Integrated Circuits for Communication
Berkeley

Two-Port Noise Analysis

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Any noisy two port can be replaced with a *noiseless* two-port and equivalent input noise sources

In general, these noise sources are correlated. For now let’s neglect the correlation.
The equivalent sources are found by opening and shorting the input.
If we leave the base of a BJT open, then the total output noise is given by

\[ \overline{i_o^2} = \overline{i_c^2} + \beta^2 \overline{i_b^2} = \overline{i_n^2} \beta^2 \]

or

\[ \overline{i_n^2} = \frac{\overline{i_c^2}}{\beta^2} + \overline{i_b^2} \approx \overline{i_b^2} \]
BJT (cont)

- If we short the input of the BJT, we have

\[
\overline{I_o^2} \approx g_m^2 \overline{V_n^2} \left( \frac{Z_\pi}{Z_\pi + r_b} \right)^2 = \beta^2 \frac{\overline{V_n^2}}{(Z_\pi + r_b)^2}
\]

\[
= \beta^2 \frac{\overline{V_{rb}^2}}{(Z_\pi + r_b)^2} + \overline{I_c^2}
\]

- Solving for the equivalent BJT noise voltage

\[
\overline{V_n^2} = \overline{V_{rb}^2} + \frac{\overline{I_c^2}(Z_\pi + r_b)^2}{\beta^2}
\]

\[
\overline{V_n^2} \approx \overline{V_{rb}^2} + \frac{\overline{I_c^2}Z_\pi^2}{\beta^2}
\]
at low frequencies...

\[ \overline{v_n^2} \approx \overline{v_{rb}^2} + \frac{i_c^2}{g_m^2} \]

\[ \overline{v_n^2} = 4kTB\overline{r_b} + \frac{2qI_C B}{g_m^2} \]

\[ \overline{i_n^2} = \frac{2qI_C}{\beta} \]
If $R_s = 0$, only the voltage noise $\overline{v_n^2}$ is important. Likewise, if $R_s = \infty$, only the current noise $\overline{i_n^2}$ is important.

Amplifier Selection: If $R_s$ is large, then select an amp with low $\overline{i_n^2}$ (MOS). If $R_s$ is low, pick an amp with low $\overline{v_n^2}$ (BJT).

For a given $R_s$, there is an optimal $\overline{v_n^2}/\overline{i_n^2}$ ratio. Alternatively, for a given amp, there is an optimal $R_s$. 

\[ V_s \]
Equivalent Input Noise Voltage

Let’s find the total output noise voltage

\[ \overline{v_o^2} = (\overline{v_n^2} A_v^2 + \overline{v_{R_s}^2} A_v^2) \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 + \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 R_s^2 \overline{i_n^2} A_v^2 \]

\[ = (\overline{v_n^2} + \overline{i_n^2} R_s^2 + \overline{v_{R_s}^2}) \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 A_v^2 \]
We see that all the noise can be represented by a single equivalent source
\[ \overline{v_{eq}^2} = \overline{v_n^2} + \overline{i_n^2 R_s^2} \]

Applying the definition of noise figure
\[ F = 1 + \frac{N_{amp,i}}{N_s} = 1 + \frac{\overline{v_{eq}^2}}{v_s^2} \]
Let $\overline{v}_{n}^{2} = 4kTR_{n}B$ and $\overline{i}_{n}^{2} = 4kTG_{n}B$. Then

$$F = 1 + \frac{R_{n} + G_{n}R_{s}^{2}}{R_{s}} = 1 + G_{n}R_{s} + \frac{R_{n}}{R_{s}}$$

