 \propto$ (Signal level)^2

Triple Beat

- Triple Beat: Apply three sine waves and observe effect of cubic non-linearity

$$\omega_1 + \omega_2 + \omega_3$$
$$\omega_1 - \omega_2 + \omega_3$$
$$\omega_1 - \omega_2 - \omega_3$$
$$\omega_1 + \omega_2 - \omega_3$$

-3 -2 -1 1 2 3

$$\vec{k}_{TB} = (0\;0\;0\;1\;1\;1)$$

1 1 1

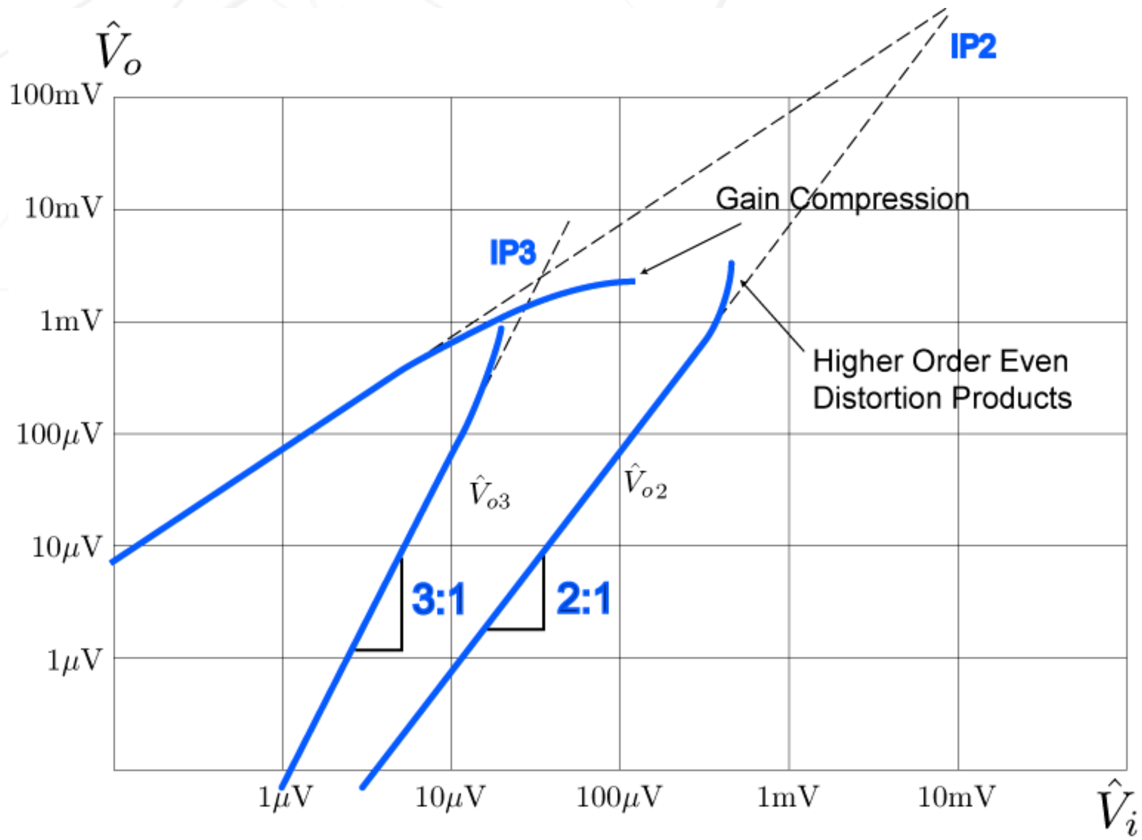
1 1 1

$$a_{k_p} = \frac{(3;\vec{k}_{TB})}{2^{3-1}} s_i^3 = \frac{3!\,s_i^3}{4} = \frac{3}{2} s_i^3$$

$$TB = \frac{3}{2}\frac{a_3}{a_1}\frac{s_i^3}{s_i} = \frac{3}{2}\frac{a_3}{a_1}s_i^2$$

Intercept Point

- <u>Intercept Point</u>: Apply a two tone input and plot output power and IM powers. The intercept point in the *extrapolated* signal power level which causes the distortion power to equal the fundamental power.



