Problem Set 4
Solutions

Please note that these are merely suggested solutions. Many of these problems can be approached in different ways.

1. (a) Notice that the first op amp is simply a noninverting amplifier with input $v_{in}$ and the second is an inverting amplifier with the same input. Denoting the respective outputs of the op amps by $v_{o1}$ and $v_{o2}$, we immediately infer that $v_{o1} = \frac{R_1 + R_2}{R_1} v_{in}$ and $v_{o2} = -\frac{R_4}{R_3} v_{in}$. If you do not see this, you can also write nodal equations at the noninverting inputs of the op amps after applying the golden rules. Now the output $v_o$ is given by the difference of the above two voltages:

$$v_o = v_{o1} - v_{o2} = \left( \frac{R_1 + R_2}{R_1} + \frac{R_4}{R_3} \right) v_{in}$$

(b) Since the supply voltages are $\pm V_s$, the maximum output that can be achieved across $R_L$ is when $v_{o1} = V_s$ and $v_{o2} = -V_s$, giving us a total output of $2V_s$. This can be potentially useful because it allows us to achieve a greater gain than is normally allowed by our supply voltages.

2. While the circuit looks intimidating, we can make massive simplifications by applying the golden rules to each op amp. We are left with only two unknown voltages as shown below:
Firstly, the first two op amps have their inverting inputs at a virtual ground due to the connection to ground at their noninverting inputs. Notice that the two 40 kΩ resistors don’t induce any voltage drop between the first pair of op amps and the second, as the currents into the op amp inputs are always 0. Because both of the middle op amps are buffers, $v_x$ and $v_y$ get carried over to their outputs. Finally, there is no voltage drop across the bottom 0.5 kΩ resistor due to lack of current (note this is NOT true for the one on the top).

With all this in mind, we can simply solve for $v_x$ and $v_y$, and this will give us access to $v_o$. Both $v_x$ and $v_y$ are given by inverting amplifiers:

$$v_x = -\frac{8 \text{kΩ}}{2 \text{kΩ}} v_1 = -4v_1$$
$$v_y = -\frac{16 \text{kΩ}}{2 \text{kΩ}} v_1 = -8v_2$$

KCL at the inverting input of the last op amp gives us

$$\frac{v_y - v_x}{0.5 \text{kΩ}} + \frac{v_y - v_o}{4 \text{kΩ}} = 0$$

Plugging in for our quantities previously solved gives us $v_o = 32v_1 - 72v_2$.

3. (a) Since the first pair of op amps are both buffers, their inputs carry over to their respective output nodes. In addition, we can define one unknown node $v_x$ at both inputs of the third op amp, by usage of the golden rules. KCL at the bottom noninverting input node gives us $v_x$ due to voltage divider:

$$v_x = \frac{R_3}{R_2 + R_3} v_2$$

KCL at the top node gives us

$$\frac{v_x - v_1}{R_2} + \frac{v_x - v_o}{R_3} = 0$$

Plugging in for $v_x$ and solving for $v_o$ gives us $v_o = \frac{R_3}{R_2} (v_2 - v_1)$.

(b) The situation here is a bit more complex, as we do not know explicitly what the outputs of the first pair of op amps are. Call them $v_{o1}$ and $v_{o2}$. If we write nodal equations at the noninverting inputs of these op amps, we get two plus the two from the last part, giving us four equations.

$$\frac{v_1 - v_{o1}}{R_1} + \frac{v_1 - v_2}{R_{\text{gain}}} = 0$$
$$\frac{v_2 - v_{o2}}{R_1} + \frac{v_2 - v_1}{R_{\text{gain}}} = 0$$
$$v_x = \frac{R_3}{R_2 + R_3} v_{o2}$$
\[
\frac{v_x - v_{o1}}{R_2} + \frac{v_x - v_o}{R_3} = 0
\]

Plug the third into the fourth equation to get rid of \(v_x\). Then solve the first two equations for \(v_{o1}\) and \(v_{o2}\) and plug them in to obtain \(v_o\) in terms of \(v_1\) and \(v_2\).

\[
v_o = \frac{R_2 + R_3}{R_2} v_x - \frac{R_3}{R_2} v_{o1} = \frac{R_3}{R_2} (v_{o2} - v_{o1})
\]

\[
v_{o1} = \frac{R_1}{R_{\text{gain}}} (v_1 - v_2) + v_1, \quad v_{o2} = \frac{R_1}{R_{\text{gain}}} (v_2 - v_1) + v_2
\]

\[
v_o = \left(1 + \frac{2R_1}{R_{\text{gain}}} \right) \frac{R_3}{R_2} (v_2 - v_1)
\]

