Two-Ports and Power Gain

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Power Gain
We can define power gain in many different ways. The power gain \( G_p \) is defined as follows

\[
G_p = \frac{P_L}{P_{in}} = f(Y_L, Y_{ij}) \neq f(Y_S)
\]

We note that this power gain is a function of the load admittance \( Y_L \) and the two-port parameters \( Y_{ij} \).
The *available power gain* is defined as follows

\[ G_a = \frac{P_{av,L}}{P_{av,S}} = f(Y_S, Y_{ij}) \neq f(Y_L) \]

The available power from the two-port is denoted \( P_{av,L} \) whereas the power available from the source is \( P_{av,S} \).

Finally, the *transducer gain* is defined by

\[ G_T = \frac{P_L}{P_{av,S}} = f(Y_L, Y_S, Y_{ij}) \]

This is a measure of the efficacy of the two-port as it compares the power at the load to a simple conjugate match.
The power gain is readily calculated from the input admittance and voltage gain

\[ P_{in} = \frac{|V_1|^2}{2} \Re(Y_{in}) \]

\[ P_L = \frac{|V_2|^2}{2} \Re(Y_L) \]

\[ G_p = \left| \frac{V_2}{V_1} \right|^2 \frac{\Re(Y_L)}{\Re(Y_{in})} \]

\[ G_p = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \frac{\Re(Y_L)}{\Re(Y_{in})} \]
To derive the available power gain, consider a Norton equivalent for the two-port where (short port 2)

\[ I_{eq} = -I_2 = Y_{21} V_1 = \frac{-Y_{21}}{Y_{11} + Y_S} I_S \]

The Norton equivalent admittance is simply the output admittance of the two-port

\[ Y_{eq} = Y_{22} - \frac{Y_{21} Y_{12}}{Y_{11} + Y_S} \]
The available power at the source and load are given by

\[ P_{av,S} = \frac{|I_S|^2}{8\Re(Y_S)} \]

\[ P_{av,L} = \frac{|I_{eq}|^2}{8\Re(Y_{eq})} \]

\[ G_a = \left| \frac{I_{eq}}{I_S} \right|^2 \frac{\Re(Y_S)}{\Re(Y_{eq})} \]

\[ G_a = \left| \frac{Y_{21}}{Y_{11} + Y_S} \right|^2 \frac{\Re(Y_S)}{\Re(Y_{eq})} \]
The transducer gain is given by

\[ G_T = \frac{P_L}{P_{av,S}} = \frac{\frac{1}{2} \Re(Y_L)|V_2|^2}{\frac{|I_S|^2}{8 \Re(Y_S)}} = 4 \Re(Y_L) \Re(Y_S) \left| \frac{V_2}{I_S} \right|^2 \]

We need to find the output voltage in terms of the source current. Using the voltage gain we have and input admittance we have

\[ \left| \frac{V_2}{V_1} \right| = \left| \frac{Y_{21}}{Y_L + Y_{22}} \right| \]

\[ I_S = V_1(Y_S + Y_{in}) \]

\[ \left| \frac{V_2}{I_S} \right| = \left| \frac{Y_{21}}{Y_L + Y_{22}} \right| \frac{1}{|Y_S + Y_{in}|} \]
|Y_S + Y_{in}| = \left| Y_S + Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} \right|

- We can now express the output voltage as a function of source current as

\[
\left| \frac{V_2}{I_S} \right|^2 = \frac{|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}
\]

- And thus the transducer gain

\[
G_T = \frac{4\mathcal{R}(Y_L)\mathcal{R}(Y_S)|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}
\]
Maximum Power Gain and the Bi-Conjugate Match
It’s interesting to note that all of the gain expression we have derived are in the exact same form for the impedance, hybrid, and inverse hybrid matrices.

In general, \( P_L \leq P_{av,L} \), with equality for a matched load. Thus we can say that \( G_T \leq G_a \).