Let’s find the optimum $R_{s}$

$$\frac{dF}{dR_{s}} = G_{n} - \frac{R_{n}}{R_{s}^{2}} = 0$$

We see that the noise figure is minimized for

$$R_{opt} = \sqrt{\frac{R_{n}}{G_{n}}} = \sqrt{\frac{\overline{v}_{n}^{2}}{\overline{i}_{n}^{2}}}$$
The major assumption we made was that $\overline{v_n^2}$ and $\overline{i_n^2}$ are not correlated. The resulting minimum noise figure is thus

$$F_{\text{min}} = 1 + G_n R_s + \frac{R_n}{R_s}$$

$$= 1 + G_n \sqrt{\frac{R_n}{G_n}} + \sqrt{\frac{G_n}{R_n}} R_n$$

$$= 1 + 2 \sqrt{R_n G_n}$$
Consider the difference between $F$ and $F_{\text{min}}$

$$F - F_{\text{min}} = G_n R_s + \frac{R_n}{R_s} - 2 \sqrt{R_n G_n}$$

$$= \frac{R_n}{R_s} (1 + \frac{G_n R_s^2}{R_n} - 2 \frac{R_s}{R_n} \sqrt{R_n G_n})$$

$$= \frac{R_n}{R_s} \left(1 + \left(\frac{R_s}{R_{\text{opt}}}\right)^2 - \frac{2R_s}{R_{\text{opt}}}\right)$$

$$= \frac{R_n}{R_s} \left|\frac{R_s}{R_{\text{opt}}} - 1\right|^2$$

$$= R_n R_s \left|G_{\text{opt}} - G_s\right|^2$$
Sometimes $R_n$ is called the noise sensitivity parameter since

$$F = F_{min} + R_n R_s |G_{opt} - G_s|^2$$

This is clear since the rate of deviation from optimal noise figure is determined by $R_n$. If a two-port has a small value of $R_n$, then we can be sloppy and sacrifice the noise match for gain. If $R_n$ is large, though, we have to pay careful attention to the noise match.

Most software packages (Spectre, ADS) will plot $Y_{opt}$ and $F_{min}$ as a function of frequency, allowing the designer to choose the right match for a given bias point.
We found the equivalent noise generators for a BJT

\[
\overline{v_n^2} = \overline{v_{rb}^2} + \frac{\overline{i_c^2}}{g_m^2} = 4kTBr_b + \frac{2qIcB}{g_m^2} \quad \overline{i_n^2} = \overline{i_b^2}
\]

The noise figure is

\[
F = 1 + \frac{4kTr_b + \frac{2qIc}{g_m^2}}{4kTR_s} + \frac{2qIcR_s^2}{\beta 4kTR_s} = 1 + \frac{r_b}{R_s} + \frac{1}{2g_mR_s} + \frac{g_mR_s}{2\beta}
\]

According to the above expression, we can choose an optimal value of \(g_mR_s\) to minimize the noise. But the second term \(r_b/R_s\) is fixed for a given transistor dimension
The device can be scaled to lower the net current density in order to delay the onset of the Kirk Effect.

The base resistance also drops when the device is made larger.
We can thus see that BJT transistor sizing involves a compromise:

- The transconductance depends only on $I_C$ and not the size (first order)
- The charge storage effects and $f_T$ only depend on the base transit time, a fixed vertical dimension.
- A smaller device has smaller junction area but can only handle a given current density before Kirk effect reduces performance
- A larger device has smaller base resistance $r_b$ but larger junction capacitance
Let’s partition the input noise current into two components, a component correlated (“parallel”) to the noise voltage and a component uncorrelated (“perpendicular”) of the noise voltage

\[ i_n = i_c + i_u \]

where we assume that \( <i_u, v_n> = 0 \) and

\[ i_c = Y_C v_n \]

We can therefore write

\[ v_{eq} = v_n (1 + Y_C Z_S) + Z_S i_u \]
Noise Figure of Two-Port

Which is a sum of uncorrelated random variables. The variance is thus the sum of the variances

$$\overline{\nu^2_{eq}} = \overline{\nu^2_n} |1 + Y_C Z_S|^2 + |Z_s|^2 \overline{i_u^2}$$

This allows us to immediately write the noise figure as

$$F = 1 + \frac{\overline{\nu^2_n} |1 + Y_C Z_S|^2 + |Z_s|^2 \overline{i_u^2}}{\overline{\nu^2_s}}$$

Let $$\overline{\nu^2_n} = 4kTBR_n$$, $$\overline{i_u^2} = 4kTBG_u$$, and $$\overline{\nu^2_s} = 4kTBR_s$$. Then

$$F = 1 + \frac{R_n |1 + Y_C Z_S|^2 + |Z_s|^2 G_u}{R_s}$$
Optimum Source Impedance