4. (a) The circuit redrawn with the transconductance amplifier model looks like the following (notice the placement of each of the components):

(b) The output current \(i_o\) is simply given by the current divider of the current \(Gv_i\):

\[
i_o = \frac{R_{\text{out}}}{R_{\text{out}} + R_L} Gv_i = \frac{10 \, \text{k}\Omega}{10 \, \text{k}\Omega + 2 \, \text{k}\Omega} 10000v_i = 8333v_i
\]

So \(G_{\text{msc}} = \frac{i_o}{v_i} = 8333\).

(c) To calculate the output, we simply need to relate the input current \(i_s\) to \(v_i\):

\[
i_o = 8333v_i = 8333(R_s||R_{\text{in}})i_s = 8333(5 \, \text{k}\Omega||100 \, \text{\Omega})(1 \, \mu\text{A}) = 0.817 \, \text{A}
\]

5. (a) The order of the amplifiers will matter due to the nonidealities of each and the loading effects that they cause. For order AB, our circuit can be redrawn:
The power input is given by the power through the input source $v_s$:

$$P_i = v_s i_s = (1V) \left( \frac{1V}{1k\Omega + 1k\Omega} \right) = 0.5 \text{ mW}$$

The power output can be found by using the current going through $R_L$:

$$P_o = i_L^2 R_L = \left( \frac{1 \text{ M}\Omega}{1 \text{ M}\Omega + 100 \Omega} \right)^2 (100 \Omega) = 9998 \left( \frac{8v_{in}}{1 \Omega + 10 \Omega} \right)^2 = 5288(0.5 \text{ V})^2 = 1322 \text{ W}$$

Now consider the order BA:

$$P_i = v_s i_s = (1V) \left( \frac{1V}{1k\Omega + 10 \Omega} \right) = 0.99 \text{ mW}$$

$$P_o = \frac{v_L^2}{R_L} = \left( \frac{100 \Omega}{1 \Omega + 100 \Omega} \frac{8v_{in}}{1 \Omega + 10 \Omega} \right)^2 / (100 \Omega) = 0.627[10i_{in}(1 \text{ M}\Omega||1k\Omega)]^2$$

$$= 6.26 \cdot 10^7 \left( \frac{1V}{1k\Omega + 10 \Omega} \right)^2 = 61.3 \text{ W}$$

Clearly, the power gain of AB is much greater, with $2.65 \cdot 10^6$ as opposed to 62000 of BA. Without the amplifiers, the power gain is smaller than 1:

$$G_P = \frac{i^2 R_L}{i v_s} = \frac{i R_L}{v_s} = \frac{v_s}{R_s + R_L} \frac{R_L}{v_s} = \frac{100}{100 + 1000} = 0.099$$

(b) Consider the power output in the absence of the second stage (replace the 10 \Omega resistor with a 100 \Omega).

$$P_o = i_L^2 R_L = \left( \frac{8v_{in}}{1 \Omega + 100 \Omega} \right)^2 (100 \Omega) = 0.627(0.5 \text{ V})^2 = 0.157 \text{ W}$$

Clearly, the second stage delivers a lot more power to the load.

(c) The voltage gain of the first stage is $\frac{1k\Omega}{1k\Omega+1k\Omega} \cdot \frac{A_{v\Omega}}{10 \Omega + 10 \Omega} = 3.64$. The gain of the cascade amplifiers is $\frac{1k\Omega}{1k\Omega+1k\Omega} \cdot \frac{A_{v\Omega}}{10 \Omega + 10 \Omega} \times A_1M\Omega||100\Omega = 363.6$. Since the cascade voltage gain is larger than the gain of the first state, the second stage will clip first when the input voltage is $\frac{\pm 10 \text{ V}}{363.6} = \pm 27.5 \text{ mV}$. 
6. (a) For the sequence BAC, we first have a noninverting amplifier, followed by an inverting amplifier, and finally a voltage follower. The gain is thus

\[ G = \left(1 + \frac{R_4}{R_3}\right) \left(-\frac{R_2}{R_1}\right) = \left(1 + \frac{8k}{4k}\right) \left(-\frac{6k}{2k}\right) = -9 \]

The load resistance seen by the voltage input is infinite, since no current flows into the noninverting input of op amp B.