The maximum transducer gain as a function of the load impedance thus occurs when the load is conjugately matched to the two-port output impedance

\[
G_{T,\text{max},L} = \frac{P_L(Y_L = Y_{out}^*)}{P_{av,S}} = G_a
\]
Likewise, since $P_{in} \leq P_{av,S}$, again with equality when the two-port is conjugately matched to the source, we have

$$G_T \leq G_p$$

The transducer gain is maximized with respect to the source when

$$G_{T,\max,S} = G_T(Y_{in} = Y_S^*) = G_p$$
Bi-Conjugate Match

- When the input and output are simultaneously conjugately matched, or a bi-conjugate match has been established, we find that the transducer gain is maximized with respect to the source and load impedance

\[ G_{T,max} = G_{p,max} = G_{a,max} \]

- This is thus the recipe for calculating the optimal source and load impedance in to maximize gain

\[ Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} = Y_S^* \]

\[ Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}} = Y_L^* \]

- Solution of the above four equations (real/imag) results in the optimal \( Y_{S,opt} \) and \( Y_{L,opt} \).
Another approach is to simply equate the partial derivatives of $G_T$ with respect to the source/load admittance to find the maximum point:

$$\frac{\partial G_T}{\partial G_S} = 0 \quad \frac{\partial G_T}{\partial G_L} = 0$$
$$\frac{\partial G_T}{\partial B_S} = 0 \quad \frac{\partial G_T}{\partial B_L} = 0$$

Again we have four equations. But we should be smarter about this and recall that the maximum gains are all equal. Since $G_a$ and $G_p$ are only a function of the source or load, we can get away with only solving two equations.
Calculation of Optimal Source/Load

- Working with available gain
  \[
  \frac{\partial G_a}{\partial G_S} = 0 \quad \quad \frac{\partial G_a}{\partial B_S} = 0
  \]
- This yields \( Y_{S,opt} \) and by setting \( Y_L = Y_{out}^* \) we can find the \( Y_{L,opt} \).
- Likewise we can also solve
  \[
  \frac{\partial G_p}{\partial G_L} = 0 \quad \quad \frac{\partial G_p}{\partial B_L} = 0
  \]
- And now use \( Y_{S,opt} = Y_{in}^* \).
Let’s outline the procedure for the optimal power gain. We’ll use the power gain $G_p$ and take partials with respect to the load. Let

\[ Y_{jk} = m_{jk} + jn_{jk} \]
\[ Y_L = G_L + jX_L \]
\[ Y_{12}Y_{21} = P + jQ = Le^{j\phi} \]
\[ G_p = \frac{|Y_{21}|^2}{D} G_L \]
\[ \Re \left( Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} \right) = m_{11} - \frac{\Re(Y_{12}Y_{21}(Y_L + Y_{22})^*)}{|Y_L + Y_{22}|^2} \]
\[ D = m_{11}|Y_L + Y_{22}|^2 - P(G_L + m_{22}) - Q(B_L + n_{22}) \]

\[ \frac{\partial G_p}{\partial B_L} = 0 = -\frac{|Y_{21}|^2 G_L}{D^2} \frac{\partial D}{\partial B_L} \]
Solving the above equation we arrive at the following solution

\[ B_{L, opt} = \frac{Q}{2m_{11}} - n_{22} \]

In a similar fashion, solving for the optimal load conductance

\[ G_{L, opt} = \frac{1}{2m_{11}} \sqrt{(2m_{11}m_{22} - P)^2 - L^2} \]

If we substitute these values into the equation for \( G_p \) (lot's of algebra ...), we arrive at

\[ G_{p, max} = \frac{|Y_{21}|^2}{2m_{11}m_{22} - P + \sqrt{(2m_{11}m_{22} - P)^2 - L^2}} \]
Notice that for the solution to exists, $G_L$ must be a real number. In other words

$$(2m_{11}m_{22} - P)^2 > L^2$$

$$(2m_{11}m_{22} - P) > L$$

$$K = \frac{2m_{11}m_{22} - P}{L} > 1$$

This factor $K$ plays an important role as we shall show that it also corresponds to an unconditionally stable two-port. We can recast all of the work up to here in terms of $K$

$$Y_{S,\text{opt}} = -j\Im(Y_{11}) + \frac{Y_{12}Y_{21} - 2\Re(Y_{11})\Re(Y_{22}) + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{22})}$$

$$Y_{L,\text{opt}} = -j\Im(Y_{22}) + \frac{Y_{12}Y_{21} - 2\Re(Y_{11})\Re(Y_{22}) + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{11})}$$

$$G_{p,\text{max}} = G_{T,\text{max}} = G_{a,\text{max}} = \frac{Y_{21}}{Y_{12}} \frac{1}{K + \sqrt{K^2 - 1}}$$
The maximum gain is usually written in the following insightful form

\[ G_{\text{max}} = \frac{Y_{21}}{Y_{12}} (K - \sqrt{K^2 - 1}) \]

For a reciprocal network, such as a passive element, \( Y_{12} = Y_{21} \) and thus the maximum gain is given by the second factor

\[ G_{r,\text{max}} = K - \sqrt{K^2 - 1} \]

Since \( K > 1 \), \( |G_{r,\text{max}}| < 1 \). The reciprocal gain factor is known as the efficiency of the reciprocal network.