Intercept/IM Calculations

- Say an amplifier has an IIP3 = -10 dBm. What is the amplifier signal/distortion (IM3) ratio if we drive it with -25 dBm?
  - Note: IM3 = 0 dB at Pin = -10 dBm
  - If we back-off by 15 dB, the IM3 improves at a rate of 2:1
  - For Pin = -25 dBm (15 dB back-off), we have therefore IM3 = 30dBc

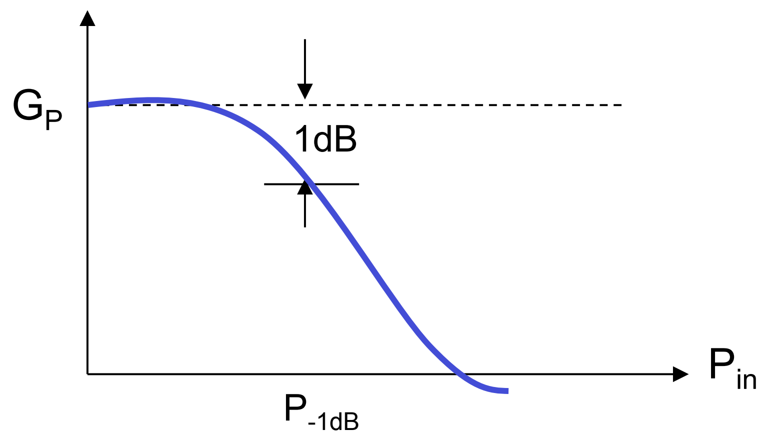
$$IM_3 = \frac{3}{4}\frac{a_3}{a_1}V_{IIP3}^{\ \ 2} = 1 \longrightarrow V_{IIP3} = \sqrt{\frac{4}{3}\left|\frac{a_1}{a_3}\right|}$$

intercept signal level

Gain Compression and Expansion

- To regenerate the fundamental for the N'th power, we need to sum k positive frequencies with k-1 negative frequencies, so $N = 2k$-1 → N must be an odd power

$$\underbrace{\omega_1 + \omega_1 + \ldots + \omega_1}_{k} \underbrace{- \omega_1 - \omega_1 - \ldots - \omega_1}_{k-1} = \omega_1$$



P1dB Compression Point

- An important specification for an amplifier is the 1dB compression point, or the input power required to lower the gain by 1dB

$$G(s_i) = a_1 + \frac{3}{4} a_3 s_i^{\,2} = a_1 \left(1 + \frac{3}{4} \frac{a_3}{a_1} s_i^2 \right)$$

$$G_0 = \lim_{s_i \to 0} G(s_i) = a_1 \qquad \frac{G_0 \cdot 10^{-0.05}}{G_0} = \left(1 + \frac{3}{4} \frac{a_3}{a_1} s_i^2 \right) = 10^{-0.05}$$

$$s_i^2 = \left(1 - 10^{-0.05}\right) \frac{4}{3} \left| \frac{a_1}{a_3} \right|$$