(b) The gain here is the same, since it is order-independent (multiplication is commutative). However, the input resistance is not 0 in this case; instead, it is \(R_1 = 2\, \text{k\Omega}\), since the source sees the inverting amplifier first. So in terms of overall gain, both orderings are equally good. However, the former is better for voltage amplification, as its input resistance is much higher.

(c) The voltage gain remains the same in both cases, since the third stage does not affect it. It is usually useful to prevent loading effects at either the input or output of an op amp, especially if it is non-ideal (although it doesn’t play any role here since all the op amps are ideal).

7. On the left side, we first combine the 10 \(\mu\text{F}\) with the 5 \(\mu\text{F}\) in series, followed by the 9 \(\mu\text{F}\) in parallel:

\[ C_{eq1} = (10 \, \mu\text{F}) || (5 \, \mu\text{F}) + 9 \, \mu\text{F} = 12.3 \, \mu\text{F} \]

On the right side, the 9 \(\mu\text{F}\) is in parallel with the 7 \(\mu\text{F}\), followed by a series connection with the 8 \(\mu\text{F}\).

\[ C_{eq2} = (9 \, \mu\text{F} + 7 \, \mu\text{F}) || (8 \, \mu\text{F}) = 5.33 \, \mu\text{F} \]

These two equivalent capacitances are in parallel with each other, as seen by \(a\) and \(b\).

\[ C_{eq} = C_{eq1} + C_{eq2} = 17.7 \, \mu\text{F} \]

8. Note that before \(t = 0\), all the current shorts through the middle branch, since there is no charge nor voltage on the capacitor. Once the switch opens, the capacitor must sustain a constant 20 mA current to satisfy KCL.

\[ v(t = 50 \, \text{ms}) = \frac{1}{C} \int_0^{50 \, \text{ms}} i(t) \, dt = \frac{1}{50 \, \mu\text{F}} \int_0^{50 \, \text{ms}} 20 \, \text{mA} \, dt = \frac{(20 \, \text{mA})(50 \, \text{ms})}{50 \, \mu\text{F}} = 20 \, \text{V} \]

\[ p(t = 50 \, \text{ms}) = v(t = 50 \, \text{ms})i(t = 50 \, \text{ms}) = (20 \, \text{V})(20 \, \text{mA}) = 0.4 \, \text{W} \]

\[ w(t = 50 \, \text{ms}) = \frac{1}{2} C v^2(t = 50 \, \text{ms}) = \frac{1}{2} (50 \, \mu\text{F})(20 \, \text{V})^2 = 0.01 \, \text{J} \]

9. The current is given by \(i(t) = C \frac{dv(t)}{dt}\). Hence it is sufficient to differentiate the given graph of voltage in segments. We thus have \(i(t) = 50 \, \mu\text{A}\) for \(0 < t < 1\), \(i(t) = 0\) for \(1 < t < 4\), \(i(t) = -37.5 \, \mu\text{A}\) for \(4 < t < 6\), \(i(t) = 0\) for \(6 < t < 9\), and \(i(t) = 25 \, \mu\text{A}\) for \(9 < t < 10\). Note that all times listed are in \(\mu\text{s}\).

The net charge and energy are both 0, since both are proportional to voltage, which is itself 0 at the end time \(t = 10 \, \mu\text{s}\). (We are only concerned with net charge and energy, which represents the total transferred.)
10. Since both capacitors are initially uncharged, conservation demands that the net charge at each node remain 0 after the voltage source is connected. We apply this principle to the node between the two capacitors. Since we are looking at the negative plate of cap 1 and the positive plate of cap 2, we must have that

\[ -q_1 + q_2 = 0 \]

Remember that the charge on parallel-plate capacitors are equal and opposite. Another relation we can construct using KVL is the following:

\[ 15 \, \text{V} = v_1 + v_2 = \frac{q_1}{C_1} + \frac{q_2}{C_2} = \frac{q_1}{12 \, \mu\text{F}} + \frac{q_2}{28 \, \mu\text{F}} \]

The above two equations give us a system that we can solve. The solutions are \( q_1 = q_2 = 0.126 \, \text{mC} \). With this we find that the voltages are \( v_1 = 10.5 \, \text{V} \) and \( v_2 = 4.5 \, \text{V} \).