If we let $Y_c = G_c + jB_c$, $Y_s = Z_s^{-1} = G_s + jB_s$, it's not to difficult to show that the optimum source impedance to minimize $F$ is given by

$$B_{opt} = B_s = -B_c$$

$$G_{opt} = G_s = \sqrt{\frac{G_u}{R_n} + G_c^2}$$

The minimum achievable noise figure is

$$F_{min} = 1 + 2G_c R_n + 2\sqrt{R_n G_u + G_c^2 R_n^2}$$

For $G_c = 0$, this reduces to our previously derived expression.
Minimum Noise $F_{min}$

- Very similar to the uncorrelated case, we have

$$F = F_{min} + \frac{R_n}{G_s}|Y_s - Y_{opt}|^2$$

- This equation states that if the source impedance $Y_s \neq Y_{opt}$, the noise figure will be larger by a factor of the “distance” squared times the factor $R_n/G_s$.

- A good device should have a low $R_n$ so that the noise match is not too sensitive.
Consider the following noise sources:

- \( R_s \) : \( \overline{v_s^2} = 4kTBR_s \)
- \( R_g \) : \( \overline{v_g^2} = 4kTBR_g \)
- \( R_{ch} \) : \( \overline{i_d^2} = 4kTBg_{d0}\gamma B \)
- \( R_L \) : \( \overline{i_L^2} = 4kTBG_L \)
Summing all the noise at the output (assume low frequency)

\[
\bar{i}_o^2 = \bar{i}_d^2 + \bar{i}_L^2 + (\bar{v}_g^2 + \bar{v}_s^2)g_m^2
\]

Which results in the noise figure

\[
F = 1 + \frac{\bar{v}_g^2}{\bar{v}_s^2} + \frac{\bar{i}_d^2 + \bar{i}_L^2}{g_m^2 \bar{v}_s^2}
\]

\[
= 1 + \frac{R_g}{R_s} + \frac{g_{d0} \gamma + G_L}{R_s g_m^2}
\]
• Assume $g_m = g_{d0}$ (long channel)

$$= 1 + \frac{R_g}{R_s} + \frac{\gamma}{g_m R_s} + \frac{G_L G_S}{g_m^2}$$

• If we make $g_m$ sufficiently large, the gate resistance will dominate the noise.

• The gate resistance has two components, the physical gate resistance and the induced channel resistance

$$R_G = R_{poly} + \delta R_{ch} = \frac{1}{3} \frac{W}{L} R^\square + \frac{1}{5} \frac{1}{g_m}$$

• The factors $1/3$ and $1/5$ come from a distributed analysis (EECS 117). They are valid for single-sided gate contacts.
To reduce the gate resistance, a multi-finger layout approach is commonly adopted. As a bonus, the junction capacitance is reduced due to the junction sharing.
If we repeat the calculation at medium frequencies, ignoring $C_{gd}$, we simply need to input refer the drain noise taking into account the frequency dependence of $G_m$

$$G_m = g_m \frac{1/(j\omega C_{gs})}{1/(j\omega C_{gs}) + R_s + R_g}$$

$$= \frac{g_m}{1 + j\omega C_{gs}(R_s + R_g)}$$
The drain noise is input referred by the magnitude squared

\[ |G_m|^2 = g_m^2 (1 + \omega^2 C_{gs}^2 (R_s + R_g)^2) \]

So the noise figure is simply given by (neglect the noise of \(R_L\))

\[ F = 1 + \frac{R_g}{R_s} + \frac{\gamma}{\alpha} (1 + \omega^2 C_{gs}^2 (R_s + R_g)^2) \]

Assume that \(R_s \gg R_g\) (good layout). The “high” frequency noise is given by

\[ F_\infty = 1 + \frac{\gamma \omega^2 C_{gs}^2 R_s^2}{\alpha g_m R_s} = 1 + \frac{\gamma}{\alpha} \left( \frac{\omega}{\omega_T} \right)^2 g_m R_s \]