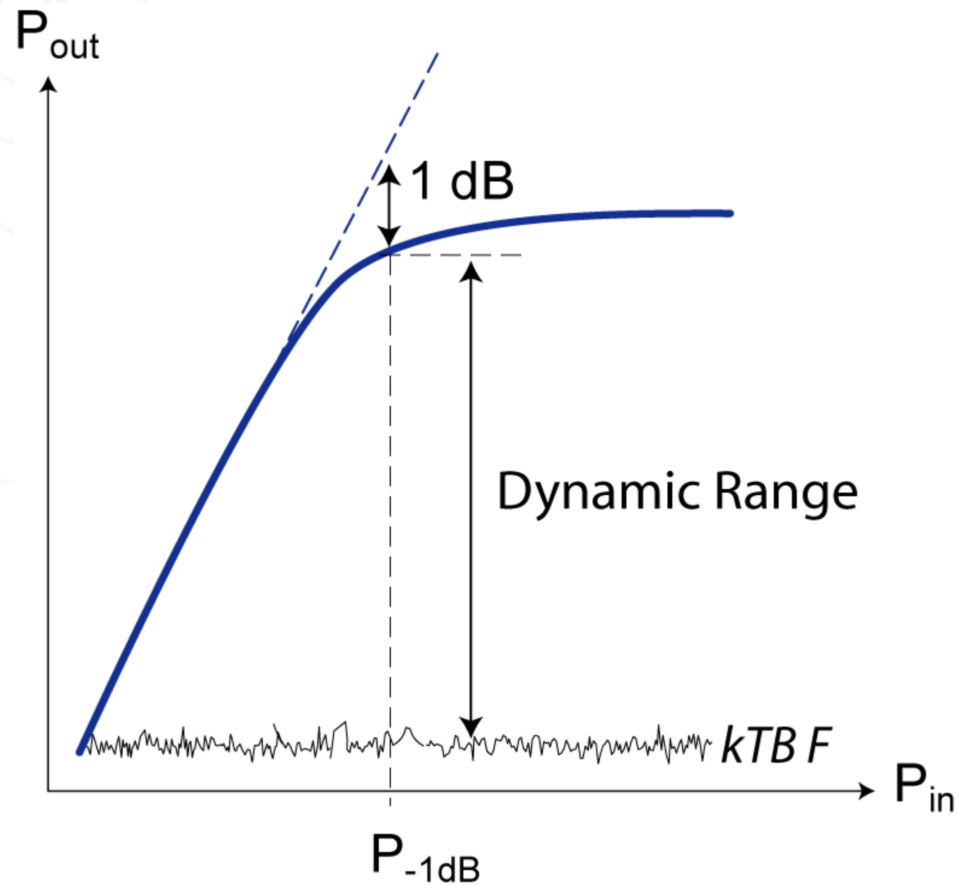
Assume $a_3/a_1 < 0$

$$s_i = \sqrt{\left(1 - 10^{-0.05}\right) \frac{4}{3} \left| \frac{a_1}{a_3} \right|} = \sqrt{\left(1 - 10^{-0.05}\right)} V_{IIP3}$$

About 9.6dB lower than IIP3

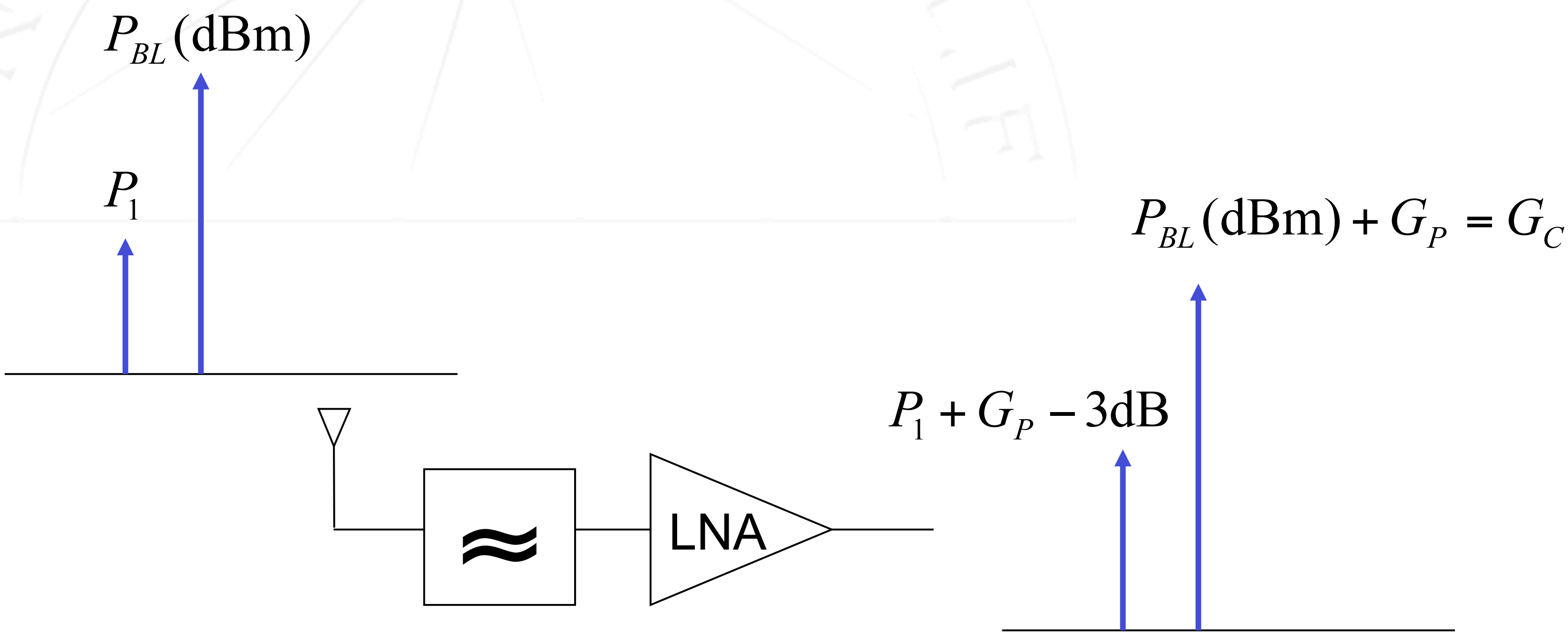
Dynamic Range

- P_{-1dB} is a convenient "maximum" signal level which sets the upper bound on the amplifier "linear" regime. Note that at this power, the IM3 ~ 20 dBc.
- The lower bound is set by the amplifier noise figure.