The first factor, on the other hand, is a measure of the non-reciprocity.
For a unilateral network, the design for maximum gain is trivial. For a bi-conjugate match

\[ Y_S = Y_{11}^* \]
\[ Y_L = Y_{22}^* \]
\[ G_{T,\text{max}} = \frac{|Y_{21}|^2}{4m_{11}m_{22}} \]
Stability of a Two-Port
Stability of a Two-Port

- A two-port is unstable if the admittance of either port has a negative conductance for a passive termination on the second port. Under such a condition, the two-port can oscillate.
- Consider the input admittance

\[
Y_{in} = G_{in} + jB_{in} = Y_{11} - \frac{Y_{12} Y_{21}}{Y_{22} + Y_L}
\]

- Using the following definitions

\[
Y_{11} = g_{11} + jb_{11} \quad \quad Y_{12} Y_{21} = P + jQ = L \angle \phi
\]

\[
Y_{22} = g_{22} + jb_{22} \quad \quad Y_L = G_L + jB_L
\]

- Now substitute real/imag parts of the above quantities into \( Y_{in} \)

\[
Y_{in} = g_{11} + jb_{11} - \frac{P + jQ}{g_{22} + jb_{22} + G_L + jB_L}
\]

\[
= g_{11} + jb_{11} - \frac{(P + jQ)(g_{22} + G_L - j(b_{22} + B_L))}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2}
\]
Taking the real part, we have the input conductance

\[ \Re(Y_{in}) = G_{in} = g_{11} - \frac{P(g_{22} + G_L) + Q(b_{22} + B_L)}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2} \]

\[ = \frac{(g_{22} + G_L)^2 + (b_{22} + B_L)^2 - \frac{P}{g_{11}}(g_{22} + G_L) - \frac{Q}{g_{11}}(b_{22} + B_L)}{D} \]

Since \( D > 0 \) if \( g_{11} > 0 \), we can focus on the numerator. Note that \( g_{11} > 0 \) is a requirement since otherwise oscillations would occur for a short circuit at port 2.

The numerator can be factored into several positive terms

\[ N = (g_{22} + G_L)^2 + (b_{22} + B_L)^2 - \frac{P}{g_{11}}(g_{22} + G_L) - \frac{Q}{g_{11}}(b_{22} + B_L) \]

\[ = \left( G_L + \left( g_{22} - \frac{P}{2g_{11}} \right) \right)^2 + \left( B_L + \left( b_{22} - \frac{Q}{2g_{11}} \right) \right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} \]
Now note that the numerator can go negative only if the first two terms are smaller than the last term. To minimize the first two terms, choose $G_L = 0$ and $B_L = -\left( b_{22} - \frac{Q}{2g_{11}} \right)$ (reactive load)

$$N_{\text{min}} = \left( g_{22} - \frac{P}{2g_{11}} \right)^2 - \frac{P^2 + Q^2}{4g_{11}^2}$$

And thus the above must remain positive, $N_{\text{min}} > 0$, so

$$\left( g_{22} + \frac{P}{2g_{11}} \right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} > 0$$

$$g_{11}g_{22} > \frac{P + L}{2} = \frac{L}{2}(1 + \cos \phi)$$
Using the above equation, we define the Linvill stability factor

\[ L < 2g_{11}g_{22} - P \]

\[ C = \frac{L}{2g_{11}g_{22} - P} < 1 \]

The two-port is stable if \( 0 < C < 1 \).
It’s more common to use the inverse of $C$ as the stability measure

$$\frac{2g_{11}g_{22} - P}{L} > 1$$

The above definition of stability is perhaps the most common

$$K = \frac{2\Re(Y_{11})\Re(Y_{22}) - \Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} > 1$$

The above expression is identical if we interchange ports 1/2. Thus it’s the general condition for stability.

Note that $K > 1$ is the same condition for the maximum stable gain derived earlier. The connection is now more obvious. If $K < 1$, then the maximum gain is infinity!