Blocking (or Jamming)

- Blocker: Any large interfering signal
- P_{BL} = Blocking level. Interfering signal level in dBm which causes a +3dB drop in gain for small desired signal



Jamming Analysis

Let: $s_i = s_1 \cos\omega_1 t + S_2 \cos\omega_2 t$

- $s_1 \cos\omega_1 t$: small desired signal
- $S_2 \cos\omega_2 t$: Large blocker

Cubic non-linearity at ω_1 $\qquad s_1^{k_1+k_{-1}} \quad S_2^{k_2+k_{-2}}$

$k_{-1} + k_1 = 1$

$k_{-2} + k_2 = 2$

$\omega_1 + \omega_2 - \omega_2 \rightarrow \omega_1$

$s_1^3 S_2^0$	Regular gain compression
$s_1^2 S_2$	Gain compression of desired signal on blocker
$s_1 S_2^2$	Gain compression of blocker on desired signal
$s_1^0 S_2^3$	Gain compression of blocker on blocker

Jamming Analysis (cont)

Count the ways: $\left(m;\vec{k}_B\right)$

$$Amplitude = \frac{s_1 S_2{}^2 \left(3;\vec{k}_B\right)}{2^{m-1}} = \frac{3}{2} s_1 S_2{}^2$$

$$\begin{matrix} -2 & -1 & +1 & +2 \end{matrix}$$

$$\vec{k}_B = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\text{Apparent gain} = a_1 \left(1 + \frac{3}{2} \frac{a_3}{a_1} S_2{}^2 \right)$$

gain w/o blocker

gain reduction or expansion due to blocker $\propto S_{BLOCKER}{}^2$

Blocking Power ~ P_{1dB}

$$20\log\left(1+\frac{3}{2}\frac{a_3}{a_1}S_2^{\ 2}\right) = -3\text{dB}$$

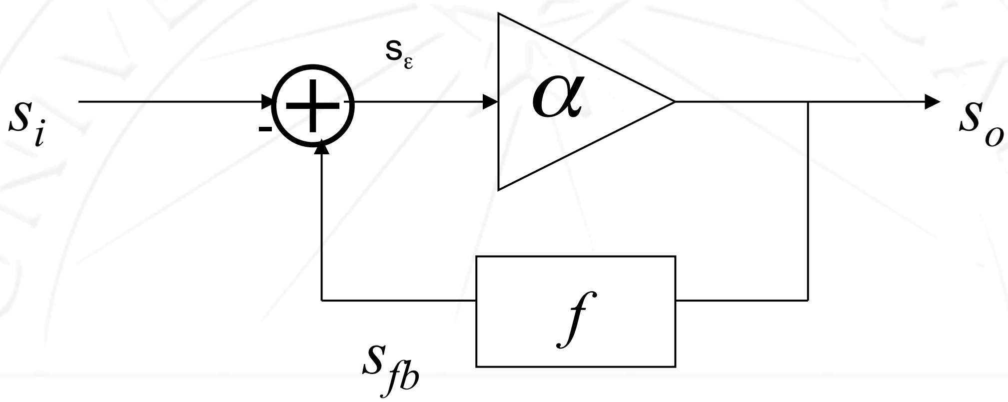
$$S_2 = \underbrace{\sqrt{\frac{4}{3}\left|\frac{a_1}{a_3}\right|}}_{V_{IIP3}} = \frac{\sqrt{0.293}}{\sqrt{2}}$$

$$P_{BL} = P_{IIP3} - 8.3\text{dB}$$

$$= P_{-1dB} + 1\text{dB}$$

Effect of Feedback on Disto

- Review from 142:



$$s_\varepsilon = s_i - s_{fb}$$

$$s_{fb} = f\, s_o$$

$$f < 1$$

$$s_o = a_1 s_\varepsilon + a_2 s_\varepsilon^{\;2} + a_3 s_\varepsilon^{\;3} + \cdots$$

$$= a_1(s_i - f\, s_o) + a_2(s_i - f\, s_o)^2 + a_3(s_i - f\, s_o)^3 + \cdots$$

$$= b_1 s_i + b_2 s_i^{\;2} + b_3 s_i^{\;3} + \cdots$$

New Non-Linear Coefficients

$$b_1 = \frac{a_1}{1 + \underbrace{a_1 f}_{\text{Loop gain } T}}$$

$$b_1 \cong \frac{1}{f} \quad (T \to \infty)$$

$$b_2 = \frac{a_2}{(1+T)^3}$$

For high loop gain, the distortion is very small. Even though the gain drops, the distortion drops with loop gain since b2 drops with a higher power.

$$b_3 = \frac{a_3(1+T) - 2a_2^{\,2} f}{(1+T)^5}$$

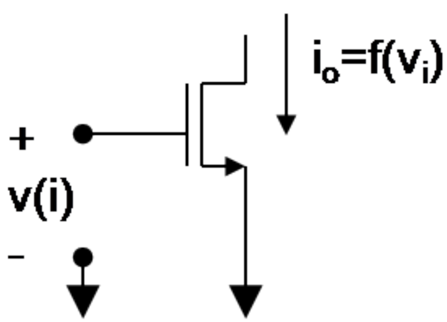
The cubic term has two components, the original cubic and a second order interaction term. If an amplifier does not have cubic, FB creates it (MOS with R_s)

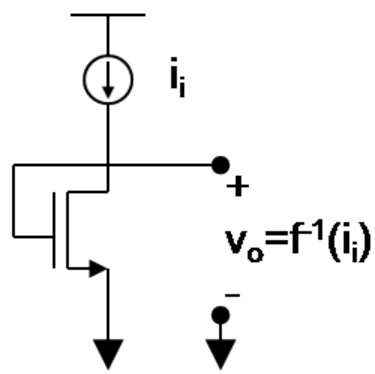
Series Inversion

$$s_a = a_1 s_b + a_2 s_b^{\ 2} + a_3 s_b^{\ 3} + \ldots \qquad s_b = b_1 s_a + b_2 s_a^{\ 2} + b_3 s_a^{\ 3} + \ldots$$

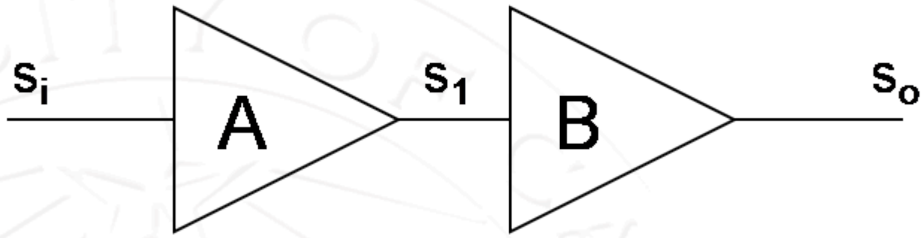
$$s_a = a_1(b_1 s_a + b_2 s_a^{\ 2} + \ldots) + a_2(b_1 s_a + b_2 s_a^{\ 2} + \ldots)^2 + \ldots$$

$$b_1 = \frac{1}{a_1} \qquad b_2 = -\frac{a_2}{a_1^{\ 3}} \qquad b_3 = \frac{2a_2^{\ 2}}{a_1^{\ 5}} - \frac{a_3}{a_1^{\ 4}}$$





Series Cascade



$$s_1 = a_1 s_i + a_2 s_i^2 + \ldots \qquad c_1 = a_1 b_1$$

$$s_0 = b_1 s_1 + b_2 s_1^2 + \ldots$$

$$\Rightarrow s_0 = c_1 s_i + c_2 s_i^2 + \ldots \qquad c_2 = b_1 a_2 + b_2 a_1^2$$

$$c_3 = b_1 a_1^3 + 2 b_2 a_1 a_2 + b_3 a_1^3$$

Second-order
interaction

IIP2 Cascade

- The cascade IIP2 is reduced due to the gain of the first stage:

$$\frac{c_2}{c_1} = \frac{b_1 a_2 + b_2 a_1^2}{b_1 a_1} = \frac{a_2}{a_1} + \frac{b_2}{b_1} a_1$$

$$\frac{1}{IIP2} = \frac{1}{IIP2^A} + \frac{a_1}{IIP2^B}$$

- To calculate the overall IIP2, simply input refer the second stage IIP2 by the *voltage gain* of the first stage.
- The overall IIP2 is a parallel combination of the first and second stage.

IIP3 Cascade

- Using the same approach, we can calculate the IIP3 of a cascade. To simplify the result, neglect the effect of second order interaction:

$$\frac{c_3}{c_1} = \frac{b_1 a_3 + b_2 a_1 a_2 2 + b_3 a_1^3}{b_a a_1} = \left(\frac{a_3}{a_1} + \frac{b_3}{b_1} a_1^2 + \frac{b_2}{b_1} 2 a_2\right)$$

$$\frac{1}{IIP3^2} = \frac{1}{IIP3_A^2} + \frac{a_1^2}{IIP3_B^2}$$

- Input refer the IIP3 of the second stage by the *power gain* of the first stage.

References

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- **Sinusoidal analysis and modeling of weakly nonlinear circuits : with application to nonlinear interference effects**, Donald D. Weiner, John F. Spina. New York : Van Nostrand Reinhold, c1980.